Optical Atomic Clocks - Problems

Quantum Sensors for Fundamental Physics school
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1 Portal Lagrangian's and the variation of fundamental constants

The Lagrangian (or Lagrangian density) and Hamiltonian are both important in classical and quantum field theory. In Quantum field theory, the Lagrangian density has a few big requirements: Terms must be functions of only the field(s) or first derivative of the field(s) (similar constraint on the classical Lagrangian), and the functional form of the Lagrangian must be invariant under boosts and rotations. The Lagrangian density for a real valued scalar field reads

$$\mathcal{L} = \frac{1}{2}c^2\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2c^4\phi^2, \tag{1}$$

with the field (ϕ) and mass (m) having units of energy (eV).

Similarly to classical theory we find the Equations of Motion (EOM) or the "Euler-Lagrange equation". In quantum field theory the Euler-Lagrange equation (relativistic) reads

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) = \frac{\partial\mathcal{L}}{\partial\phi_a}, \tag{2}$$

with the subscript 'a' enumerating over other possible fields involved. This leads to the famous "Klein-Gordon equation" (left for the reader to derive),

$$\partial_\mu\partial^\mu\phi + m^2c^2\phi = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2c^2\right)\phi = 0. \tag{3}$$

In Quantum Electrodynamics (QED) the Lagrangian density reads

$$\mathcal{L}_{QED} = \bar{\Psi}\left(i\hbar c\gamma^\mu\partial_\mu - mc^2\right)\Psi - q\bar{\Psi}\gamma^\mu\Psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{4}$$

with Ψ representing the bispinor (fermions) fields, γ^μ being the gamma matrices, A_μ is the vector potential, J^μ is the current density, and $F_{\mu\nu}$ is the Faraday Tensor.

Problem 1.1. *When dealing with exotic physics beyond the standard model, the hypothesized interaction Lagrangian between standard model fields and the exotic field(s) is simply added to Eq. (4). Suppose the exotic field is described by a linear scalar portal with Lagrangian density,*

$$\mathcal{L}_{int} = -\frac{1}{4}\Gamma\phi F_{\mu\nu}F^{\mu\nu}. \tag{5}$$

Here Γ is the coupling constant that quantifies the exotic field interaction strength.

What is the new Lagrangian density?

Problem 1.2. *Now suppose our exotic field is a dark matter field such that our interaction Lagrangian assumes a quadratic scalar portal,*

$$\mathcal{L}_{int} = -\left(\Gamma_f m_f c^2 \bar{\Psi}_f \Psi_f + \frac{1}{4}\Gamma_\alpha F_{\mu\nu}F^{\mu\nu}\right)\phi\phi^*. \tag{6}$$

Here Γ_f and Γ_α are the coupling constants that quantify the DM interaction strength to the fermions and the fine structure respectively, m_f is the mass of the fermions, Ψ_f and $\bar{\Psi}_f$ are the standard model fermion fields.

What is the new Lagrangian density?

Problem 1.3. *Consider the Dirac Lagrangian density*

$$\mathcal{L}_D = \bar{\Psi}\left(i\hbar c\gamma^\mu \partial_\mu - mc^2\right)\Psi. \tag{7}$$

Now suppose we have a Lagrangian for an exotic field given by a quadratic scalar portal,

$$\mathcal{L}_{int} = -\Gamma mc^2 \bar{\Psi}\Psi\phi\phi^*, \tag{8}$$

with ϕ being a scalar field, Ψ is the Dirac field (electrons), and Γ being the coupling constant. The Dirac Hamiltonian is given by

$$H_D = c\alpha \cdot \boldsymbol{p} + \beta mc^2 + V_{int}, \tag{9}$$
$$i\hbar\frac{\partial}{\partial t}\Psi = H_D\Psi.$$

Here α is not the fine structure constant but are matrices given by,

$$\gamma_0\gamma^i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ \boldsymbol{\sigma}_i & 0 \end{pmatrix} = \boldsymbol{\alpha}^i \tag{10}$$

Given the interaction Lagrangian density what is the effective potential V_{int} that is added to the Dirac Hamiltonian? Why does this lead to a variation of the fundamental constant m?

Hint:

> *Consider the Euler-Lagrange equations, and multiplying by γ_0 to get to the Hamiltonian form in Eq. (9)*

2 Sensitivity of networks of quantum sensors to exotic fields: angular and velocity resolutions

To detect exotic fields such as dark matter or exotic low-mass fields, typically networks of quantum sensors such as atom interferometers, magnetometers, or atomic clocks are used. Application specific Templates are created to reflect the exotic physics of the interaction with the standard model and compared to resulting data. As an example, suppose the exotic parameters of interest are direction ($\hat{n}$) and speed (v). Then we need to generate templates that span the continuous parameter space of directions $\hat{n}$ and speeds v. This leads to the question of angular and velocity resolutions of the network. In Fig. 1, we have an example template for a dark matter transient propagating through a circular network in the direction towards sensor 1. Here the diameter of the quantum sensor network is D. The angular resolution is roughly the ratio of the temporal resolution Δ_t to the field burst propagation time through the quantum sensor network D/v.

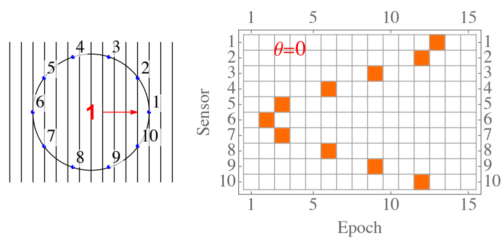

Figure 1: Construction of the template corresponding to a vertically-oriented line sweeping from the left (polar angle $\theta = 0$). Sensor 6 is affected first and sensor 1 — last.

Problem 2.1. *Suppose we consider two different line sweeps in Fig. 1 with a constant speed v but differ by the incident direction: θ and $\theta' = \theta + \Delta_\theta$. For small tilt angles Δ_θ both line sweeps will result in the same template. What is the critical value for Δ_θ for which we will generate a new template?*
Hint:

> *Consider each sensor in the center of each line sweep and rotate the circular network Δ_θ about the center to generate a new template.*

Problem 2.2. *Now suppose we consider two different line sweeps keeping the direction θ of the exotic field constant but differ by the incident speed: v and $v' = v + \Delta_v$. The number of line slice sweeps for velocity v is given by*

$$\frac{D}{v\Delta_t}. \tag{11}$$

The number of line sweeps for v' is given by

$$\frac{D}{(v + \Delta_v)\Delta_t}. \tag{12}$$

What is the critical value (velocity resolution) Δ_v that will result in a new template?
Hint:

> *Consider the difference between the two number of line sweeps.*