Tutorial 2: Neutrino Oscillations

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Derivation of oscillation probability

$$\mathscr{L}_{\mathrm{CC}} = \sum_{\alpha=e,\mu,\tau} \left[\frac{g}{\sqrt{2}} \overline{\nu_{\alpha L}} \gamma^{\mu} \ell_{\alpha L} W^{+}_{\mu} + \frac{g}{\sqrt{2}} \overline{\ell_{\alpha L}} \gamma^{\mu} \nu_{\alpha L} W^{-}_{\nu} \right]$$

$$\nu_{\alpha L} = \sum_{j} U_{\alpha j}\, \nu_{jL}$$

Fields with definite masses m_j

PMNS mixing matrix

$$|\nu_\alpha\rangle = \sum_j U^*_{\alpha j} |\nu_j\rangle$$

States with definite masses m_j

Let's treat $|\nu_j\rangle$ as plane waves. Then, at a distance L and time t after production

$$|\nu_\alpha(t, L)\rangle = \sum_j U^*_{\alpha j}\, e^{-iE_j t + i p_j L} |\nu_j\rangle$$

A neutrino detector measures the neutrino flavour β

$$\langle \nu_\beta | = \sum_j U_{\beta j} \langle \nu_j |$$

Derivation of oscillation probability

The amplitude for transition $\nu_\alpha \to \nu_\beta$

$$\mathscr{M}_{\alpha\beta} = \langle \nu_\beta | \nu_\alpha(t, L) \rangle = \sum_{j,k} U^*_{\alpha j} U_{\beta k} \, e^{-iE_j t + i p_j L} \underbrace{\langle \nu_k | \nu_j \rangle}_{\delta_{jk}} = \sum_j U^*_{\alpha j} U_{\beta j} \, e^{-iE_j t + i p_j L}$$

The oscillation probability

$$P_{\alpha\beta} = |\mathscr{M}_{\alpha\beta}|^2 = \sum_{j,k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} \, e^{-i(E_j - E_k)t + i(p_j - p_k)L}$$

In all practical cases neutrinos travel almost at the speed of light, so

$$p_j \approx p_k \equiv p \approx E \qquad \text{and} \qquad t \approx L$$

$$E_j = \sqrt{p^2 + m_j^2} \approx p + \frac{m_j^2}{2E}$$

$$P_{\alpha\beta} = \sum_{j,k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} \, e^{-i\frac{\Delta m^2_{jk} L}{2E}} \qquad \Delta m^2_{jk} \equiv m_j^2 - m_k^2$$

2-flavour case

Exercise 1. Using the obtained formula, *i.e.*

$$P_{\alpha\beta} = \sum_{j,k} U^*_{\alpha j} U_{\beta j} U_{\alpha k} U^*_{\beta k} \, e^{-i\frac{\Delta m^2_{jk} L}{2E}} \qquad \Delta m^2_{jk} \equiv m_j^2 - m_k^2$$

and taking into account that in the case of two neutrino mixing

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

derive the expression for the oscillation probability $P_{e\mu}$ of $\nu_e \to \nu_\mu$

(Define $\Delta m^2 \equiv m_2^2 - m_1^2$)

Hint: Use the trigonometric identities

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = 2\cos(\theta)^2 - 1$$

2-flavour case

Exercise 1. Solution

$$P_{e\mu} = U_{e1}^2 U_{\mu 1}^2 + U_{e2}^2 U_{\mu 2}^2 + U_{e1} U_{\mu 1} U_{e2} U_{\mu 2} \left[e^{i\frac{\Delta m^2 L}{2E}} + e^{-i\frac{\Delta m^2 L}{2E}} \right]$$

$$= 2\cos^2\theta \sin^2\theta - 2\cos^2\theta \sin^2\theta \cos\left(\frac{\Delta m^2 L}{2E}\right)$$

$$= \frac{1}{2}\sin^2 2\theta \left[1 - \cos\left(\frac{\Delta m^2 L}{2E}\right)\right]$$

$$= \sin^2(2\theta)\ \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

$$\boxed{P_{e\mu} = \sin^2(2\theta)\ \sin^2\left(\frac{\Delta m^2 L}{4E}\right)}$$

Experimentally relevant units

Exercise 2. Express the argument of the second sine in terms of experimentally relevant units, *i.e.* find # in

$$\frac{\Delta m^2 L}{4E} = \# \frac{\Delta m^2\left(\mathrm{eV}^2\right)\, L\,(\mathrm{m})}{E\,(\mathrm{MeV})}$$

Hint: Use the relation

$$\hbar c = 1 \approx 197\ \mathrm{fm}\cdot\mathrm{MeV}$$

$$(1\ \mathrm{fm} = 10^{-15}\ \mathrm{m}\ \ \text{and}\ \ 1\ \mathrm{MeV} = 10^{6}\ \mathrm{eV})$$

Experimentally relevant units

Exercise 2. Solution

For $\Delta m^2 = 1\ \mathrm{eV}^2, L = 1\ \mathrm{m}$ and $E = 1\ \mathrm{MeV}$

$$\frac{1}{4} \cdot \frac{1\,\mathrm{eV}^2}{1\ \mathrm{MeV}} \cdot 1\,\mathrm{m} = \frac{1}{4} \cdot \frac{10^{-12}\,\mathrm{MeV}^2}{1\,\mathrm{MeV}} \cdot \frac{10^{15}}{197\,\mathrm{MeV}} = \frac{1000}{4 \cdot 197} \approx 1.27$$

Thus

$$\frac{\Delta m^2 L}{4E} \approx 1.27\,\frac{\Delta m^2\left(\mathrm{eV}^2\right)\ L\,(\mathrm{m})}{E\,(\mathrm{MeV})}$$

Oscillation length

Exercise 3.

