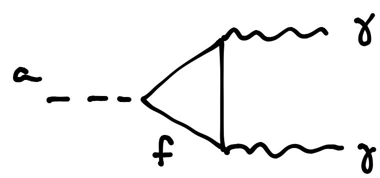
1. **Parametric resonance:** Consider an axion-like particle (ALP) coupled to photons:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2} m_\phi^2 \phi^2 - \frac{1}{4} g_{\phi\gamma\gamma} \phi F^{\mu\nu} \tilde{F}_{\mu\nu}$$

If this ALP is dark matter observed today, it has to be stable on long time scales.

(a) It is possible for the ALP to decay photons $\phi \to \gamma\gamma$ perturbatively through the following process:



with a perturbative decay rate

$$\Gamma_{\text{pert}} = \frac{g_{\phi\gamma\gamma}^2 m_\phi^3}{64\pi} \ .$$

The ALP becomes cosmologically unstable when $\Gamma_{\text{pert}} > H_0$, where H_0 is the present-day Hubble parameters $H_0 \simeq 10^{-42}$ GeV. For a typical dark matter ALP with mass $m_a = 10^{-6}$ eV, what is the largest allowed coupling before the ALP becomes unstable? Astrophysical bounds for an ALP of mass 10^{-6} eV restrict $g_{\phi\gamma\gamma} \lesssim 7\times 10^{-11}$ GeV^{-1}. Given the bound, should we be concerned about perturbative decay?

Solution:

$$\Gamma_{\text{pert}} = \frac{g_{\phi\gamma\gamma}^2 m_\phi^3}{64\pi}$$
$$\Gamma_{\text{pert}} < H_0$$
$$\Rightarrow g_{\phi\gamma\gamma} < \sqrt{\frac{64\pi H_0}{m_\phi^3}} \simeq \sqrt{\frac{64\pi \times 10^{-42}\ \text{GeV}}{(10^{-6})^3 \text{eV}^3 \times 10^{-27}\ \text{GeV}^3}} \simeq 450\ \text{GeV}^{-1}$$

which is many orders of magnitude larger than what's been ruled out.

(b) One can also estimate the decay rate $\phi \to \gamma\gamma$ by considering the growth of the photon field in a time-dependent ALP background. The equation of motion for the photon field in Fourier space is:

$$\ddot{A} + \left(k^2 - g_{\phi\gamma\gamma} k \dot{\phi}\right) A = 0$$

We know from lecture that the ALP dark matter field oscillates, which can be written as

$$\phi(t) = \phi_0 \sin(m_\phi t) \ .$$

i. Rewrite this equation of motion for the photon as the Mathieu equation:

$$\frac{d^2A}{dx^2}+[\lambda-2q\cos(2x)]\,A=0$$

Solution:

$$2x = m_\phi t \quad \Rightarrow \quad 0=\frac{m_\phi^2}{4}\frac{d^2A}{dx^2}+\left(k^2-g_{\phi\gamma\gamma}k\phi_0 m_\phi\cos(2x)\right)A$$

$$0=\frac{d^2A}{dx^2}+\left(\frac{4k^2}{m_\phi^2}-\frac{4}{m_\phi}g_{\phi\gamma\gamma}k\phi_0\cos(2x)\right)A$$

$$\lambda=\frac{4k^2}{m_\phi^2} \quad \Rightarrow \quad 0=\frac{d^2A}{dx^2}+\left(\lambda-\frac{4}{m_\phi}g_{\phi\gamma\gamma}k\phi_0\cos(2x)\right)A$$

$$q=\frac{2}{m_\phi}g_{\phi\gamma\gamma}k\phi_0 \quad \Rightarrow \quad 0=\frac{d^2A}{dx^2}+(\lambda-2q\cos(2x))\,A$$

ii. The full set of solutions to this differential equation contains an exponentially growing solution $A\sim e^{\eta_\lambda x}$, where the rate η_λ is

$$\eta_\lambda=\beta_\lambda\frac{2q}{\sqrt{\lambda}}$$

where β_λ is some $\mathcal{O}(1)$ number. This corresponds to the exponential solution $A\sim e^{\eta_k t}$. Find η_k in terms of $g_{\phi\gamma\gamma}$, ϕ_0, and m_ϕ. Estimate the rate of exponential growth, η_k for a typical $m_a=10^{-6}$ eV ALP with $g_{a\gamma\gamma}=10^{-11}$ GeV^{-1}. The rate of photon growth is related to enhanced ALP decay on the timescale

$$\tau\sim 1/\eta_k$$

Would this allow the ALP to be dark matter?

