

Variation of Fundamental Constants with Atomic Clocks

Hoang Bao Tran Tan

September 2021

In this tutorial, we explore the dependence of atomic transition frequencies on the fine structure constant α and the electron-proton mass ration m_e/m_p .

1. In this problem, we consider atomic transitions between levels with difference principle quantum numbers or between fine structure multiplets which have the same principle quantum number but different angular momenta. The frequencies associated with these transitions are of the order of 10^{15} Hz, i.e., in the far left end of the visible spectrum. The clocks using these transitions are thus referred to as ‘optical clocks’.

Let us begin with the case of a hydrogenlike atom/ion consisting of a single electron moving in the Coulomb field of an infinitely heavy nucleus of charge Ze . The Schrödinger formular for the electron’s nonrelativistic energy reads

$$E_n = -\frac{Z^2}{n^2} \text{Ry}, \quad (1)$$

while the relativistic energy is given by the Dirac formula

$$E_{nj} = m_e c^2 \left[1 + \left(\frac{Z\alpha}{n - |\kappa| + \gamma} \right)^2 \right]^{-1/2}. \quad (2)$$

Here,

$$\text{Ry} = \frac{m_e e^4}{32\pi^2 \epsilon_0^2 \hbar^2} \quad (3)$$

is the Rydberg constant, $n = 1, 2, \dots$ is the principal quantum number and j is the electron’s total angular momentum. The quantum number κ is defined via j and the orbital angular momentum l as $\kappa = (l - j)(2j + 1)$ and $\gamma = \sqrt{\kappa^2 - (Z\alpha)^2}$.

- (a) Use Eqs. (1) and (2) to verify that the leading relativistic (fine structure) correction to the energy level is given by

$$\Delta_{nj}^{\text{fs}} = -E_n \frac{(Z\alpha)^2}{n} \left(\frac{1}{j + 1/2} - \frac{3}{4n} \right). \quad (4)$$

Hint: It might be a good idea to use Mathematica or Maple to do the expansion.

Equations (1) and (4) suggest that the dependence of an electronic energy level on α may be written as

$$E_{nj}(\alpha) = m_e c^2 + E_{nj}^0 + q_{nj} \left[\left(\frac{\alpha}{\alpha_0} \right)^2 - 1 \right], \quad (5)$$

where α_0 is the current value of the fine structure constant, $1/\alpha_0 = 137.035999084(51)$, and E_{nj}^0 is the energy corresponding to this value of α .

In Eq. (5), E_{nj}^0 is independent of α . This may appear odd at first sight, since one can rewrite the definition of Ry as

$$\text{Ry} = \alpha^2 m_e c^2 / 2, \quad (6)$$

in which indicates that all terms in Eq. (5) are proportional to α^2 .

However, the point here is that because energy is a dimensionful quantity, a value of its measurement depends on the units we are using. In other words, we actually measure the ratio of the energy in

question with respect to another reference energy, e.g., Ry. A natural reference energy to choose is that between the ground state hyperfine doublet of Cs, which defines the second. We will consider this choice of reference in Exercise 2.

For this exercise, we simply note that if we take the ratio of two energies $E_{n_1j_1}/E_{n_2j_2}$ then the units cancel out and we can use the expansion (5). In other words, when investigating the effects of varying fundamental constants, we need not worry about units and their dependence on the fundamental constants.

(b) Using Eqs. (1) and (4), find the expressions for E_{nj}^0 and q_{nj} in terms of Z , n and j .

Although the results we obtained so far were for a hydrogenlike atom/ion, they hold for multielectron atoms as well. Therefore, let us now consider the transition between two levels n_1j_1 and n_2j_2 in an arbitrary atom. Clearly, the transition energy $E \equiv E_{n_2j_2} - E_{n_1j_1}$ may be put in the form

$$E = E_0 + q \left[\left(\frac{\alpha}{\alpha_0} \right)^2 - 1 \right], \quad (7)$$

where $E_0 \equiv E_{n_2j_2}^0 - E_{n_1j_1}^0$ and $q \equiv q_{n_2j_2} - q_{n_1j_1}$.

The parameter q links the variation δE of the transition energy E to the variation $\delta\alpha$ of α .

(c) Using Eq. (7), show that

$$\left. \frac{\delta E}{E} \right|_{\alpha=\alpha_0} = K \frac{\delta\alpha}{\alpha_0}, \quad (8)$$

and derive the formula for the dimensionless sensitivity factor K .

As discussed in question 1b, in an experimental setting, we measure the ratio $R = E_1/E_2 = \nu_1/\nu_2$, between two transition energies (or frequencies, $\nu = E/h$).

