QSFP 2021

"Atom interferometry and gravitational wave detection"

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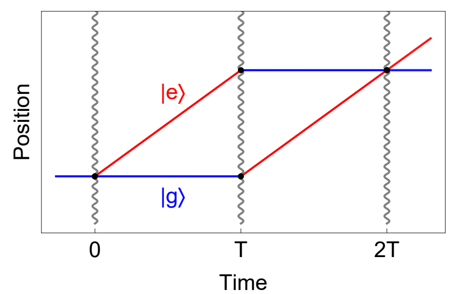
Tutorial 1: Atom interferometer accelerometer phase shift

Here we will calculate the phase shift of a light-pulse atom interferometer subject to a uniform gravitational acceleration g. The trajectories of the atomic ground and excited states are shown in blue and red, respectively, while the light pulses are shown in gray at times 0, T, and $2T$. The interferometer uses a three-pulse



Mach-Zehnder sequence ($\frac{\pi}{2} - \pi - \frac{\pi}{2}$) with time T between pulses and beamsplitter momentum $\hbar k$ at each atom-light interaction point (i.e., single-photon atom optics). Use the non-relativistic phase shift analysis and ignore any delays due to the finite speed of light.

1. First we solve for the phase shift using perturbation theory. Work out the propagation phase difference between interferometer arms, treating gravity as a perturbation (i.e., calculate the atom propagation phase along *unperturbed* trajectories shown in the figure).
   Hint: The atom propagation phases $\phi_{u,l}^{prop}$ along the upper and lower arms of the interferometer (u, l) are given by $\phi_{u,l}^{prop} = \frac{1}{\hbar}\int_0^{2T} L_{u,l}(t)\, dt$, where $L_{u,l}$ is the perturbing Lagrangian.

2. Now calculate the phase shift again, this time without using perturbation theory. Unlike in part 1, this means that, for example, the trajectories in the lab frame are parabolas. For simplicity, analyze the problem in the freely falling frame.
   Hint: The atom propagation phase in the free-falling frame is zero (discuss why) - consider the acceleration of atoms towards the laser and the associated laser phase imprinted on the interferometer arms.

3. Find an expression for the acceleration sensitivity (in $g/\sqrt{\mathrm{Hz}}$) assuming the phase noise is limited by atom shot noise. Assume an average detected atom flux of n atoms per second.

4. Now assume the interferometer uses large momentum transfer (LMT) atom optics, with momentum $N\hbar k$ at each interaction. Show that the above phase shift due to gravity is N times larger.
   Hint: Consider your solutions to parts 1 and 2 - how does LMT change this picture?

5. Calculate the phase shift (to lowest order) due to a gravity gradient. Assume the strength of gravity varies linearly as $g(z) = g + T_{zz}z$, where T_{zz} is the gravity gradient.
   Hint: Consider the gravitational potential energy in the presence of a uniform gravity gradient, and use a perturbative approach.

Tutorial 1 Solutions

Solution to part 1:

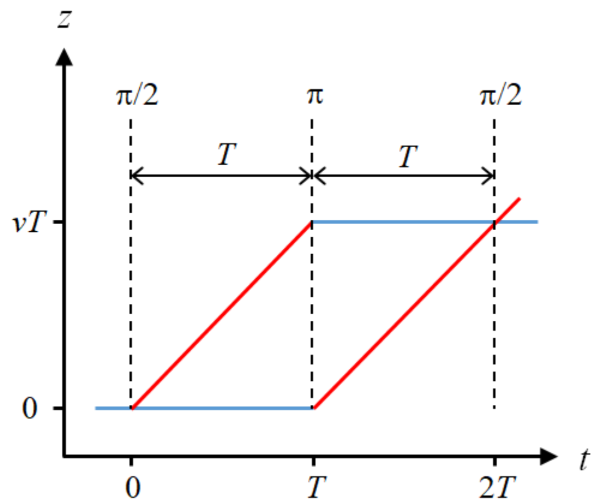


Figure 1: Pulse sequence and timing for the Mach-Zehnder interferometer configuration.

The atom propagation phases $\phi_{u,l}^{prop}$ along the upper and lower arms of the interferometer (u, l) are given by

$$\phi_{u,l}^{prop} = \frac{1}{\hbar}\int_0^{2T} L_{u,l}(t)\,dt \tag{1}$$

For the Mach-Zehnder interferometer in a uniform gravitational field, we have $L(z, t) = mgz(t) + E_i(t)$, where $E_i(t)$ is the internal energy of the atom. We can choose an energy scale such that the atomic ground state energy is $E_g = 0$ and the excited energy is E_e. The trajectories $z_{u,l}(t)$ along each arm are given by:

$$z_u(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ v_r t & \text{for } 0 < t < T \\ v_r T & \text{for } t \geq T \end{cases} \tag{2}$$

$$z_l(t) = \begin{cases} 0 & \text{for } t \leq T \\ v_r(t - T) & \text{for } T < t < 2T \\ v_r T & \text{for } t \geq 2T \end{cases} \tag{3}$$

where $v_r \equiv \hbar k/m$ is the recoil velocity of the atom when it absorbs the photon with momentum $\hbar k$.

