QSFP 2021

“Atom interferometry and gravitational wave detection”

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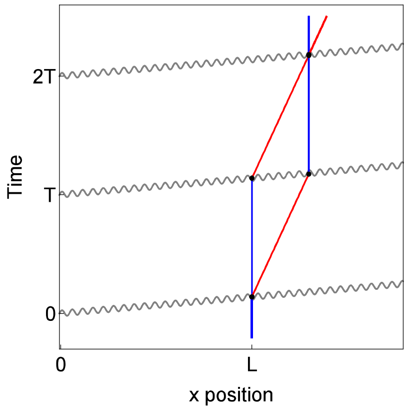
Tutorial 2: Gravitational wave sensitivity

Here we will calculate the atom interferometer phase shift due to a gravitational wave. In this problem we will make use of the relativistic treatment of atom interferometry. Consider the gravitation wave metric in the TT gauge:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 + h(t)(dx^2 - dy^2) \tag{40}$$

with $h(t) \equiv h\cos\left(\omega(t - \frac{z}{c}) + \phi_0\right)$ the dimensionless stain. In proposed detectors such as MAGIS, a pair of atom interferometers separated by a large baseline are used to make a differential measurement that suppresses laser technical noise. In this problem, we will begin by analyzing the phase shift of a single atom interferometer at position $x = L$ as shown in the figure below. A laser source is assumed to be located



at $x = 0$ on the left and emits pulse that propagate to the right at times 0, T, and $2T$ to implement the $\frac{\pi}{2} - \pi - \frac{\pi}{2}$ sequence. The paths of the light pulses are shown in gray. The resulting atom paths are shown for both the ground (blue) and excited (red) states. Assume the excited state has energy $\hbar\omega_A$ with respect to the ground state. The following trigonometric identities are useful for part 4:

$$2\sin(a+b) - \sin(a+2b) - \sin(a) = 4\sin^2\left(\frac{b}{2}\right)\sin(a+b)$$
$$\sin(A+B) - \sin(A) = 2\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2} + A\right)$$

1. Find the null geodesic trajectory $x(t)$ for light propagating along the x-direction, to first order in the strain h, assuming initial conditions $x(t_0) = x_0$.

2. Invert the previous result to find the arrival time $t(x, x_0, t_0)$ of the light pulse at position x, assuming the pulse propagates along x from an initial position x_0, t_0. As before, use $h \ll 1$.

3. Calculate the atom propagation phases $\phi_{u,l} = \int_{u,l} L dt$ along the upper and lower interferometer arms, assuming three laser pulses interact with the atoms as shown in the figure and described in the following:

   (a) The first pulse begins at $x_0 = 0,\ t_0 = 0$, and when it reaches the atoms it puts the upper arm of the interferometer into the excited state while leaving the lower arm in the ground state

   (b) The middle pulse begins at $x_0 = 0,\ t_0 = T$ and excites the lower arm while de-exciting the upper arm

(c) The final pulse begins at $x_0 = 0,\ t_0 = 2T$, de-exciting the lower arm while leaving the upper arm in the ground state.

Hint: The atom propagation phase accumulates as $\omega_A \Delta t_{u,l}$ along each arm, for time intervals $\Delta t_{u,l}$ spent in the excited state. Hint 2: when calculating the intersection points of the light and atom paths, you can neglect the motion of the atom due to the recoil velocity $v_r = \hbar k/m$ (i.e. assume that the atom position is fixed at $x = L$ for both the upper and the lower arm). What condition must hold for this approximation to be valid?

4. Show that, to first order in L, the interferometer phase is $\Delta\phi = 2hL\omega_A \sin^2(\omega T/2)\cos(\phi_0 + \omega T + \pi)$ (that is, assuming $L\omega/c \ll 1$).

5. Find the strain sensitivity $\overline{h}(\omega)$ (in strain per $\sqrt{\mathrm{Hz}}$) assuming a phase noise amplitude spectral density of $\overline{\delta\phi}$ (in rad/$\sqrt{\mathrm{Hz}}$). What atom flux (in atoms/second) is required to achieve a phase resolution of $\overline{\delta\phi} = 10^{-3}$ rad/$\sqrt{\mathrm{Hz}}$, assuming the atom shot noise limit?

