From scattering amplitudes to Newton's potential…and beyond

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with

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based on 1905.05657 [hep-th], 1911.10108 [hep-th], 2006.02375, 2012.06548 [hep-th], 2104.11206 [hep-th] + in progress

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SAGEX
Scattering Amplitudes: from Geometry to Experiment

Menu

- Newtonian potential from scattering amplitudes
  - Deflection angle…
- General relativity as an effective theory
  - extract predictions at low energy
- Classical GR corrections from quantum scattering
  - count powers of $\hbar$, Newtonian potential at higher powers of G
- Add higher-derivative interactions: cubic, quartic in R
  - potentials, deflection angles from the eikonal phase matrix
- Approach based on a new heavy-mass effective theory
  - compact trees for easy-to-integrate loops from a new colour/kinematics duality

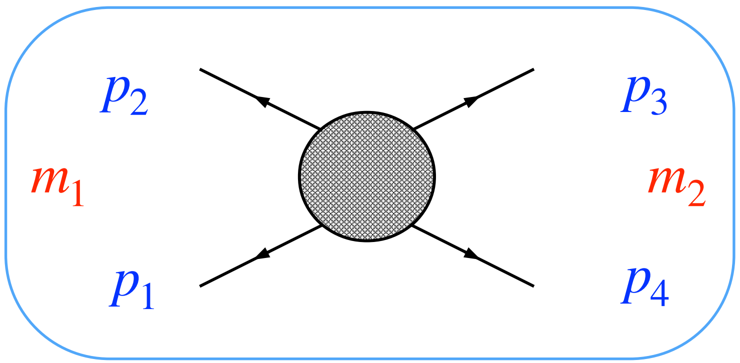
Newton potential from amplitudes

- Extract from two-to-two scattering of heavy particles
  - Connection to amplitudes suggested by Iwasaki in 1971
  - Classical + quantum correction at $O(G^2)$ from one-loop Feynman diagrams
  - Iwasaki pointed out the "erroneous belief" that only tree diagrams contribute to the classical potential
    (e.g. in R.P. Feynman, Acta Phys. Polon. 24 (1963), 697)
  - He also noted that "it is unclear whether one can obtain finite physically meaningful results from fourth-order Feynman diagrams since the quantum theory of gravity is unrenormalisable"
  - However, "it will be shown that in spite of the unrenormalisability we can obtain a finite physically meaningful potential"

The potential at tree level and one loop

Kinematic setup

- Elastic scattering of two massive scalars (COM frame):



$$p_1^\mu = -(E_1, \vec{p} - \vec{q}/2)\,, \qquad p_4^\mu = -(E_4, -\vec{p} + \vec{q}/2)$$

$$p_2^\mu = (E_2, \vec{p} + \vec{q}/2)\,, \qquad p_3^\mu = (E_3, -\vec{p} - \vec{q}/2)$$

$$E_1 = E_2 = \sqrt{m_1^2 + \vec{p}^{\,2} + \frac{\vec{q}^{\,2}}{4}}\,, \quad E_3 = E_4 = \sqrt{m_2^2 + \vec{p}^{\,2} + \frac{\vec{q}^{\,2}}{4}}$$

  - $-\vec{p}_1, \; -\vec{p}_4$ incoming
  - $\vec{q} = \vec{p}_1 + \vec{p}_2$ = momentum transfer, lives in a 2d space: $\vec{p} \cdot \vec{q} = 0$

$$s = (p_1 + p_2)^2 = -\vec{q}^{\,2}\,, \qquad t = (p_1 + p_4)^2\,, \qquad u = (p_1 + p_3)^2$$

- Static limit: $s = -\vec{q}^{\,2}, \quad t \simeq (m_1 + m_2)^2, \quad u \simeq (m_1 - m_2)^2 - s$

Dyson i (the i from the Schrödinger equation)

- Definition of the potential:

$$\langle f|S|i\rangle = \delta_{fi} + (2\pi)^4\delta^{(4)}(p_f - p_i)\, A_{fi}(\vec{q},\vec{p}) = -i(2\pi)\delta(E_f - E_i)\langle f|\tilde{V}|i\rangle$$

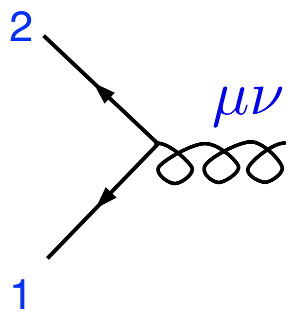
$$V(\vec{x}) \;=\; i\int \frac{d^3q}{(2\pi)^3}\, e^{\frac{i\vec{q}\cdot\vec{x}}{\hbar}}\, \frac{A_{fi}(\vec{q},\vec{p})}{4E_1E_4}$$

  - Note: potential is not uniquely defined!
  - Depends on the choice of coordinates
  - Observables (e.g. deflection angles) better quantities to compute

- Next, compute four-point scattering amplitude in GR

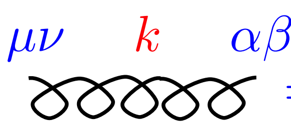
Tree level is Newton's law

- Feynman rules
  - $\phi\phi h$ vertex:



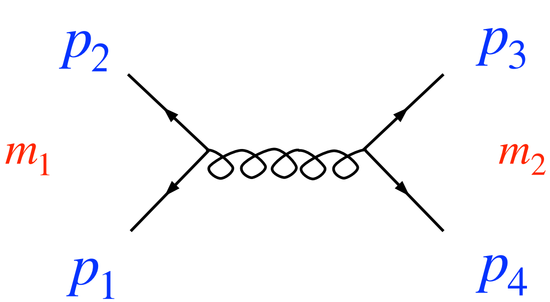
$$= i\left(\frac{\kappa}{2}\right)\left[-\eta^{\mu\nu}(p_1\cdot p_2+m^2)+p_1^\mu p_2^\nu+p_2^\mu p_1^\nu\right]$$

  - de Donder propagator:



$$= \frac{i}{k^2}\frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}\right)$$

- Amplitude:
  - $A(1_{m_1},2_{m_1},3_{m_2},4_{m_2}) =$



$$= i\left(\frac{\kappa}{2}\right)^2\frac{1}{4s}\left[s^2-(t-u)^2+8\,m_1^2m_2^2\right]$$

- In the static limit:
  - $s = -\vec{q}^{\,2}, \quad t \simeq (m_1 + m_2)^2$
  - $A \simeq i\,G\,\dfrac{16\pi\, m_1^2 m_2^2}{\vec{q}^{\,2}}$ with $G := \kappa^2/(32\pi)$ Newton's constant

- Static potential: $\widetilde{V}_{\rm static}(\vec{q}) := i\,\dfrac{A(\vec{q})}{4m_1 m_2}$
  - Finally using $\displaystyle\int \frac{d^3q}{(2\pi)^3}\frac{e^{-i\vec{q}\cdot\vec{r}}}{\vec{q}^{\,2}} = \frac{1}{4\pi r}$ we get

$$V_{\rm static} = -\frac{G m_1 m_2}{r}$$

Newton's law from scattering amplitudes!

What's next?

- Corrections in G and in $\hbar$
  - Relevant/irrelevant for experiments
  - Long history of calculations, usually in disagreement
    - First attempt at $O(G^2)$ by Iwasaki (1971)
    - Definitive calculation at 2PM:
      Bjerrum-Bohr, Donoghue, Holstein (2003)

n PM = G^n

$$V_{\text{static}} = -\frac{Gm_1m_2}{r}\left[1 + 3\frac{G(m_1+m_2)}{r} + \frac{41}{10\pi}\frac{G\hbar}{r^2}\right] + \mathcal{O}(G^3)$$

  - Rederived from amplitudes by Bjerrum-Bohr, Donoghue, Vanhove (classical + quantum) and Neill & Rothstein (classical)
  - Amplitude derivations free of potential mistakes due to evaluation of a large set of (separately non gauge-invariant) Feynman diagrams

Gravity as an effective field theory

- Start with Einstein-Hilbert Lagrangian + matter…

$$\mathcal{L} = \sqrt{-g}\Big[-\frac{2}{\kappa^2}R + \frac{1}{2}\sum_{i=1}^{2}(D^\mu\phi_i D_\mu\phi_i - m_i^2\phi_i^2)\Big]$$

- …+ higher-derivative corrections. Appear already…
  - …in the effective action for closed strings (Tseytlin 1986)
  - quadratic terms in the curvature appear as one-loop counterterms in gravity coupled to matter (pure gravity renormalisable at one loop)

$$\Delta\mathcal{L}^{(1)}_{\text{c.t.}} \sim \frac{1}{\epsilon}R^2$$

('t Hooft & Veltman 1974)

  - At two loops, cubic counterterms also appear

$$\Delta\mathcal{L}^{(2)}_{\text{c.t.}} \sim \frac{1}{\epsilon}R^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\lambda}R^{\rho\lambda}{}_{\mu\nu}$$

(Goroff & Sagnotti 1985)

- Treat GR as an effective theory

$$\mathcal{L} = \sqrt{-g}\Big[-\frac{2}{\kappa^2} R + a\,R^2 + b\,R^{\mu\nu}R_{\mu\nu} + c\,R^{\mu\nu\alpha\beta}R_{\alpha\beta\rho\lambda}R^{\rho\lambda}{}_{\mu\nu} + \cdots \Big]$$

  - must include all higher-derivative interactions (modulo field redefinitions)
  - what about non-renormalisability?

- Focus on low energy
  - Low-energy predictions reliable even if theory is non-renormalisable
  - Original application to phenomenological Lagrangians for the pion S-matrix (Weinberg 1979)
  - Applied to gravity by Donoghue (1994)

Key points

- Shift attention away from the UV to the IR
  - UV: we don't know what is the ultimate theory…
  - IR: we know gravitons and their interactions
  - low-energy gravitons propagate long distances
  - Signature of long-range effects is non-analyticity
    - Typical one-loop terms: $1/\sqrt{-q^2}$ and $\log(-q^2)$ (in Fourier space)
  - long-range effects dominate over analytic contributions from propagation of massive modes
    - analytic terms give rise to localised (short-range) contributions

- Summary: find non-analytic terms in the amplitudes
  - unitarity cuts!

