The phase diagram of QCD with isospin chemical potential

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Contents

1. Introduction

QCD at finite isospin chemical potential

QCD at finite chemical potential ($N_f = 2$):
u quark: μ_u d quark: μ_d

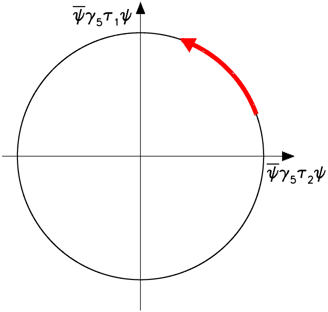
- Can be decomposed in baryon and isospin chemical potentials:
  $\mu_B = 3(\mu_u + \mu_d)/2$ and $\mu_I = (\mu_u - \mu_d)/2$
- Non-zero μ_I introduces an asymmetry between isospin ± 1 particles
  Positive μ_I: $\Rightarrow$ More protons than neutrons!
- Such situations occur regularly in nature:
  - Within nuclei with # neutrons > # protons.
  - Within neutron stars.
  - ...
- However: Usually $\mu_I \ll \mu_B$.

Here: consider $\mu_B = 0$!

Symmetry breaking

Introduction of μ_I: $D \rightarrow D + \mu_I \gamma_0 \tau_3$

$\Rightarrow$ Breaks $SU_V(2)$ explicitly to $U_{\tau_3}(1)$.



- $U_{\tau_3}(1)$ broken spontaneously by a charged pion condensate

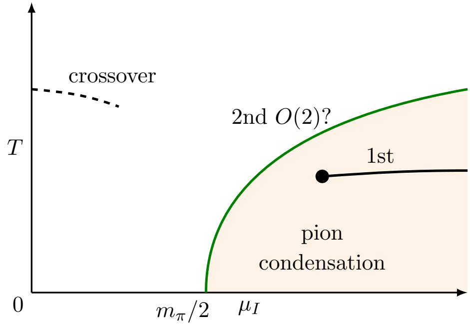
$$\langle \bar{\psi}\gamma_5\tau_{1,2}\psi \rangle$$

- At $T = 0$:
  This happens when $\mu_I = m_\pi/2$!

- 1 Goldstone mode appears!

Expected phase diagram

Exploring the phase diagram using χPT at finite μ_I: [Son, Stephanov, PRL86 (2001)]



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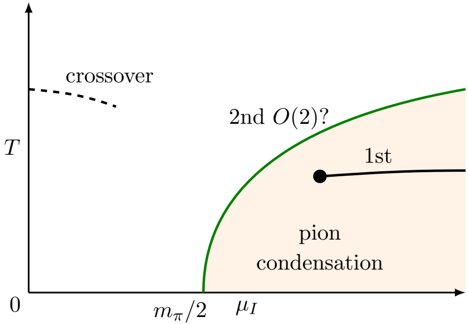

First lattice simulations ($N_t = 4$, $N_f = 2$, $m_\pi > m_\pi^{\mathrm{phys}}$):

1st order deconfinement and 2nd order curve join?

$\Rightarrow$ Existence of tri-critical point? [Kogut, Sinclair, PRD66 (2002); PRD70 (2004)]

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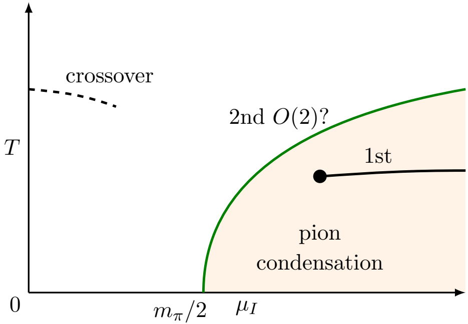

First lattice simulations (4×8^3, $N_f = 8$, $m_\pi > m_\pi^{\rm phys}$ – 1st order region):

Transition becomes weaker with μ_I!

