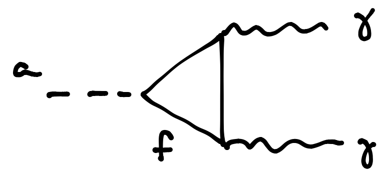
1. **Parametric resonance:** Consider an axion-like particle (ALP) coupled to photons:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2} m_\phi^2 \phi^2 - \frac{1}{4} g_{\phi\gamma\gamma} \phi F^{\mu\nu} \tilde{F}_{\mu\nu}$$

If this ALP is dark matter observed today, it has to be stable on long time scales.

(a) It is possible for the ALP to decay photons $\phi \to \gamma\gamma$ perturbatively through the following process:



with a perturbative decay rate

$$\Gamma_{\text{pert}} = \frac{g_{\phi\gamma\gamma}^2 m_\phi^3}{64\pi} \, .$$

The ALP becomes cosmologically unstable when $\Gamma_{\text{pert}} > H_0$, where H_0 is the present-day Hubble parameters $H_0 \simeq 10^{-42}$ GeV. For a typical dark matter ALP with mass $m_a = 10^{-6}$ eV, what is the largest allowed coupling before the ALP becomes unstable? Astrophysical bounds for an ALP of mass 10^{-6} eV restrict $g_{\phi\gamma\gamma} \lesssim 7\times 10^{-11}$ GeV^{-1}. Given the bound, should we be concerned about perturbative decay?

(b) One can also estimate the decay rate $\phi \to \gamma\gamma$ by considering the growth of the photon field in a time-dependent ALP background. The equation of motion for the photon field in Fourier space is:

$$\ddot{A} + \left(k^2 - g_{\phi\gamma\gamma} k \dot{\phi}\right) A = 0$$

We know from lecture that the ALP dark matter field oscillates, which can be written as

$$\phi(t) = \phi_0 \sin(m_\phi t) \, .$$

i. Rewrite this equation of motion for the photon as the Mathieu equation:

$$\frac{d^2 A}{dx^2} + \left[\lambda - 2q\cos(2x)\right] A = 0$$

ii. The full set of solutions to this differential equation contains an exponentially growing solution $A \sim e^{\eta_\lambda x}$, where the rate η_λ is

$$\eta_\lambda = \beta_\lambda \frac{2q}{\sqrt{\lambda}}$$

where β_λ is some $\mathcal{O}(1)$ number. This corresponds to the exponential solution $A \sim e^{\eta_k t}$. Find η_k in terms of $g_{\phi\gamma\gamma}$, ϕ_0, and m_ϕ. Estimate the rate of exponential growth,

η_k for a typical $m_a = 10^{-6}$ eV ALP with $g_{a\gamma\gamma} = 10^{-11}$ GeV^{-1}. The rate of photon growth is related to enhanced ALP decay on the timescale

$$\tau \sim 1/\eta_k$$

Would this allow the ALP to be dark matter?

(c) This process, however, is happening in an expanding universe. In order for tachyonic instability to truly enhance $a \to \gamma\gamma$ decay, has to be on a short timescale compared to the time scale at which the modes are redshifting out of the resonance (see [1] for more details). Compare

$$\Delta t_{\text{growth}} \sim \frac{1}{2\eta_k}$$

with

$$\Delta t_{\text{redshift}} \sim \frac{\delta k}{k_*} \frac{1}{H_0} \ .$$

Use $\delta k \sim g_{\phi\gamma\gamma}\phi_0 m_\phi/2$ and $k_* = m_\phi/2$ (resulting from fully solving the Mattieu equation). Use $g_{\phi\gamma\gamma} = 10^{-11}$ GeV^{-1}, $m_\phi = 10^{-6}$ eV .

2. The equation of motion for a massive spin-0 boson around a spinning black hole is closely analogous to the quantum mechanics of the hydrogen atom, since both gravity and electromagnetism have $1/r$ potentials. The "gravitational fine structure constant" is

$$\alpha_G = G_N M \mu_S \ ,$$

where G_N is Newton's constant $G_N = 6.7 \times 10^{-39}$ GeV^{-2}, M is the mass of the black hole, and μ_S is the boson mass.

(a) By analogy with the Bohr atom, what is the typical distance of the $n = 1$ level from the black hole? Compute this numerically for a black hole with $M = 10 M_\odot$ and $\mu_S = 10^{-12}$ eV.

(b) Solving the full equations of motion gives a set of quasinormal modes which can have both real and imaginary parts, where the imaginary part represents exponentially growing or decaying modes depending on the sign. The quasinormal modes are [2]:

$$\omega_{nlm} \sim \mu_S - \frac{\mu_S}{2}\left(\frac{\alpha_G}{l+n}\right)^2 + \frac{i}{\gamma_{nlm} G_N M}\left(m\frac{a}{\sqrt{G_N} M} - 2\mu_S r_+\right)(\alpha_G)^{4l+5}, \quad \alpha_G \ll 1 \ ,$$

where M is the mass of the BH, μ_S is the mass of the scalar field, and a is the Kerr rotation parameter (the spin angular momentum of the BH over its mass J/M). γ_{nlm} is a coefficient that results from solving the EOM and depends on the quantum numbers n, l, m (Note that m is the magnetic quantum number, not a mass). Here, r_+ is the event horizon for a rotating (Kerr) black hole:

$$r_+ = G_N M + \sqrt{G_N^2 M^2 - G_N a^2}$$

When the imaginary part of ω_{nlm} is positive, the field grows exponentially $\phi \sim e^{\mathrm{Im}(\omega_{nlm})t}$. What is the time scale of this instability? Use $M = 10M_{\odot}$, $\mu_S = 10^{-11}$ eV. Assume $\{n, l, m\} = \{1, 1, 1\}$ and using $\gamma_{111} = 48$ (the dominant unstable mode, see [3]). Leave your answer in terms of a dimensionless black hole spin angular momentum parameter

$$\frac{a}{\sqrt{G_N}M} \, .$$

Then plug in $a/(\sqrt{G}M) = 1$, which corresponds to an extremal BH with maximum spin.

(c) When these quasinormal modes are real, the superradiant condition is saturated. What is the critical rotation parameter a_{crit} such that this happens? Find the corresponding dimensionless spin angular momentum parameter $a_{\text{crit}}/(\sqrt{G_N}M)$ using the numerical inputs from part (b).

References

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