Classical physics from loops??

(Donoghue & Holstein; Iwasaki…; Kosower, Maybee, O'Connell)

- Loop expansion is not an $\hbar$ expansion
  - Itykson-Zuber, chapter 6.2.1:
  - "The loopwise perturbative expansion, i.e. the expansion according to the increasing number of independent loops of connected Feynman diagrams, may be identified with an expansion in powers of $\hbar$…
  - …we leave aside the factor of $\hbar$ that gives the mass term a correct dimension. In other words, the Klein-Gordon equation should read

$$\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\phi = 0$$

  - ….indicating that the mass is of quantum origin. This phenomenon is disregarded in the sequel."

- **Need also $\hbar$ from masses**
  - counting powers of $\hbar$ from propagators and vertices not enough

- **Propagator:**

No $\hbar$!

$$\langle 0|T\big(\phi(x)\phi(0)\big)|0\rangle \;=\; \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{i\,\hbar}{k^2 - \left(\frac{mc}{\hbar}\right)^2 + i\epsilon}$$

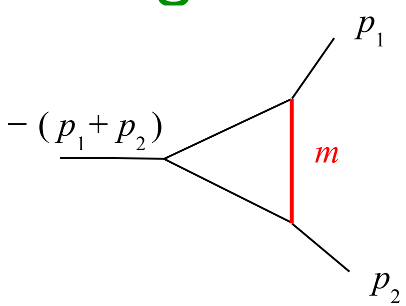
  - k is the wave four-vector, so that the loop momentum is $\ell = \hbar\, k$

- **Classical effects from quantum loops:**
  - Consider the combination: $\sqrt{\frac{m^2}{-q^2}} = \frac{1}{\hbar}\frac{m}{\sqrt{-k^2}}$
  - may cancel $\hbar$ from the loop expansion, giving a classical effect!
  - it appears in familiar integral functions

- Triangle with one internal mass

$p_1 = \hbar\, k_2$

$p_2 = \hbar\, k_2$



$$= I_3(s;m) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - (\frac{mc}{\hbar})^2)(k-k_1)^2(k+k_2)^2}$$

$$= -\frac{i}{32}\left[\frac{1}{(mc/\hbar)\sqrt{-k_{12}^2}} + \frac{\log(-k_{12}^2/(mc/\hbar)^2)}{\pi^2\,(mc/\hbar)^2}\right] + \mathcal{O}(\sqrt{k_{12}^2})$$

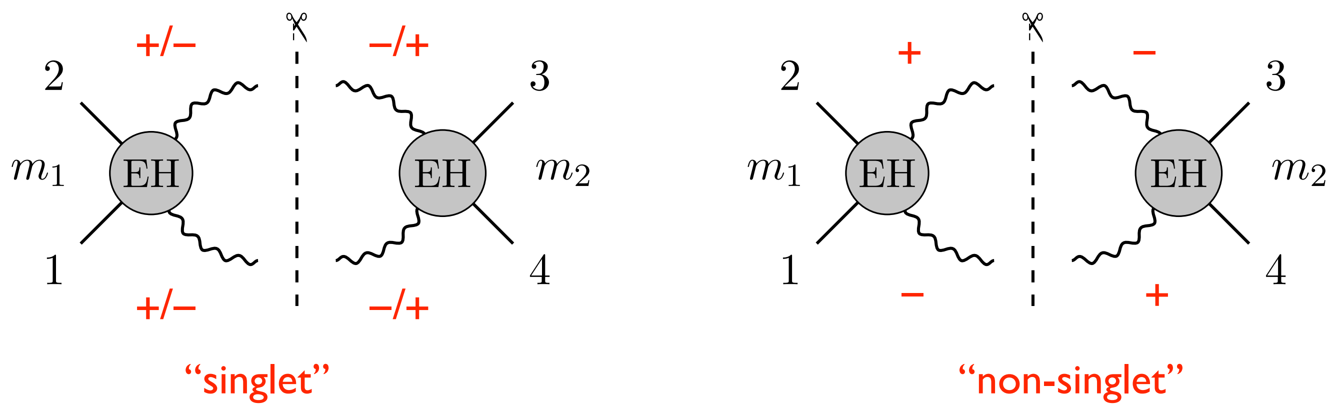
- Note:
  - The two terms have different powers of $\hbar$
  - Second term is O($\hbar$) compared to the first one: classical and quantum (when combined with everything)

- General (one-loop) rule:
  - $\sqrt{-q^2}$ terms: classical
  - $\log(-q^2)$ terms: quantum
  - systematic study by Kosower, Maybee O'Connell

Summarising:

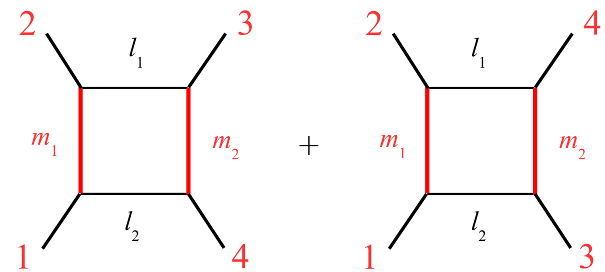
- Use amplitudes to compute the potential
- Focus on non-analytic terms
  - From long-range propagation of two or more massless particles
  - at low energy, this dominates over the analytic terms from the propagation of massive modes
  - Need to look only at the discontinuity in the q^2-channel
    - reconstructed from the cut diagrams in the corresponding channel
    - massless particles in the cut
    - $1/\sqrt{-q^2}$ classical; $\log(-q^2)$ quantum
  - Ideal calculation for amplitudeologists — don't need even to reconstruct the amplitude from different cuts!