- Taking into account that the dependence of $P_{e\mu}$ on L/E is periodic, derive the expression for the oscillation length $L_{\rm osc}$ associated with Δm^2
- Express $L_{\rm osc}$ in meters for Δm^2 in $\rm eV^2$ and E in $\rm MeV$

$$P_{e\mu} = \sin^2(2\theta)\,\sin^2\left(\frac{\Delta m^2 L}{4E}\right) \approx \sin^2(2\theta)\,\sin^2\left(1.27\,\frac{\Delta m^2\left(\mathrm{eV}^2\right)\,L\,(\mathrm{m})}{E\,(\mathrm{MeV})}\right)$$

Oscillation length

Exercise 3. Solution

- The period in $\dfrac{\Delta m^2 L}{4E}$ is π. Thus, we have

$$\left| \frac{\Delta m^2 L_{\rm osc}}{4E} \right| = \pi \quad \Rightarrow \quad L_{\rm osc} = \frac{4\pi E}{|\Delta m^2|}$$

- Alternatively

$$\left| 1.27 \frac{\Delta m^2 \left(\mathrm{eV}^2\right) \; L_{\rm osc}\,(\mathrm{m})}{E\,(\mathrm{MeV})} \right| = \pi \quad \Rightarrow \quad L_{\rm osc} = \frac{\pi}{1.27} \ \mathrm{m} \ \frac{E\,(\mathrm{MeV})}{\left|\Delta m^2 \left(\mathrm{eV}^2\right)\right|}$$

Therefore

$$L_{\rm osc} \approx 2.48 \ \mathrm{m} \ \frac{E\,(\mathrm{MeV})}{\left|\Delta m^2 \left(\mathrm{eV}^2\right)\right|}$$

Typical oscillation lengths

Exercise 4. Using the obtained formula, *i.e.*

$$L_{\text{osc}} \approx 2.48 \text{ m} \frac{E\,(\text{MeV})}{\left|\Delta m^2\left(\text{eV}^2\right)\right|}$$

compute the oscillation length for

- reactor neutrinos with $E = 1\ \text{MeV}$, assuming $\Delta m^2 = 7.4 \cdot 10^{-5}\ \text{eV}^2$
- accelerator neutrinos with $E = 1\ \text{GeV}$, assuming $\Delta m^2 = 2.5 \cdot 10^{-3}\ \text{eV}^2$

What are representative neutrino oscillation experiments?

Where would you place the detector to measure oscillations driven by $\Delta m^2 = 10\ \text{eV}^2$ with $E \sim 2.5\ \text{GeV}$?

Typical oscillation lengths

Exercise 4. Solution

- $$L_{\rm osc} \approx 2.48 \text{ m} \frac{1 \text{ MeV}}{7.4 \cdot 10^{-5} \text{ eV}^2} \approx 33.5 \text{ km}$$ JUNO
- $$L_{\rm osc} \approx 2.48 \text{ m} \frac{10^3 \text{ MeV}}{2.5 \cdot 10^{-3} \text{ eV}^2} = 992 \text{ km}$$ NOvA

The optimal distance $L \sim L_{\rm osc}$

$$L_{\rm osc} \approx 2.48 \text{ m} \frac{2500 \text{ (MeV)}}{10 \text{ (eV}^2)} = 620 \text{ m}$$

Oscillation maxima

Exercise 5. Assuming $\Delta m^2 = 2.5 \cdot 10^{-3}\ \mathrm{eV}^2$ and $L = 1300\ \mathrm{km}$, derive the energies of the first and second oscillation maxima

$$P_{e\mu} = \sin^2(2\theta)\ \sin^2\left(1.27\,\frac{\Delta m^2\left(\mathrm{eV}^2\right)\ L\left(\mathrm{km}\right)}{E\,(\mathrm{GeV})}\right)$$

Oscillation maxima

Exercise 5. Solution

$P_{e\mu}$ is maximal if the argument of the second sine is $\dfrac{\pi k}{2}, \quad k = 1, 3, 5, \ldots$

Therefore, we find

$$E_{\max} = \frac{2 \cdot 1.27}{\pi k} \text{ GeV} \cdot \Delta m^2 \left(\text{eV}^2\right) \cdot L\left(\text{km}\right) \approx \frac{0.8}{k} \text{ GeV} \cdot \Delta m^2 \left(\text{eV}^2\right) \cdot L\left(\text{km}\right)$$

$$E_{\max} \approx 2.6 \text{ GeV} \quad \text{for} \quad k = 1$$

$$E_{\max} \approx 0.87 \text{ GeV} \quad \text{for} \quad k = 3$$