We can now calculate the atom propagation phase along each arm, as:

$$\phi_u^{prop} = \frac{1}{\hbar}\int_0^{T} [mgv_r t + E_e]\,dt + \frac{1}{\hbar}\int_T^{2T} mgv_r T\,dt \tag{4}$$

$$= \frac{1}{\hbar}\left[E_e T + \frac{1}{2}mgv_r T^2 + mgv_r T^2\right] \tag{5}$$

$$= \frac{1}{\hbar}\left[E_e T + \frac{3}{2}mgv_r T^2\right] \tag{6}$$

and for the lower arm:

$$\phi_l^{prop} = \frac{1}{\hbar}\int_0^T 0\,dt + \frac{1}{\hbar}\int_T^{2T}\left[mgv_r(t-T)+E_e\right]\,dt \tag{7}$$

$$= \frac{1}{\hbar}\left[E_eT + \frac{1}{2}mgv_r(2T)^2 - \frac{1}{2}mgv_rT^2 - mgv_rT^2\right] \tag{8}$$

$$= \frac{1}{\hbar}\left[E_eT + \frac{1}{2}mgv_rT^2\right] \tag{9}$$

which gives the result for the atom propagation phase difference:

$$\Delta\phi = \phi_u^{prop} - \phi_l^{prop} \tag{10}$$

$$= \frac{1}{\hbar}mgv_rT^2 \tag{11}$$

$$= \frac{1}{\hbar}mg\frac{\hbar k}{m}T^2 \tag{12}$$

$$= kgT^2 \tag{13}$$

Alternatively, the phase can be calculated by taking advantage of the geometry of the unperturbed trajectories, noting that the solution to the integral

$$\Delta\phi^{prop} = \frac{mg}{\hbar}\int_0^{2T}\left[z_u(t)-z_l(t)\right]\,dt \tag{14}$$

is just given by the area of the parallelogram formed by the upper and lower paths, equal to v_rT^2. This returns the same result for the phase shift,

$$\Delta\phi^{prop} = \frac{mg}{\hbar}v_rT^2 = \frac{mg}{\hbar}\frac{\hbar k}{m}T^2 = kgT^2. \tag{15}$$

Solution to part 2:

In the free-falling frame, the atom propagation phase $\Delta\phi^{prop} = 0$. This is because the gravitational potential energy of the atom is zero, and by the same set of steps as part 1 above, the phase accumulated due to the atom's internal energy will be the same ($E_eT/\hbar$) in both the upper and lower arms.

In the free-falling frame, the interferometer signal results from the acceleration of the atoms towards the laser. In the accelerating frame, the laser frequency $\omega_L(t)$ experiences a time-dependent Doppler shift[1],

$$\omega_L(t) = \omega_L\left(1+\frac{v_{\text{frame}}(t)}{c}\right) = \omega_L\left(1+\frac{gt}{c}\right), \tag{16}$$

which, for a plane wave propagating along z, and assuming without loss of generality that $\phi(0,0)=0$, implies a laser phase[2]:

$$\phi_L(z,t) = \phi_L(z,0) + \int_0^t \omega_L(t)\,dt \tag{17}$$

$$= -kz + \omega_Lt + \frac{gt^2\omega_L}{2c} \tag{18}$$

Now we can calculate the laser phases imprinted on the upper and lower arms of the interferometer, for the atom exiting the lower output port.

The atoms on the upper path absorb a photon at $z=0, t=0$, then undergo stimulated emission of a photon at $z=v_rT, t=T$, before continuing in a straight line through to the lower output port. The laser phase imprinted on the upper arm is therefore:

$$\phi_u^{Laser} = \phi_L(0,0) - \phi_L(v_rT,T) \tag{19}$$

$$= kv_rT - \omega_LT - \frac{gT^2\omega_L}{2c} \tag{20}$$

[1]This equation is strictly the non-relativistic Doppler shift, valid because $v_{\text{frame}} \ll c$.

[2]Strictly speaking, the wavevector k also experiences a Doppler effect, to first order given by $k(t) \approx k(0)(1+v_{\text{frame}}/c)$. However, the laser phase resulting from the wavevector Doppler effect is suppressed by a factor v/c with respect to the leading-order term.