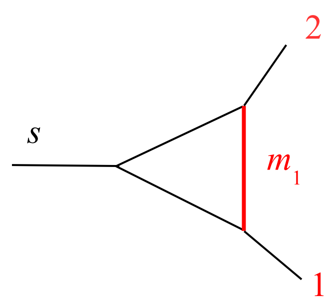
- One loop: cut in the $s = -\vec{q}^{\,2}$ channel

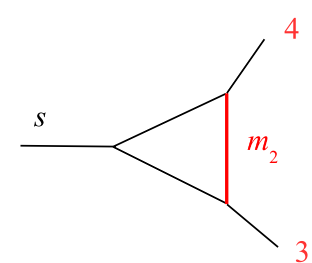


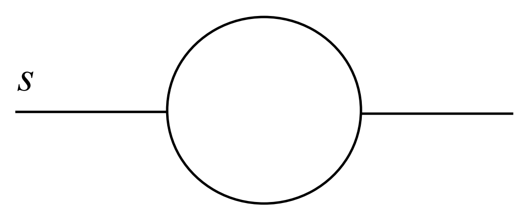
- Massless gravitons in the loop
- Two cases: singlet and non-singlet (two internal helicity assignments)
- Can use four-dimensional amplitudes since any rational terms do not produce terms with discontinuities
- Tree amplitudes can be generated with BCJ, KLT, Feynman diagrams…

- Static potential: result of reduction & static limit
  - $s \to 0\,, \quad t \to (m_1 + m_2)^2$

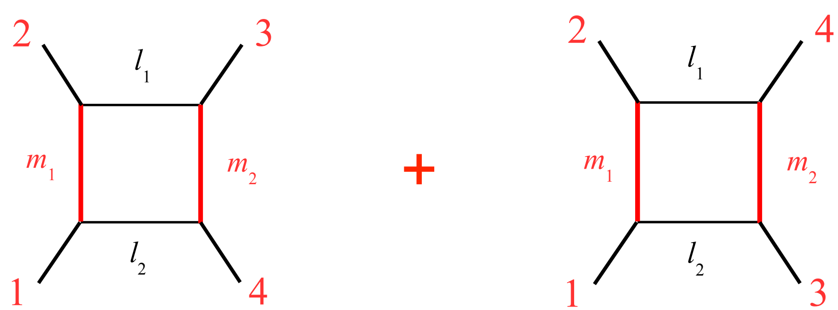
Boxes: $4\, m_1^4 m_2^4$ []static limit

Triangles: [$-12 m_1^4 m_2^2$ 

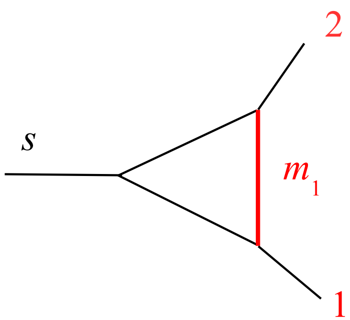
$-12 m_1^2 m_2^4$ ]static limit

Bubbles: $\frac{73}{15} m_1^2 m_2^2$ []static limit

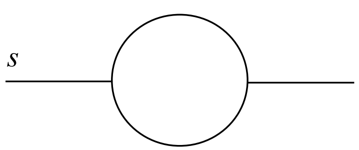
- ## Limits on functions:
  - blue= classical   red=quantum



$$\rightarrow \frac{1}{16\pi\, s\, m_1\, m_2} \cdot \log(-s)$$



$$\rightarrow -\frac{i}{32}\left[\frac{1}{m_1\sqrt{-s}} + \frac{\log(-s)}{\pi^2 m_1^2}\right]$$



$$\rightarrow -\frac{i}{16\,\pi^2}\log(-s)$$

- **The potential:**
  - Multiply amplitude by $i/(4m_1 m_2)$
  - Reinstate one-loop prefactor of $(\kappa/2)^4 = 64\,\pi^2 G^2$
  - Tree level:
    $$V^{(0)}_{\rm static} = G\frac{4\pi m_1 m_2}{s}$$ classical
  - From one loop:
    $$V^{(1)}_{\rm static} = G^2\left[\frac{6\pi^2 m_1 m_2(m_1+m_2)}{\sqrt{-s}} + \hbar\,\frac{41}{5} m_1 m_2 \log(-s)\right]$$

one loop, classical — one loop, quantum

- **Final result:** (Iwasaki 1971, Bjerrum-Bohr, Donoghue, Holstein 2003)

$$V_{\rm static} = -\frac{G m_1 m_2}{r}\left[1 + 3\frac{G(m_1+m_2)}{r} + \frac{41}{10\pi}\frac{G\hbar}{r^2}\right] + \mathcal{O}(G^3)$$