$\Rightarrow$ Existence of tri-critical point? [de Forcrand, Stephanov, Wenger, PoS LAT2007]

2. Simulation setup and λ extrapolation

Lattice action

[G. Endrődi, PRD90 (2014)]

- Gauge action: Symanzik improved
- Mass-degenerate u/d quarks:

  Fermion matrix: [Kogut, Sinclair, PRD66 (2002); PRD70 (2004)]

  $$M = \begin{pmatrix} D(\mu) & \lambda\gamma_5 \\ -\lambda\gamma_5 & D(-\mu) \end{pmatrix}$$

  $D(\mu)$: staggered Dirac operator with $2\times$-stout smeared links

  λ: small explicit breaking of residual symmetry

  - Necessary to observe spontaneous symmetry breaking at finite V.
  - Serves as a regulator in the pion condensation phase.
- Strange quark: rooted staggered fermions (no chemical potential)
- Quark masses are tuned to their physical values.
- Lattice sizes: $6\times16^3, 24^3, 32^3$, $8\times24^3, 32^3, 40^3$, $10\times28^3, 40^3$

  ...

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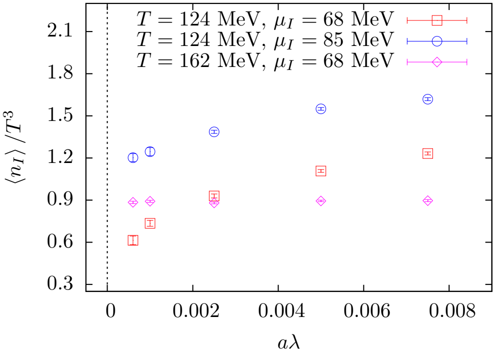
λ-extrapolations

For physical results: λ needs to be removed!

Problem: dependence on λ is not known! (at least for most of the observables)



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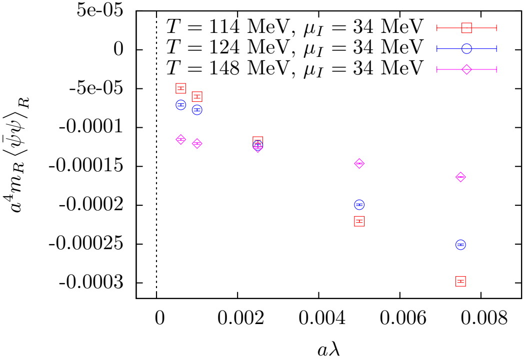

λ-extrapolations

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Problem: dependence on λ is not known! (at least for most of the observables)

Best possibility for model independence:

- Use a (cubic) spline extrapolation.
  - Fix one of the external points.
  - Leave the associated outer deriatives free.
    (additional free parameters)
- To stabilise the extrapolation:

  Need to assume that last two points lie on a (cubic) curve!

Remaining systematic effect: Position of nodepoints influences the result!

λ-extrapolations

Possible solution: Perform a "spline Monte-Carlo" [see S. Borsanyi]

- Average "all" splines with a similarly good description of the data.

  Allow for changes of # of nodes and node positions.
- Splines are weighted according to some suitable "action" S.
  Two possibilities:
  - Use the Akaike information criterion: $S_{\mathrm{AIC}} = 2N_P + \chi^2$
  - Use the negative goodness of the fit: $S_{\mathrm{GOD}} = P(\chi^2, N_{\mathrm{dof}}) - 1$

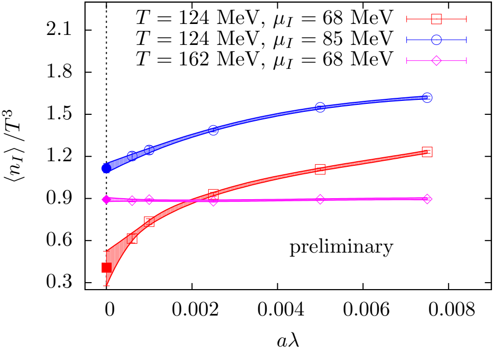
  $P(\chi^2, N_{\mathrm{dof}}) = \frac{\gamma(\chi^2/2, N_{\mathrm{dof}}/2)}{\Gamma(N_{\mathrm{dof}}/2)}$ – cumulative χ^2 distribution function
  (γ: lower incomplete gamma fct.)
- Problem: oscillating solutions

