Optical Atomic Clocks - Solutions 1

Quantum Sensors for Fundamental Physics school
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1 The standard quantum limit (SQL)

1.1

To solve this problem, you must use the rotation operators $R_{\hat{z}}(\delta\omega\, T)$ and $R_{\hat{y}}(-\pi/2)$ to rotate the Bloch vector to its final orientation. The free precession $\delta\omega\, T$ is caused by the laser's detuning from the transition, whereas the final $pi/2$ rotation around the $\hat{y}$ axis is caused by the application of another laser pulse to the atom.

Starting with the state $|\Psi_1\rangle$ as given:

$$|\Psi_f\rangle = R_{\hat{y}}(\pi/2)\, R_{\hat{z}}(\delta\omega\, T)\; |\Psi_1\rangle \tag{1}$$

where

$$\begin{aligned} R_{\hat{y}}(\pi/2) &= e^{\frac{-i\pi}{4}(\hat{y}\cdot\vec{\sigma})} && (2)\\ &= \frac{1}{\sqrt{2}}\,(\mathbb{1} - i\vec{\sigma_y}) && (3) \end{aligned}$$

and

$$\begin{aligned} R_{\hat{z}}(\delta\omega\, T) &= e^{\frac{i\delta\omega\, T}{2}(\hat{z}\cdot\vec{\sigma})} && (4)\\ &= \left(\cos\frac{\delta\omega\, T}{2}\mathbb{1} - i\sin\frac{\delta\omega\, T}{2}\vec{\sigma_z}\right) && (5) \end{aligned}$$

From here, use matrix multiplication or ket expansion to show that

$$\begin{aligned} |\Psi_f\rangle &= R_{\hat{y}}(\pi/2)\, R_{\hat{z}}(\delta\omega\, T)\; |\Psi_1\rangle && (6)\\ &= R_{\hat{y}}(\pi/2)\, \left(\cos\frac{\delta\omega\, T}{2}\mathbb{1} - i\sin\frac{\delta\omega\, T}{2}\vec{\sigma_z}\right) |\Psi_1\rangle && (7)\\ &= R_{\hat{y}}(\pi/2)\, e^{-\frac{i\delta\omega\, T}{2}}\,\frac{|0\rangle + e^{i\delta\omega\, T}\,|1\rangle}{\sqrt{2}} && (8)\\ &= e^{-\frac{i\delta\omega\, T}{2}}\,\frac{(1 - e^{i\delta\omega\, T})\,|0\rangle + (1 + e^{i\delta\omega\, T})\,|1\rangle}{2} && (9) \end{aligned}$$

1.2

Here we simply collapse the wavefunction found above:

$$\begin{aligned}
|\langle 1|\Psi_f\rangle|^2 &= \frac{1}{4}(1+e^{i\delta\omega T})(1+e^{-i\delta\omega T}) \\
&= \frac{2+e^{i\delta\omega T}+e^{-i\delta\omega T}}{4} \\
&= \frac{1+\cos(\delta\omega T)}{2} \qquad (10) \\
&= \cos^2\left(\frac{\delta\omega T}{2}\right) \qquad (11)
\end{aligned}$$

1.3

The mean and variance of a random variable X which is distributed binomially with n measurements with probability p of yielding an excitation are

$$E[X] = np, \quad \sigma_X^2 = np(1-p)$$

Since the atoms are uncorrellated, each collapse of their wavefunction can be treated as an independant measurement. There are therefore $n = N_m N_a$ measurements. The question asks about the variance of the **mean** probability, so the answer is:

$$\sigma_p = \sqrt{\frac{p(1-p)}{N_a N_m}} \qquad (12)$$

1.4

Here, we begin by differentiating the sensitivity function obtained in question 1.2:

$$\begin{aligned}
\frac{dp}{d\delta\omega} &= \frac{d}{d\delta\omega}\left[\cos^2\left(\frac{\delta\omega T}{2}\right)\right] \\
&= \frac{d}{d\delta\omega}\left[\frac{1+\cos(\delta\omega T)}{2}\right] \\
&= -\frac{T}{2}\sin(\delta\omega T) \qquad (13)
\end{aligned}$$

We then use

$$\begin{aligned}
\sigma_{\delta\omega} &= \sigma_p . \left|\frac{d\delta\omega}{dp}\right| \\
&= \sigma_p / \left|\frac{dp}{d\delta\omega}\right| \qquad (14)
\end{aligned}$$

and, substituting eqs. (10) and (12), obtain

$$\begin{aligned}\sigma_{\delta\omega} &= \sqrt{\frac{\cos^2\left(\frac{\delta\omega T}{2}\right)\left(1-\cos^2\left(\frac{\delta\omega T}{2}\right)\right)}{N_a N_m}} \Big/ \frac{T}{2}\sin(\delta\omega\, T) \\ &= \frac{\sin(\frac{\delta\omega T}{2})\cos(\frac{\delta\omega T}{2})}{\sqrt{N_a N_m}} \Big/ \frac{T}{2}\sin(\delta\omega\, T) \\ &= \frac{1}{T\sqrt{N_m N_a}} \qquad (15)\end{aligned}$$

Finally, use the fact that one measurement takes a time T to write $N_m = t/T$, and conclude that

$$\sigma_{\delta\omega} = \frac{1}{\sqrt{N_a T t}} \quad . \qquad (16)$$