(d) Show that R changes in response to the variation of α as

$$\left. \frac{\delta R}{R} \right|_{\alpha=\alpha_0} = \Delta K \frac{\delta\alpha}{\alpha_0}, \quad (9)$$

and derive the formula for the dimensionless sensitivity factor ΔK .

2. In this exercise we consider the more traditional ‘microwave clocks’ which use transitions with frequencies of the order of GHz. The most notable example is perhaps ^{133}Cs clocks which define the second.

Atomic transitions in the GHz range often happen between hyperfine levels, which arise due to the interaction between the nuclear multipoles, e.g., magnetic dipole and electric quadrupole, and the fields created by the atomic electrons. In this exercise, we will consider the nuclear magnetic dipole only.

The magnetic dipole of a nucleus $\boldsymbol{\mu}_I$ is proportional to the nuclear spin \mathbf{I}

$$\boldsymbol{\mu}_I = g\mu_N\mathbf{I}, \quad (10)$$

where g is the nuclear gyromagnetic factor, $\mu_N = \frac{e\hbar}{2m_p}$ is the nuclear magneton and m_p is the proton mass.

Due to the interaction between $\boldsymbol{\mu}_I$ and the electrons, the electrons’ total angular momentum \mathbf{J} is no longer a good quantum number. Instead, electronic states are now eigenfunctions of the operator $\mathbf{F} \equiv \mathbf{I} + \mathbf{J}$. In other words, a state with a definite J at the fine structure level splits further into states with $F = |J - I|, \dots, J + I$.

Let us now consider again the case of a hydrogenlike atom/ion. We may write the interaction between the electron and the nuclear magnetic moment $\boldsymbol{\mu}_I$ as

$$V_{\text{hfs}} = \frac{\mu_0}{4\pi} \frac{e\boldsymbol{\mu}_I \cdot (\mathbf{r} \times \boldsymbol{\alpha})}{r^3}, \quad (11)$$

where \mathbf{r} is the electron's position vector and $\boldsymbol{\alpha}$ are its alpha matrices (this interaction Hamiltonian comes from the minimal coupling prescription for the Dirac equation: $\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A}$ where $\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\boldsymbol{\mu}_I \times \mathbf{r}}{r^3}$ is the vector potential created by $\boldsymbol{\mu}_I$).

Using perturbation theory, we find that the hyperfine shift corresponding to the level E_{nj} is given by

$$\Delta_{njF}^{\text{hfs}} = \frac{\mu_0 e}{(4\pi)^2} \frac{\mu_I}{I} \frac{m_e^2 c^3}{\hbar^2} \frac{(Z\alpha)^3 \kappa}{j(j+1)} \frac{2\kappa(\gamma + n_r) - N}{N^4 \gamma(4\gamma^2 - 1)} [F(F+1) - I(I+1) - j(j+1)], \quad (12)$$

where $n_r = n - |\kappa|$ and $N = \sqrt{n_r^2 + 2n_r\gamma + \kappa^2}$.

(a) Show that Eq. (12) may be put in the form

$$\Delta_{njF}^{\text{hfs}} = A_{\text{hfs}} g_I \frac{m_e}{m_p} \alpha^2 F_{\text{hfs}}(\alpha) \text{Ry}, \quad (13)$$

where A_{hfs} is a number depending on the particular atom but not α and $F_{\text{hfs}}(\alpha)$ is a relativistic correction factor specific to each hyperfine transition.

(b) Show that F_{hfs} has no linear term in α . Then show that the change in the transition energy E between two levels in the same hyperfine multiple in response to α variation satisfies

$$\left. \frac{\delta E}{E} \right|_{\alpha=\alpha_0} = (K+2) \frac{\delta\alpha}{\alpha_0}, \quad (14)$$

where K is a quantity similar to that which you found in problem 1c.

Hint: It might be a good idea to use Mathematica or Maple to do the expansion

(c) Relate the change of E on the change of m_e/m_p .

Although we derived these results for a hydrogenlike atom/ion, they apply equally well for multi-electron atoms. As with problem 1, we are interested in how the ratio $R = E_1/E_2$ of two transition energies depends on the variation of α and m_e/m_p . We consider two separate cases: **(1)** both E_1 and E_2 are optical and **(2)** one where E_1 is optical but E_2 is microwave. Note that if we take E_2 to be the standard ground state hyperfine splitting in ^{133}Cs then R is the 'absolute' measurement of E_1 in the normal sense.

(d) Derive the sensitivity of R with respect to α and m_e/m_p in case **(1)**.

(e) Derive the sensitivity of R with respect to α and m_e/m_p in case **(2)**.