Tutorial 2 Solutions

Solution to part 1:

The light follows null geodesics which satisfy $ds = 0$. We consider one-dimensional light paths propagating along the x direction, so we take $dy = dz = 0$. We assume paths with $z = 0$ without loss of generality. Note that any deflection in z as a function of time will be $\sim \mathcal{O}(h)$, and can therefore only effect the x geodesic at second order in h. Thus motion in directions other than x can be safely ignored.

$$0 = c^2 dt^2 - dx^2 + h(t)dx^2 \tag{41}$$

The coordinate velocity fo the light is then found to be

$$\frac{dx}{dt} = \frac{c}{\sqrt{1-h(t)}} \approx c(1 + \tfrac{1}{2}h(t))$$

where the approximation takes advantage of the fact that $|h(t)| \ll 1$.

$$\frac{dx}{dt} = c(1 + \tfrac{1}{2}h\cos(\omega t + \phi_0)) \tag{42}$$

Integrating, assuming $x(t_0) = x_0$:

$$x(t) = c(t - t_0) + \frac{hc}{2\omega}\Big(\sin(\omega t + \phi_0) - \sin(\omega t_0 + \phi_0)\Big) + x_0 \tag{43}$$

Solution to part 2:

Inverting this equation to solve for $t(x)$:

$$t(x) = t_0 + \frac{1}{c}(x - x_0) - \frac{h}{2\omega}\Big(\sin(\omega\, t(x) + \phi_0) - \sin(\omega t_0 + \phi_0)\Big) \tag{44}$$

$$\approx t_0 + \frac{1}{c}(x - x_0) - \frac{h}{2\omega}\Big(\sin(\tfrac{\omega}{c}(x - x_0) + \omega t_0 + \phi_0) - \sin(\omega t_0 + \phi_0)\Big) \tag{45}$$

Note that above we used $t(x) \approx t_0 + \frac{1}{c}(x - x_0)$ inside the sine, which is sufficient up to first order h.

Solution to part 3:

Each light pulse starts at $x_0 = 0$. Assuming the atom is at $x = L$, and ignoring any motion of the atom due to recoil effects, the arrival times of the light determine the total time each arm spends in the excited state:

$$\phi_u = \omega_A\Big[t(x = L; t_0 = T) - t(x = L; t_0 = 0)\Big] \tag{46}$$

$$\phi_l = \omega_A\Big[t(x = L; t_0 = 2T) - t(x = L; t_0 = T)\Big] \tag{47}$$

where, for example, the notation $t(x = L; t_0 = T)$ indicates the time of arrival at position $x = L$ for the null geodesic launched at time $t_0 = T$ from $x_0 = 0$. Here we implicitly take the position of the atom to be fixed at $x = L$ for the duration of the interferometer. This is an approximation that neglects the motion of the atom due to the recoil velocity v_r. This approximation is valid so long as the size of the interferometer is small compared to the baseline ($v_r T \ll L$), so that the affect of the GW strain is largest for the light paths. This is the correct limit for most practical proposals for GW detection.

Expanding each arrival time using the result of part 2, we find,

$$\phi_u = \omega_A\Big[T - \tfrac{h}{2\omega}\sin(\phi_0) + \tfrac{h}{2\omega}\sin(\phi_0 + \tfrac{\omega L}{c}) + \tfrac{h}{2\omega}\sin(\phi_0 + \omega T) - \tfrac{h}{2\omega}\sin(\phi_0 + \tfrac{\omega L}{c} + \omega T)\Big]$$

$$\phi_l = \omega_A\Big[T - \tfrac{h}{2\omega}\sin(\phi_0 + \omega T) + \tfrac{h}{2\omega}\sin(\phi_0 + \tfrac{\omega L}{c} + \omega T) + \tfrac{h}{2\omega}\sin(\phi_0 + 2\omega T) - \tfrac{h}{2\omega}\sin(\phi_0 + \tfrac{\omega L}{c} + 2\omega T)\Big]$$

The propagation phase difference is

$$\begin{aligned}\Delta\phi &= \phi_u - \phi_l \\ &= -\frac{h\omega_A}{2\omega}\Big[\Big(2\sin(\phi_0 + \tfrac{\omega L}{c} + \omega T) - \sin(\phi_0 + \tfrac{\omega L}{c} + 2\omega T) - \sin(\phi_0 + \tfrac{\omega L}{c})\Big) \\ &\qquad - \Big(2\sin(\phi_0 + \omega T) - \sin(\phi_0 + 2\omega T) - \sin(\phi_0)\Big)\Big]\end{aligned}$$