Scales in the problem

- Recast $V(r) = -\frac{Gm_1 m_2}{r}\left[1 + \frac{3(r_{S,1}+r_{S,2})}{2\,r} + \frac{41}{10\pi}\frac{\ell_P^2}{r^2}\right] + \mathcal{O}(G^3)$

- Planck length $\ell_P = \sqrt{\hbar\, G/c^3} \sim 3 \times 10^{-35} m$

- Schwarzschild radius of a classical source: $r_S = \frac{2\,GM}{c^2}$
  - Human body (~ 70 kg): $10^{-25}\, m$
  - Earth (6×10^{24} kg) : $10^{-2}\, m$
  - Sun (2×10^{30} kg): $3 \times 10^{3}\, m$
  - Cygnus X-1 (~$14.8\, M_\odot$) $41 \times 10^{3}\, m$
  - Sagittarius A* SMBH (8×10^{36} kg): $10^{10}\, m$

Higher-derivative modifications

Cubic corrections to Newton's potential

(Brandhuber, GT; Emond, Moynihan)

- Add terms to EH action cubic in curvature:

$$S = -\frac{2}{\kappa^2}\int d^4x\sqrt{-g}\Big[R + \frac{\alpha'^2}{48} I_1 + \frac{\alpha'^2}{24} G_3\Big]$$

  - $I_1 := R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}$ $\qquad G_3 := I_1 - 2R^{\mu\nu\alpha}{}_{\beta}R^{\beta\gamma}{}_{\nu\sigma}R^{\sigma}{}_{\mu\gamma\alpha}$
  - Only two independent combinations to consider (Metsaev & Tseytlin)
  - Added with coefficient from bosonic strings, but treated separately

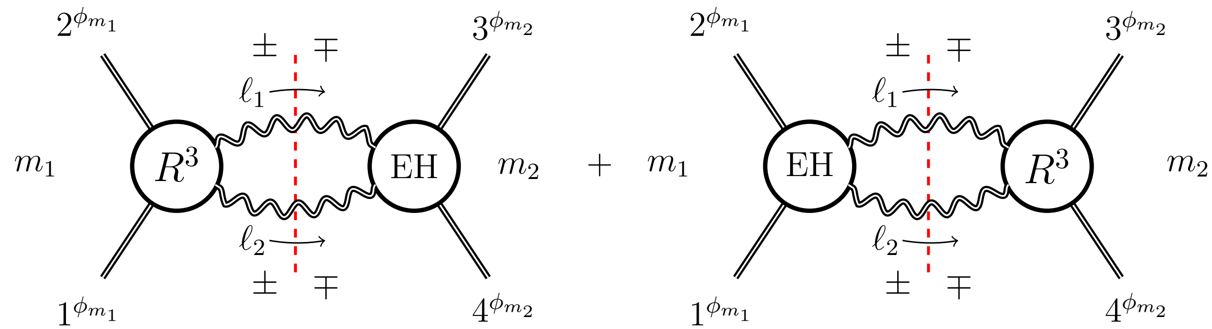
- Special features of these couplings:
  - I_1: generates three-point all-plus/all-minus amplitudes: (Dixon, Broedel)

$$A_{I_1}(1^{++},2^{++},3^{++}) = -i\left(\frac{\kappa}{2}\right)\left(\frac{\alpha'}{4}\right)^2([12][23][31])^2$$

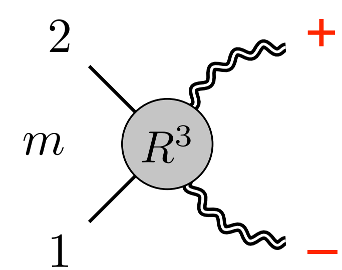
  - G_3: vanishing three- & four-point graviton amplitudes, topological in 6D

- Next: compute the potential from cuts

- Two diagrams:



  - note absence of non-singlet channel since



$= 0$

- Relevant four-point amplitudes:

$$A_{\mathrm{EH}}(1^{\phi_{m_1}}, 2^{\phi_{m_1}}, \ell_1^{--}, \ell_2^{--}) = -\left(\frac{\kappa}{2}\right)^2 m_1^4 \frac{\langle \ell_1\, \ell_2\rangle^2}{[\ell_1\, \ell_2]^2}\left[\frac{i}{(\ell_1+p_1)^2 - m_1^2} + \frac{i}{(\ell_1+p_2)^2 - m_1^2}\right]$$

$$A_{I_1}(-\ell_1^{++}, -\ell_2^{++}, 3^{\phi_{m_2}}, 4^{\phi_{m_2}}) = \left(\frac{\kappa}{2}\right)^2\left(\frac{\alpha'}{4}\right)^2 \frac{4i}{s_{12}}[\ell_1\, \ell_2]^4\,(\ell_1\cdot p_3)(\ell_2\cdot p_3)$$

$$A_{G_3}(-\ell_1^{++}, -\ell_2^{++}, 3^{\phi_{m_2}}, 4^{\phi_{m_2}}) = \frac{i}{2}\left(\frac{\kappa}{2}\right)^2\left(\frac{\alpha'}{4}\right)^2 [\ell_1 \ell_2]^4 (s+2m^2)$$