The atoms on the lower path absorb a photon at $z = 0, t = T$, then undergo stimulated emission of a photon at $z = v_r T, t = 2T$ to exit from the lower output port. The laser phase imprinted on the lower arm is therefore

$$\phi_l^{Laser} = \phi_L(0,T) - \phi_L(v_r T, 2T) \tag{21}$$

$$= \omega_L T + \frac{gT^2\omega_L}{2c} + kv_r T - 2\omega_L T - \frac{g(2T)^2\omega_L}{2c} \tag{22}$$

$$= kv_r T - \omega_L T - \frac{3}{2}\frac{gT^2\omega_L}{c} \tag{23}$$

The difference in laser phase is therefore given by:

$$\Delta\phi = \phi_u^{Laser} - \phi_l^{Laser} \tag{24}$$

$$= \frac{gT^2\omega_L}{c} \tag{25}$$

$$= kgT^2 \tag{26}$$

where in the last step we have used the relation $k = \omega_L/c$.
Solution to part 3:

$$\frac{\delta g}{g} = \frac{\delta\phi}{\Delta\phi}$$

$$\delta g = \frac{\delta\phi}{kT^2}$$

At the point of maximum sensitivity (on the equator of the Bloch sphere), the phase noise is set by the signal-to-noise ratio (SNR) of the detected atom population:

$$\delta\phi = \frac{1}{\text{SNR}}$$

Assuming atom shot noise, the quantum projection noise (per shot) of the population is $1/\sqrt{N}$, where N is the number of atoms per shot. Since the signal has size N, we have $\text{SNR} = N/\sqrt{N} = \sqrt{N}$.

Assuming each shot has duration τ, we have $N = n\tau$ in terms of the average atom flux. The acceleration sensitivity per shot is

$$\delta g = \frac{1}{kT^2\sqrt{N}} = \left(\frac{1}{kT^2\sqrt{n}}\right)\frac{1}{\sqrt{t}}$$

The acceleration amplitude noise spectral density is then

$$\overline{\delta g} = \frac{1}{kT^2\sqrt{n}}$$

Solution to part 4

There are two approaches. First, using the approach of part 1, the LMT atom optics result in a larger recoil velocity of Nv_r. As a result, the area of the interferometer is larger: $A = (Nv_r T)T$, leading to

$$\Delta\phi = NkgT^2$$

Alternatively, using the approach in part 2, the LMT atom optics are realized by replacing each of the original three light pulses with pulse sequences that transfer N photons of momentum. Each light pulse imprints a laser phase to the atom. Assuming the N pulses arrive over a time short compared to the interferometer pulse spacing $\sim T$, the phases of all the pulses in a given sequence are approximately the same (the atom is at approximately the same position). As a result we have

$$\phi_u^{Laser} = N\phi_L(0,0) - N\phi_L(Nv_r T, T) \tag{27}$$

$$\phi_l^{Laser} = N\phi_L(0,T) - N\phi_L(Nv_r T, 2T) \tag{28}$$

Using the results from part 2 above,

$$\phi_u^{Laser} = Nkv_rT - N\omega_L T - N\frac{gT^2\omega_L}{2c} \tag{29}$$

$$\phi_l^{Laser} = Nkv_rT - N\omega_L T - N\frac{3}{2}\frac{gT^2\omega_L}{c} \tag{30}$$

$$\Delta\phi = \phi_u^{Laser} - \phi_l^{Laser} = NkgT^2 \tag{31}$$

Solution to part 5

The gravitational potential energy in the presence of a uniform gravity gradient is

$$U(z) = mgz + \frac{1}{2}mT_{zz}z^2 \tag{32}$$

As a check, the associated force if $F = -\partial_z U = -mg - mT_{zz}z$ and so we see the gravitational acceleration $a = g + T_{zz}z$ varies linearly with gradient T_{zz}.

To calculate the phase shift due to a gravity gradient, we treat $\delta U = \frac{1}{2}mT_{zz}z^2$ as a perturbation and integrate the perturbing Lagrangian over the unperturbed paths shown in Fig. .

$$\phi_u^{prop} = \frac{1}{\hbar}\int_u \delta U dt = \frac{1}{\hbar}\int_0^{2T} \frac{1}{2}mT_{zz}z_u^2(t)\, dt \tag{33}$$

$$\phi_l^{prop} = \frac{1}{\hbar}\int_l \delta U dt = \frac{1}{\hbar}\int_0^{2T} \frac{1}{2}mT_{zz}z_l^2(t)\, dt \tag{34}$$

Using Eqs. 2,

$$\phi_u^{prop} = \frac{mT_{zz}}{2\hbar}\left(\int_0^T (v_r t)^2\, dt + \int_T^{2T} (v_r T)^2\, dt\right) \tag{35}$$

$$\phi_l^{prop} = \frac{mT_{zz}}{2\hbar}\left(\int_0^T (0)^2\, dt + \int_T^{2T} (v_r(t-T))^2\, dt\right) \tag{36}$$

Notice that two of these integrals are equal (corresponding to the excited state portions of each path), so they cancel in the phase difference:

$$\Delta\phi = \phi_u^{prop} - \phi_l^{prop} = \frac{mT_{zz}}{2\hbar}\int_T^{2T} (v_r T)^2\, dt \tag{37}$$

$$= \frac{mT_{zz}}{2\hbar}v_r^2 T^3 \tag{38}$$

Using $v_r = \hbar k/m$,

$$\Delta\phi = \frac{\hbar k^2}{2m}T_{zz}T^3 \tag{39}$$