Optical Atomic Clocks - Problems 1

Quantum Sensors for Fundamental Physics school
15th September 2021
Charles Baynham

1 The standard quantum limit (SQL)

Atomic clocks use the interaction of light with a transition in a reference atomic system to measure the frequency of that transition, ω_0.

One way to do this is the Ramsey method: treating the atom as a 2-level qubit, a coherent state is prepared by allowing a group of N_a atoms prepared in an initial state

$$|\Psi_{\text{initial}}\rangle = |0\rangle^{\otimes N_a} \tag{1}$$

to interact with a "probe" electric field generated by a laser oscillating at angular frequency ω. This is timed such that the interaction rotates the Bloch vector by $\pi/2$, creating a superposition of the ground and excited states. Over a period of time T (the *Ramsey time*) these states evolve a phase difference proportional to the probing laser's detuning from the atomic transition. By reading out this phase, the operator of the clock deduces information about the detuning of their laser from the atomic transition, allowing them to realise a frequency standard.

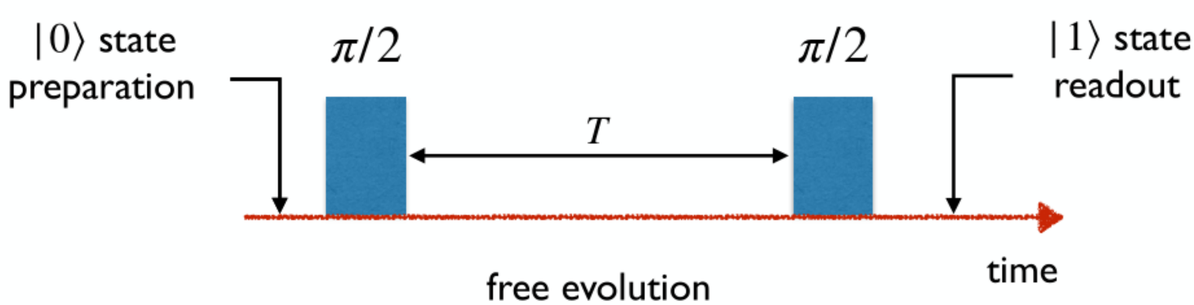

Figure 1: Laser pulses applied during a Ramsey sequence. See fig. 2 for how the pulses / evolution time affect the atomic state. [figure from A. Derevianko talk 1, QSFP2021]

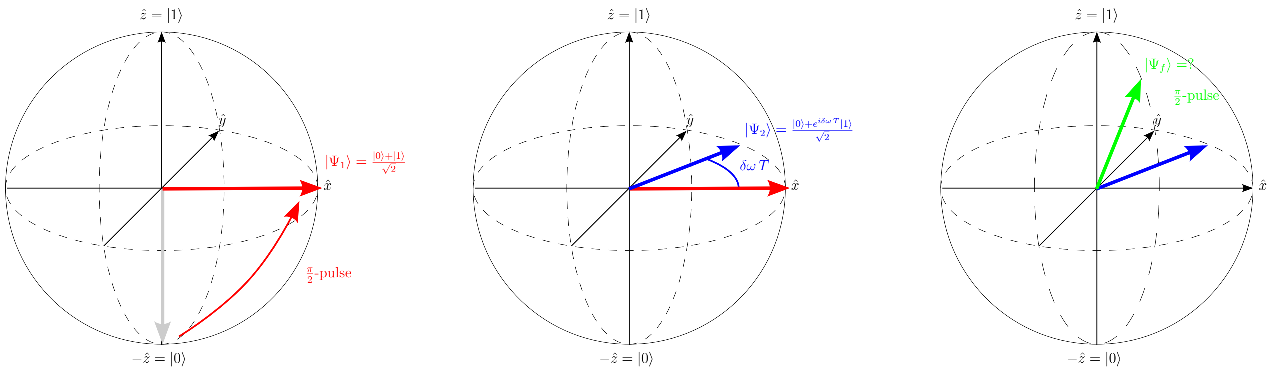

Figure 2: A 2-pulse Ramsey sequence. *(left)* The atom is prepared in state $|0\rangle$ before interaction with a laser rotates the state around the y-axis to $|\Psi_1\rangle$. *(centre)* Detuning of the laser from the atomic transition results in a precession around the z-axis and a phase difference between the two states ($|\Psi_2\rangle$). *(right)* To read out this phase difference, a final $\pi/2$ pulse converts it into a populations difference by rotating around the y-axis again ($|\Psi_f\rangle$).

However, there is a limit to how much information can be obtained this way, which results from the Heisenburg uncertainy principle.[1] In this question we explore the Standard Quantum Limit – the limit of stability for quantum systems operating with uncorrellated states.

Problem 1.1. *First, we simplify the problem to consider only one atom. After the first $\pi/2$ pulse, the atom is in the state*

$$|\Psi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad . \tag{2}$$

Find the final state $|\Psi_f\rangle$ after the atom is allowed to freely precess for time T, then rotated by another $\pi/2$ pulse by using the rotation operators $R_{\hat{z}}(\delta\omega T)$ for the precession and $R_{\hat{y}}(\pi/2)$ for the $\pi/2$ pulse.

Hint:

> *A rotation of the Bloch vector by an angle θ around the unit vector $\hat{n}$ can be effected by using the rotation operator*

$$\begin{aligned} R_{\hat{n}}(\theta) &= e^{-\frac{i\theta}{2}(\hat{n}\cdot\vec{\sigma})} \\ &\equiv \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)\hat{n}\cdot\vec{\sigma} \end{aligned} \tag{3}$$

> *where*

[1] A good way to see this which also offers hints on how this limit can be circumvented is to consider the commutation relations of the collective spin operators, $J_\alpha = \frac{1}{2}\sum_{i=0}^{N_a}\sigma_{i,\alpha}$. We do not have time to cover this in this tutorial, but see Budker, D., et. al. (2004) for an example.

$$\vec{\sigma} \overset{\text{def}}{=} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}, \quad \sigma_{\text{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0| + |0\rangle\langle 1|,$$

$$\sigma_{\text{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\,|1\rangle\langle 0| - i\,|0\rangle\langle 1|, \quad \sigma_{\text{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Problem 1.2. *Show that when the atom's state is measured, the probability to find it in the excited state is*

$$|\langle 1|\Psi_f\rangle|^2 = \cos^2\left(\frac{\delta\omega T}{2}\right) \tag{4}$$

Problem 1.3. *Writing the probability of being in the excited state as* $p \overset{\text{def}}{=} |\langle 1|\Psi_f\rangle|^2$*, what would be the variance of the mean excitation,* σ_p^2*, if this experiment was repeated with* N_a *atoms,* N_m *times?*

Hint 1: You can treat each atom as a separate measurement. To see this, realise that both the initial state [eq. (1)] and the rotation operators $R_{\hat{n}}(\theta)^{\otimes N}$ *are separable.*

Hint 2: Consider a binomial distribution: feel free to check Wikipedia for the mean and variance of this distribution!

Problem 1.4. *You have derived the variance in mean excitation observed for an atomic clock. However, when operating a clock, the job of the experimenter is to deduce the atomic frequency. Determine the relationship between uncertainty in excitation fraction and uncertainty in frequency by using*

$$\sigma_{\delta\omega} = \sigma_p \cdot \left|\frac{d\delta\omega}{dp}\right| \quad .$$

Express your result in terms of the total time t for which the atomic clock is operated, to find the Standard Quantum Limit (SQL):

$$\sigma_{\delta\omega} = \frac{1}{\sqrt{N_a T t}} \tag{5}$$