Note absence of pole

- Proceed as before

- **Result for I_1 :**

$$V_{\rm cl}(\vec{r},\vec{p}) = \frac{(\alpha' G)^2}{r^6} \, \frac{3(m_1+m_2)}{32 E_1 E_4} \left[(t-m_1^2-m_2^2)^2 - 4m_1^2 m_2^2\right]$$
$$\simeq \frac{(\alpha' G)^2}{r^6} \left[\frac{3}{8}\frac{(m_1+m_2)^3}{m_1 m_2}\,\vec{p}^{\,2}\right]$$

$$V_{\rm qu}(\vec{r},\vec{p}) = \frac{(\alpha' G)^2}{r^7} \left\{-\frac{15}{4\pi}\frac{\left[(t-m_1^2-m_2^2)^2 - 2m_1^2 m_2^2\right]}{E_1 E_4}\right\}$$
$$\simeq \frac{(\alpha' G)^2}{r^7} \left\{-\frac{15}{4\pi}\left[2m_1 m_2 + \vec{p}^{\,2}\left(8 + 3\frac{m_1^2+m_2^2}{m_1 m_2}\right)\right]\right\}$$

- **Comments:**
  - $V = V_{\rm cl} + \hbar\, V_{\rm qu}$, keep also "post-Newtonian" corrections
  - $1/r^6$ (classical) and $1/r^7$ (quantum) corrections to Newton's potential
  - $V_{\rm cl}$ vanishes in the static limit $\vec{p} \to \vec{0}$ (with $E_{1,4} \to m_{1,2}$)

- Result for G_3 :

$$V_{G_3} \;=\; 12\,\frac{(\alpha' G_N)^2}{r^6}\,\frac{(m_1 m_2)^2}{E_1 E_4}\,\left[(m_1+m_2) \;-\; \frac{\hbar}{r}\,\frac{10}{\pi}\right]$$

- Comment:
  - Curiously non-vanishing in the static limit $(E_1 \to m_1, E_4 \to m_2)$

Observables

- Deflection angle of massless scalars, photons and gravitons passing by a heavy scalar of mass m
  - In EH: classical part universal, but not the quantum part (Bjerrum-Bohr, Donoghue, Holstein, Planté, Vanhove; Chi)
  - particles follow geodesics regardless of their species
  - For cubic couplings: we find universality of the quantum parts for scalars and photons (Brandhuber, GT)
  - More delicate discussion for gravitons (Accettulli Huber, Brandhuber, De Angelis, GT)
- Strategy of the calculation
  - compute amplitude discontinuities due to low-energy gravitons
  - find the deflection angle using standard techniques: from eikonal phase matrix or from the potential
  - no need to assume that helicity of bent particle stays unchanged!

S-matrix in the eikonal approximation

Amati, Ciafaloni & Veneziano (1987); Kabat & Ortiz (1992); ….Weinberg (1965)…

- Work in the limit: $m \gg \omega \gg \sqrt{\vec{q}^{\,2}}$
  - ω = energy of massless scattered particle
- S-matrix in impact parameter space:

$$S_{\mathrm{eik}} = e^{i(\delta_0+\delta_1+\cdots)} \;=\; 1 + \tilde{A}^{(0)}_{\omega} + \tilde{A}^{(1)}_{\omega^2} + \tilde{A}^{(1)}_{\omega} + \cdots$$

  - $\tilde{A}(\vec{b}) \;=\; \frac{1}{4m\omega}\int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{i\vec{q}\cdot\vec{b}}\, A(\vec{q})$ $\qquad$ $b =$ impact parameter
- Eikonal phases: $\delta_0 = -\,i\,\tilde{A}^{(0)}_{\omega}\,,\ \delta_1 = -\,i\,\tilde{A}^{(1)}_{\omega}$
  - Consistency condition: $\tilde{A}^{(1)}_{\omega^2} = \frac{1}{2}(\tilde{A}^{(0)}_{\omega})^2$ or $\tilde{A}^{(1)}_{\omega^2} = -\frac{(\delta_0)^2}{2}$
  - At each order in G and large ω, terms growing faster than ω simply exponentiate divergent-in-energy terms from lower loops orders

Deflection angle & time delay

- Deflection angle from eikonal phase (matrix):
  (Amati, Ciafaloni, Veneziano; …Bjerrum-Bohr, Donoghue, Holstein, Planté, Vanhove)

$$\theta = \frac{1}{\omega}\frac{\partial}{\partial b}(\delta_0 + \delta_1 + \cdots)$$

- Deflection angle from the potential:
  (Donoghue & Holstein 1985)

$$\theta = -\frac{b}{\omega}\int_{-\infty}^{+\infty} du \, \frac{V'(b\sqrt{1+u^2})}{\sqrt{1+u^2}}$$

  - Results are identical (Bjerrum-Bohr, Donoghue, Holstein, Planté, Vanhove)