  $\Rightarrow$ Include some measure δ for oscillations

  Full action: $S = S_{\mathrm{AIC/GOD}} + f \times \delta$ (parameter f needs to be tuned)

λ-extrapolations

Results with S_{AIC} and $f = 10.0$:



3. Pion condensation phase

The pion condensate

Pion condensation $\Rightarrow$ Non-zero pion condensate

$$\langle \pi \rangle = \frac{T}{V} \frac{\partial \log Z}{\partial \lambda}$$

- Renormalisation:
  - additive divergences vanish for $\lambda \to 0$
  - multiplicative renormalisation: $Z_\pi = Z_\lambda^{-1} = Z_{m_{ud}}^1$

  Fully renormalised version (normalised):

$$\Sigma_\pi = \frac{m_{ud}}{m_\pi^2 F_\pi^2} \langle \pi \rangle$$

- Problem: λ extrapolation is very steep!

  Transition point is defined by onset of $\Sigma_\pi \neq 0$ (demands high accuracy)!

$\Rightarrow$ Another formalism is needed!

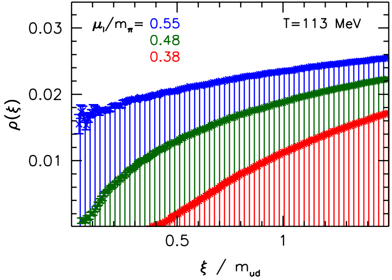
New method: Singular value representation

Rewrite the pion condensate as: $\pi = i\mathrm{Tr}(M^{-1}\gamma_5\tau_2) = \mathrm{Tr}[2\lambda/(D^\dagger D + \lambda^2)]$

Represented in terms of singular values $(D^\dagger D\psi_i = \xi_i^2\psi_i)$:

$$\pi = \frac{T}{V}\sum_i \frac{2\lambda}{\xi_i^2+\lambda^2} = \int d\xi\,\rho(\xi)\,\frac{2\lambda}{\xi^2+\lambda^2} \quad \overset{\lambda\to 0}{\to} \quad \pi\rho(0)$$

(derived in [Kanazawa, Wettig, Yamamoto, JHEP1112 (2011)] – analogue to Banks-Casher)

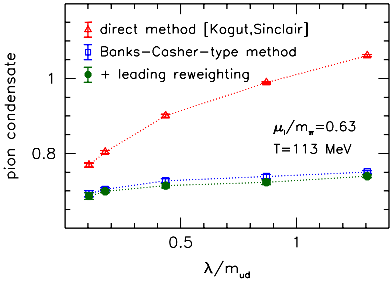


Improvent in λ-extrapolation

On top: Perform a leading order reweighting:

$$\langle \pi \rangle_{\mathrm{rew}} = \langle \pi W_\lambda \rangle \,/\, \langle W_\lambda \rangle \qquad W_\lambda = \exp[-\lambda V_4 \pi + \mathcal{O}(\lambda^2)]$$

Resulting extrapolation mostly flat:



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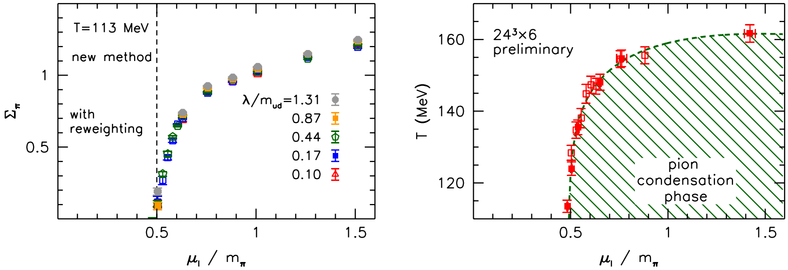
Combination allows for an accurate extraction of phase boundary:



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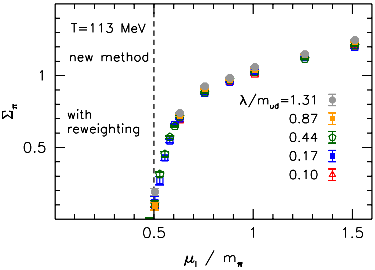

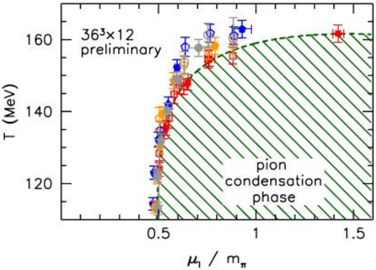

4. Small isospin chemical potential

Definition of the transition point

Investigate the finite temperature transition (crossover) for $\mu_I < \mu_I^C$.

Transition temperature T_C is defined by the behaviour of $\langle\bar{\psi}\psi\rangle$:

- Standard: Use the inflection point of the condensate.
- Easier alternative for $\mu_I < \mu_I^C$:

  Use the point where subtracted condensate reaches a certain value.
  (that value has to be known from $\mu = 0$ – Silver Blaze)

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Here: Use subtracted u/d condensate renormalised by the quark mass:

$$m_R \left\langle\bar{\psi}\psi\right\rangle_R = m_{u/d}\Big(\left\langle\bar{\psi}\psi\right\rangle - \left\langle\bar{\psi}\psi\right\rangle\big|_{T=0,\mu_I=0}\Big)$$

Value at the transition (in continuum): $m_R \left\langle\bar{\psi}\psi\right\rangle_R = -7.407\,10^{-5}\ \mathrm{GeV}^4$

[BW: Borsanyi *et al*, JHEP1009 (2010)]

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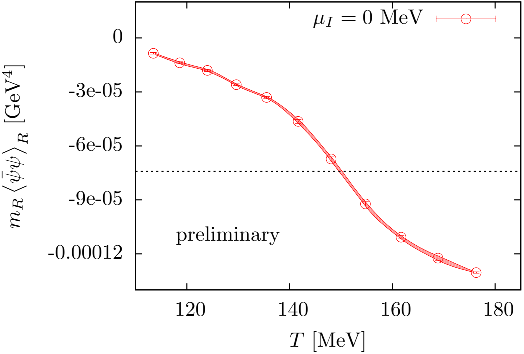

Curves: Simple spline interpolation.

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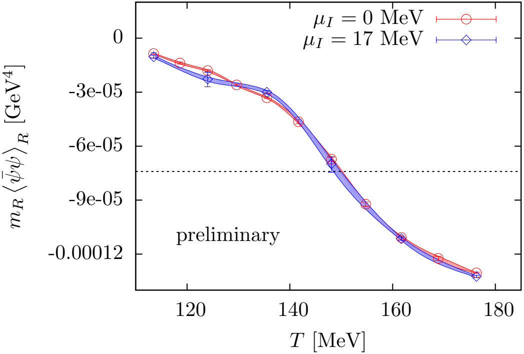

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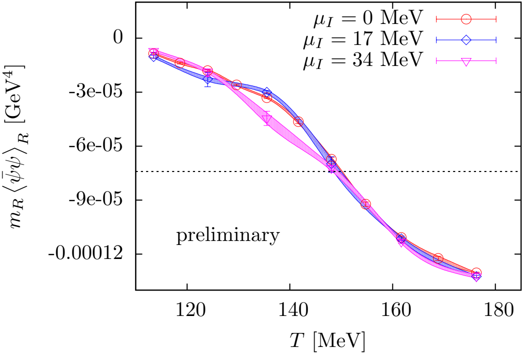