Using the identity

$$2\sin(a+b) - \sin(a+2b) - \sin(a) = 4\sin^2\left(\frac{b}{2}\right)\sin(a+b)$$

for each set of terms inside the square brackets,

$$\Delta\phi = -\frac{h\omega_A}{2\omega}\Big[4\sin^2\left(\frac{\omega T}{2}\right)\sin(\phi_0 + \tfrac{\omega L}{c} + \omega T) - 4\sin^2\left(\frac{\omega T}{2}\right)\sin(\phi_0 + \omega T)\Big] \tag{48}$$

$$= -\frac{2h\omega_A}{\omega}\sin^2\left(\frac{\omega T}{2}\right)\Big[\sin(\phi_0 + \tfrac{\omega L}{c} + \omega T) - \sin(\phi_0 + \omega T)\Big] \tag{49}$$

Next, using the identity

$$\sin(A+B) - \sin(A) = 2\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2} + A\right)$$

inside the square brackets,

$$\Delta\phi = -\frac{4h\omega_A}{\omega}\sin^2\left(\frac{\omega T}{2}\right)\sin\left(\frac{\omega L}{2c}\right)\cos(\phi_0 + \tfrac{\omega L}{2c} + \omega T) \tag{50}$$

The propagation phase above is in fact the (leading order) phase response to a gravitational wave. Note that any separation phase can be neglected here for the same reason that we were able to neglect recoil corrections to the atom paths: the size of the interferometer is assumed to be small compared to L, making tidal effects across the interferometer negligible. For a single atom interferometer, there is a laser phase contribution arising from the phase of three light pulses: $\Delta\phi_{\text{laser}} = \phi_1 - 2\phi_2 + \phi_3$. In practice, gravitational wave detection requires a gradiometer, where the science signal is given by the phase difference between two atom interferometers on opposite sides of the baseline. In this case, the laser phase would be the same for both interferometers (since they share the same null geodesic light pulses), so $\Delta\phi_{\text{laser}}$ cancels in the differential measurement. Thus the propagation phase contains the complete GW sensitivity.

Solution to part 4:

Assuming $L\omega/c \ll 1$, we can expand to first order in L,

$$\Delta\phi \approx -\frac{4h\omega_A}{\omega}\frac{\omega L}{2c}\sin^2\left(\frac{\omega T}{2}\right)\cos(\phi_0 + \omega T) \tag{51}$$

$$= \frac{2hL\omega_A}{c}\sin^2\left(\frac{\omega T}{2}\right)\cos(\phi_0 + \omega T + \pi) \tag{52}$$

Solution to part 5:

The initial phase of the gravitational wave can vary shot-to-shot as the wave evolves in time. Letting $\phi_0 \equiv \omega t$, the observed phase shift as a function of time is

$$\Delta\phi(t) = \frac{2L\omega_A}{c}\sin^2\left(\frac{\omega T}{2}\right)h(t) \tag{53}$$

where here $h(t) \equiv \cos(\omega t + \omega T + \pi)$. Computing the rms of each side,

$$\left\langle \Delta\phi^2(t)\right\rangle = \left(\tfrac{2L\omega_A}{c}\sin^2\left(\tfrac{\omega T}{2}\right)\right)^2\left\langle h^2(t)\right\rangle \tag{54}$$

The rms phase and rms strain can be related to their respective amplitude spectral densities:

$$\left\langle \Delta\phi^2(t)\right\rangle = \int \overline{\delta\phi}(\omega)^2 d\omega \tag{55}$$

$$\left\langle h^2(t)\right\rangle = \int \overline{h}(\omega)^2 d\omega \tag{56}$$

Substituting in these expressions implies

$$\overline{\delta\phi}(\omega) = \tfrac{2L\omega_A}{c}\sin^2\left(\tfrac{\omega T}{2}\right)\overline{h}(\omega) \tag{57}$$

The strain sensitivity (in strain per $\sqrt{\text{Hz}}$) is then

$$\overline{h}(\omega) = \frac{\overline{\delta\phi}}{\frac{2L\omega_A}{c}\sin^2\left(\frac{\omega T}{2}\right)} \tag{58}$$

Assuming atom shot noise, the phase noise amplitude spectral density is $\overline{\delta\phi} = \frac{1}{\sqrt{n}}$ where n is the atom flux (in atoms/second). Therefore, a flux of $n = 10^6$ atoms/s corresponds to a phase resolution of $\overline{\delta\phi} = \frac{1}{\sqrt{n}} = 10^{-3}$ rad/$\sqrt{\text{Hz}}$.