- Shapiro time delay (Shapiro 1964)

$$t = \frac{\partial}{\partial \omega}(\delta_0 + \delta_1 + \cdots)$$

  - Time advance leads to violation of causality

- In Einstein-Hilbert theory:
  - Deflection angle: (tree level: Einstein 1911, up to a famous factor of 2)

$$\theta_{\rm EH} = -\frac{4Gm}{b}\left(1 + G\frac{15\pi}{16}\frac{m}{b} + \cdots\right)$$

- Shapiro time delay (Shapiro 1964)

$$t_{\rm EH} = 4Gm\left(\log\frac{b_0}{b} + \frac{15\pi}{16}\frac{Gm}{b} + \cdots\right)$$

  - As usual, the time delay is defined as the difference between the time delays measured by an observer at b and one at $b_0 \gg b$

- Amplitude result for I_1
  - $$A^{(1)} = \mathcal{D}\left[(m^2 s\,\omega)^2 I_3(s;m) \;+\frac{3}{2}(ms\,\omega)^2 I_2(s)\right]$$
  - Same for photons and massless scalars (gravitons discussed later)
  - up to irrelevant phases (contained in $\mathcal{D}$)
- Amplitude result for G_3 : zero!
- Comments
  - Integrands and PV reductions look completely different
  - Only after eikonal limit is taken miracles occur and the two expressions coincide!
  - Would be nice to find a way to get the eikonal result directly!

On to the bending angle

- Compute deflection angle
  - From eikonal phase or from potential
- Result is: (Brandhuber, GT)

$$\theta = (\alpha' G)^2 \, \frac{3}{64} \Big(15\pi \, \frac{m^2}{b^6} - \hbar \frac{1024}{\pi} \, \frac{m}{b^7} \Big)$$

- Comments:
  - suppressed by a factor of $(\alpha')^2/b^4$ compared to GR:

$$\theta_{\rm EH} = -\Big(G \, \frac{4\,m}{b} + G^2 \, \frac{15\pi}{4} \, \frac{m^2}{b^2} + \cdots \Big) = -\Big(2\,\frac{r_S}{b} + \frac{15\pi}{16} \Big(\frac{r_S}{b}\Big)^2 + \cdots \Big)$$

  - universality of classical and quantum parts
  - in EH only classical part is universal; quantum corrections differentiate

Graviton bending

(Accettulli Huber, Brandhuber, De Angelis, GT)

- The story is more delicate/interesting
  - at tree level the amplitude where the helicity of the scattered graviton flips dominates in energy due to the nature of R^3 coupling
  - Eikonal phase matrix appears already at tree level!

$$\delta \sim \begin{pmatrix} \tilde{A}(\phi,\phi,h^{++},h^{--}) & \tilde{A}(\phi,\phi,h^{++},h^{++}) \\ \tilde{A}(\phi,\phi,h^{--},h^{--}) & \tilde{A}(\phi,\phi,h^{--},h^{++}) \end{pmatrix}$$

  - $\tilde{A} =$ amplitude in impact parameter space
  - closely related to earlier work of Camanho, Edelstein, Maldacena and Zhiboedov (CEMZ) on tree-level four-point graviton scattering

- **Leading eikonal matrix**

$$\delta_0 \;\rightarrow\; \left(\frac{\kappa}{2}\right)^2 \frac{m\omega}{2\pi} \begin{pmatrix} -\frac{1}{2\epsilon} - \log|\vec{b}| & \left(\frac{\alpha'}{4}\right)^2 \frac{3}{\bar{b}^4} \\ \left(\frac{\alpha'}{4}\right)^2 \frac{3}{b^4} & -\frac{1}{2\epsilon} - \log|\vec{b}| \end{pmatrix}$$

$$b = \frac{b_1 + i b_2}{2}, \quad \bar{b} = \frac{b_1 - i b_2}{2}$$
$$(\vec{b} \cdot \hat{z} = 0)$$

  - Eigenvalues:

$$\delta_0^{(1,2)} \;\rightarrow\; \left(\frac{\kappa}{2}\right)^2 \frac{m\omega}{2\pi}\left[-\frac{1}{2\epsilon} - \log|\vec{b}| \pm \left(\frac{\alpha'}{4}\right)^2 \frac{48}{|\vec{b}|^4}\right]$$

  - Compare to CEMZ
    - $m\omega$ replaced by ω^2

- Exponentiation checked: $\tilde{A}^{(1)}_{\omega^2} = -(\delta_0)^2/2$
- Subleading eikonal matrix

$$\delta_{1,R^3} = \left(\frac{\kappa}{2}\right)^4 \frac{1}{256\pi} \frac{m^2\omega}{|\vec{b}|} \begin{pmatrix} c_1 \left(\frac{\alpha'}{4}\right)^2 \frac{1}{|\vec{b}|^4} & \left(\frac{\alpha'}{4}\right)^2 \frac{c_2}{\vec{b}^4} \\ \left(\frac{\alpha'}{4}\right)^2 \frac{c_2}{b^4} & c_1 \left(\frac{\alpha'}{4}\right)^2 \frac{1}{|\vec{b}|^4} \end{pmatrix}$$

$$c_1 = -9$$
$$c_2 = \frac{1365}{16}$$

- Result for deflection:

$$\theta^{(1,2)} = -\frac{4Gm}{b}\left[1 + \frac{15\pi}{16}\frac{Gm}{b^2} \pm \left(\frac{\alpha'}{4}\right)^2 \frac{192}{b^4} + \frac{5\pi}{16}(-9 \pm 1365)\left(\frac{\alpha'}{4}\right)^2 \frac{Gm}{b^5}\right]$$