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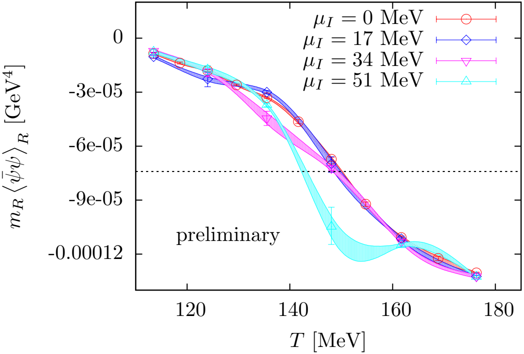

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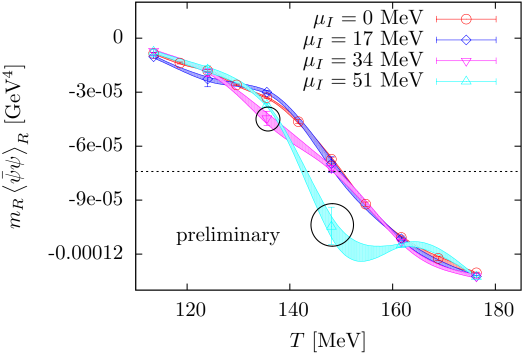
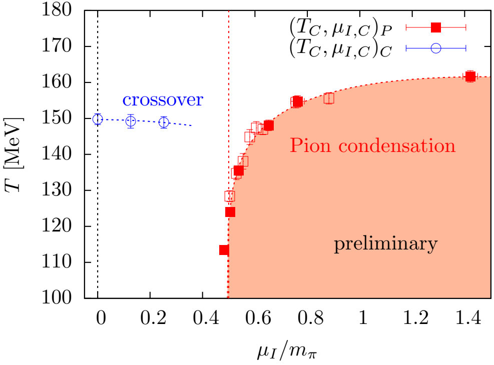


Curves: Simple spline interpolation.

Biggest problem: λ-extrapolation!

Phase diagram for 6×24^3



Phase diagram: Open questions

- Where is the meeting point between crossover and pion condensation boundary?
- What is the order of the transition on the boundary?

  Presence of a tri-critical point?
- What happens in the $\mu_I \to \infty$ limit?
- More generally:
  Are the deconfinement transition and the boundary of the pion condensation phase equivalent?

5. Comparison to Taylor expansion around $\mu_I = 0$

Taylor expansion around $\mu_I = 0$

Simulations at finite μ_B suffer from a sign problem!

One of the most important tools to obtain information at finite μ_B:
Taylor expansion around $\mu_B = 0$.

However: Range of applicability at a given order is unknown!

Here: test Taylor expansion method using our data for $\mu_I \neq 0$

- As an observable we use the isospin density (analogue to Baryon density):

$$\langle n_I \rangle = \frac{T}{V} \frac{\partial \log Z}{\partial \mu_I}$$

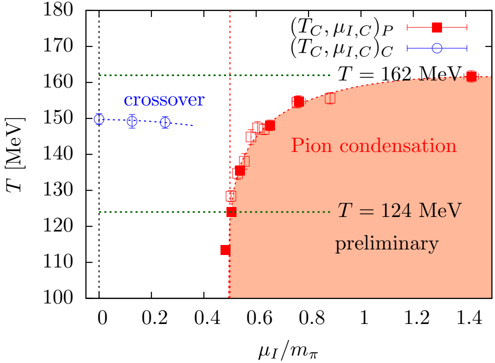
- Associated Taylor expansion (follows from expansion of pressure p/T^4):

$$\frac{\langle n_I \rangle}{T^3} = c_2 \left(\frac{\mu_I}{T} \right) + \frac{c_4}{6} \left(\frac{\mu_I}{T} \right)^3$$

Take values from Budapest-Wuppertal [BW: Borsanyi *et al*, JHEP1201 (2012)]

Comparison to data at finte μ_I

Compare data for 6×24^3 lattice:



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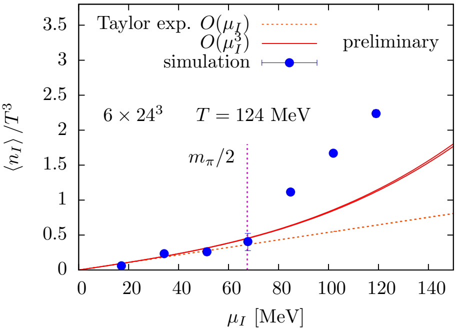

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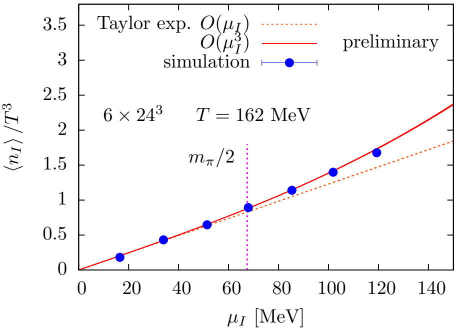

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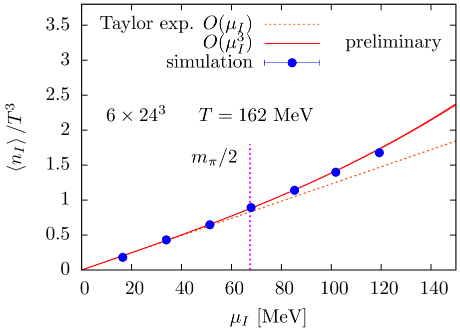
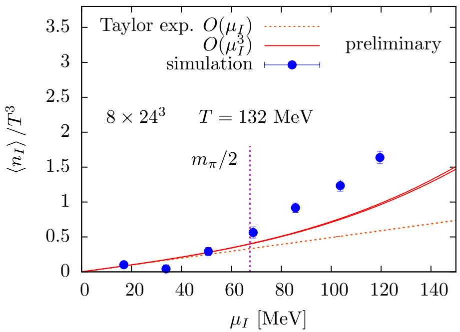


Note: Here compare to coefficients from 6×18^3 lattices.

(But finite size effects in coefficients negligible!)

Comparison to data at finte μ_I

Compare data for 8×24^3 lattice, $T < T_C$:



Here: Coefficients computed on the same lattice size!

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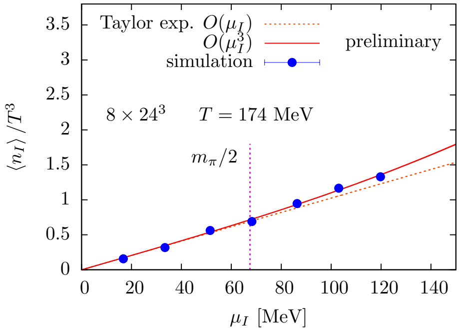

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Comparison to data at finte μ_I

- For $T < T_C$:

  Good agreement between expansion to $O(\mu_I^3)$ and data for $\mu_I < \mu_I^C$.

  Note: For $\mu_I > \mu_I^C$ the system is in another (pion condensation) phase.
  $\Rightarrow$ We do not expect agreement between expansion and data.
- For $T > T_C$:

  Good agreement between all data and expansion to $O(\mu_I^3)$
- Generally: $O(\mu_I^5)$ contributions appear to be neligible!
- It would be interesting to simulate at larger values of μ_I for $T > T_C$ to see for how long the agreement persists.

Summary and Perspectives

- We have investigated the phase structure of QCD at finite isospin chemical potential μ_I.
- Biggest issue: Full control of λ-extrapolations (need to be improved).
- We have mapped the transition to the pion condensation phase using the pion condensate.
  (new λ-extrapolation method helps a lot!)
- The crossover temperatures starting from $\mu_I = 0$ decrease slightly at finite μ_I.
- Results from Taylor expansion to $O(\mu_I^3)$ agree well with results looked at so far.
  (except for results in the pion condensation phase – as expected)

To do:

- Perform continuum limit and look at thermodynamic limit.
- Determine the order of the transitions to the pion condensation phase.

  Presence of a tricritical point?
- There are plenty of other interesting things to do with this theory!

Thank you for your attention!