Solution: From the previous part

$$
\begin{aligned}
2x &= m_\phi t \\
\lambda &= \frac{4k^2}{m_\phi^2} \\
q &= \frac{2}{m_\phi} g_{\phi\gamma\gamma} k \phi_0 \\
\Rightarrow \eta_\lambda x &= \beta_\lambda \frac{2q}{\sqrt{\lambda}} x = \beta_\lambda \times \frac{4}{m_\phi} g_{\phi\gamma\gamma} k \phi_0 \times \frac{m_\phi}{2k} \times \left(\frac{1}{2} m_\phi t\right) \\
&= \beta_\lambda \frac{1}{m_\phi} g_{\phi\gamma\gamma} \phi_0 m_\phi (m_\phi t) \\
&= \beta_\lambda g_{\phi\gamma\gamma} \phi_0 m_\phi t \equiv \eta_k t \\
\Rightarrow \eta_k &= \beta_\lambda g_{\phi\gamma\gamma} \phi_0 m_\phi \\
\eta_k &\simeq 1 \times 10^{-11} \text{ GeV}^{-1} \times \frac{\sqrt{2\rho_{\text{DM}}}}{m_\phi} \times m_\phi \\
&= 10^{-11} \text{ GeV}^{-1} \times \sqrt{2 \times 0.3 \text{ GeV/cm}^3 \times c^3 \times \hbar} = 3 \times 10^{-8} \text{ s}^{-1}
\end{aligned}
$$

This corresponds to

$$
\tau \sim 1/\eta_k = 3 \times 10^7 \text{ s} \sim 1 \text{ year}
$$

which is way too fast for DM to decay!

(c) This process, however, is happening in an expanding universe. In order for tachyonic instability to truly enhance $a \to \gamma\gamma$ decay, has to be on a short timescale compared to the time scale at which the modes are redshifting out of the resonance (see [1] for more details). Compare

$$
\Delta t_{\text{growth}} \sim \frac{1}{2\eta_k}
$$

with

$$
\Delta t_{\text{redshift}} \sim \frac{\delta k}{k_*} \frac{1}{H_0} .
$$

Use $\delta k \sim g_{\phi\gamma\gamma} \phi_0 m_\phi / 2$ and $k_* = m_\phi/2$ (resulting from fully solving the Mattieu equation). Use $g_{\phi\gamma\gamma} = 10^{-11}$ GeV^{-1}, $m_\phi = 10^{-6}$ eV .

Solution:

$$\begin{aligned}\Delta t_{\rm growth} &\sim \frac{1}{2\eta_k} \simeq 1.5\times 10^7 \text{ s} \\ \Delta t_{\rm redshift} &\sim \frac{\delta k}{k_*}\frac{1}{H_0} \\ &\sim \frac{g_{a\gamma\gamma}\phi_0 m_\phi/2}{m_\phi/2}\frac{1}{10^{-42}\text{ GeV}} \\ &= \frac{1/2\left(3\times 10^{-8}\text{ s}^{-1}\right)}{1/2\times 10^{-6}\times 10^{-9}\text{ GeV}}\frac{\hbar^2}{10^{-42}\text{ GeV}} \\ &= \frac{3\times 10^{-8}\text{ s}^{-1}}{10^{-15}\text{ GeV}}\frac{\hbar^2}{10^{-42}\text{ GeV}} = 13\text{ s}\end{aligned}$$

and so

$$\Delta t_{\rm growth} \gg \Delta t_{\rm redshift}$$

and there is not enough time for the ALP to decay through the tachyonic instability.

2. The equation of motion for a massive spin-0 boson around a spinning black hole is closely analogous to the quantum mechanics of the hydrogen atom, since both gravity and electromagnetism have $1/r$ potentials. The "gravitational fine structure constant" is

$$\alpha_G = G_N M \mu_S \ ,$$

where G_N is Newton's constant $G_N = 6.7\times 10^{-39}$ GeV^{-2}, M is the mass of the black hole, and μ_S is the boson mass.

(a) By analogy with the Bohr atom, what is the typical distance of the $n = 1$ level from the black hole? Compute this numerically for a black hole with $M = 10M_\odot$ and $\mu_S = 10^{-12}$ eV.

Solution:
The Bohr radius is

$$a_0 = \frac{1}{m_e \alpha}$$

so by analogy, in this case

$$r = \frac{1}{\mu_S(G_N M \mu_S)} = \frac{1}{G_N M \mu_S^2} \text{ which agrees with Eq. (4) of [2]}$$

$$\alpha_G = G_N M \mu_S = 6.7\times 10^{-39} \text{ GeV}^{-2} \times 10 M_\odot \times 10^{-12} \text{ eV}, \qquad M_\odot = 2\times 10^{30} \text{ kg}$$

$$= 0.075 \ll 1$$

$$r = \frac{1}{6.8\times 10^{-39} \text{ GeV}^{-2}\times 10 M_\odot \times (10^{-12} \text{ eV})^2}$$

$$= 7\times 10^{48} \text{ kg}^{-1} \times \hbar/c = 2.6\times 10^6 \text{ m}$$

(b) Solving the full equations of motion gives a set of quasinormal modes which can have both real and imaginary parts, where the imaginary part represents exponentially growing or decaying modes depending on the sign. The quasinormal modes are [3]:

$$\omega_{nlm} \sim \mu_S - \frac{\mu_S}{2}\left(\frac{\alpha_G}{l+n}\right)^2 + \frac{i}{\gamma_{nlm} G_N M}\left(m\frac{a}{\sqrt{G_N}M} - 2\mu_S r_+\right)(\alpha_G)^{4l+5}, \quad \alpha_G \ll 1 ,$$

where M is the mass of the BH, μ_S is the mass of the scalar field, and a is the Kerr rotation parameter (the spin angular momentum of the BH over its mass J/M). γ_{nlm} is a coefficient that results from solving the EOM and depends on the quantum numbers n, l, m (Note that m is the magnetic quantum number, not a mass). Here, r_+ is the event horizon for a rotating (Kerr) black hole:

$$r_+ = G_N M + \sqrt{G_N^2 M^2 - G_N a^2}$$

When the imaginary part of ω_{nlm} is positive, the field grows exponentially $\phi \sim e^{\text{Im}(\omega_{nlm})t}$. What is the time scale of this instability? Use $M = 10M_\odot$, $\mu_S = 10^{-11}$ eV. Assume $\{n, l, m\} = \{1, 1, 1\}$ and using $\gamma_{111} = 48$ (the dominant unstable mode, see [4]). Leave your answer in terms of a dimensionless black hole spin angular momentum parameter

$$\frac{a}{\sqrt{G_N}M} .$$

Then plug in $a/(\sqrt{G}M) = 1$, which corresponds to an extremal BH with maximum spin.

Solution:

$$\tau \sim \frac{1}{\text{Im}(\omega_{nlm})} = \frac{\gamma_{nlm} G_N M}{\frac{m}{\sqrt{G_N}}\frac{a}{M} - 2\mu_S r_+}(\alpha_G)^{-4l-5}$$

$$= \frac{48\cdot 6.7\times 10^{-39} \text{ GeV}^{-2}\cdot 10 M_\odot \times \hbar c^2}{\frac{a}{\sqrt{G}M} - 2\cdot 10^{-12}\text{GeV}/\hbar\, r_+}$$

$$= \frac{3\times 10^7 \text{ s}}{\frac{a}{\sqrt{G_N}M} - 3\times 10^4 \text{ s}^{-1} r_+}$$

Going a bit further, we can write this in terms of a/M alone using the expression for r_+:

$$\tau = \frac{3\times 10^7 \text{ s}}{\frac{a}{\sqrt{G_N}M} - 3\times 10^4 \text{ s}^{-1}\left(G_N M + G_N M\sqrt{1-a^2/(G_N M^2)}\right)}$$

$$\tau = \frac{2\times 10^8 \text{ s}}{1-6.6\left(\frac{a}{\sqrt{G_N}M}\right)+\sqrt{1-\left(\frac{a}{\sqrt{G_N}M}\right)}}$$

$$\tau\Big|_{\frac{a}{\sqrt{G_N}M}=1} = 3.6\times 10^7 \text{ s}$$

(c) When these quasinormal modes are real, the superradiant condition is saturated. What is the critical rotation parameter a_{crit} such that this happens? Find the corresponding dimensionless spin angular momentum parameter $a_{\text{crit}}/(\sqrt{G_N}M)$ using the numerical inputs from part (b).

Solution:

$$0 = \frac{am}{\sqrt{G_N}M} - 2\mu_S r_+$$

$$a_{\text{crit}} = 2\mu_S r_+ \sqrt{G_N}M$$

$$\frac{a_{\text{crit}}}{\sqrt{G_N}M} = 2\mu_S r_+$$

$$\frac{a_{\text{crit}}}{\sqrt{G_N}M} = 2\mu_S G_N M\left(1+\sqrt{1-\frac{a_{\text{crit}}^2}{G_N M^2}}\right)$$

$$\frac{a_{\text{crit}}}{\sqrt{G_N}M} = 2\alpha_G\left(1+\sqrt{1-\frac{a_{\text{crit}}^2}{G_N M^2}}\right)$$

$$\alpha_G = G_N M\mu_S = 0.075$$

$$\frac{a_{\text{crit}}}{\sqrt{G_N}M} = 0.15$$

References

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[3] R. Brito, V. Cardoso, and P. Pani, "Superradiance: New Frontiers in Black Hole Physics," *Lect. Notes Phys.* **906** (2015) pp.1–237, arXiv:1501.06570 [gr-qc].

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