- Result for time delay / advance:

$$t^{(1,2)} = 4Gm\left\{\log\frac{b_0}{b} + \frac{15\pi}{16}\frac{Gm}{b} + \left(\frac{\alpha'}{4}\right)^2\left[\pm 48\frac{1}{b^4} + \frac{\pi}{16}(-9\pm 1365)\frac{Gm}{b^5}\right]\right\}$$

  - CEMZ argued that causality violation occurs for small enough b
  - time advance overrides Shapiro's time delay, leading to superluminal effects / causality violations
  - In their approach, theory treated as fundamental — causality restored by adding an infinite tower of massive particles: string theory!
  - In an EFT approach, breakdown occurs near where we stop trusting our predictions: $b \lesssim (\alpha')^{\frac{1}{4}} \sim \Lambda^{-1}$
  - Superluminality effects unresolvable within the regime of validity of the EFT (De Rham & Tolley; Accettulli Huber, Brandhuber, De Angelis, GT)
  - No causality issues for photons or scalars, nor from the G_3 interaction

Comments, and more results

- Quadratic terms in the curvature do not contribute
  - in bosonic strings (Tseytlin; Deser & Redlich)…
  - …plus scalars (Accettulli-Huber, Brandhuber, De Angelis, GT)
  - Field redefinitions, amplitude techniques
- Bending in bosonic string theory (Brandhuber, GT)

$$S_B \ = \ -\frac{2}{\kappa^2}\int d^4x\sqrt{-g}\Big[R-2(\partial\Phi)^2-\frac{1}{12}|dB|^2+\frac{\alpha'}{4}e^{-2\Phi}G_2+\alpha'^2e^{-4\Phi}\Big(\frac{1}{48}I_1+\frac{1}{24}G_3\Big)+\mathcal{O}(\alpha'^3)\Big]$$

  - $G_2 := R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} - 4R^{\alpha\beta}R_{\alpha\beta} + R^2$ Gauß-Bonnet combination
  - $I_1 := R^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}$ $\qquad G_3 := I_1 - 2R^{\mu\nu\alpha}{}_{\beta}R^{\beta\gamma}{}_{\nu\sigma}R^{\sigma}{}_{\mu\gamma\alpha}$
  - Two insertions of G_2 can produce a new four-graviton amplitude by contracting the two dilatons

- Result and comparison to R^3 case:

$(G_2)^2$:

$$\theta = (\alpha' G)^2 \Big\{ -\frac{1575\,\pi}{64}\frac{m^2}{b^6} + \hbar\frac{64}{\pi}\Big[-21\log\big(b/(2r_0)\big) + \frac{229}{4}\Big]\frac{m}{b^7}\Big\}$$

$r_0 := (\mu e^{\gamma_E})^{-1}$

R^3 :

$$\theta = (\alpha' G)^2\,\frac{3}{64}\Big(15\pi\,\frac{m^2}{b^6} - \hbar\frac{1024}{\pi}\,\frac{m}{b^7}\Big)$$

- $(G_2)^2$ classical (quantum) deflection larger by a factor of ~30 (~80)

Massive dilaton:

$$\theta = (\alpha' G)^2\,\frac{\omega^2}{M_\phi^2}\Big[\frac{1575\,\pi}{64}\,\frac{m^2}{b^6} - \hbar\frac{1536}{\pi}\,\frac{m}{b^7}\Big]$$

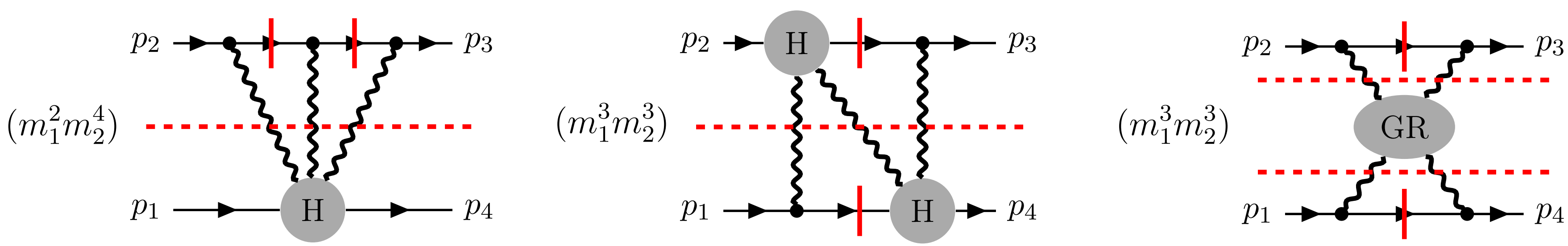
- large suppression factor ω^2/M_ϕ^2

Heavy-mass effective theory

(Brandhuber, Chen, GT, Wen)

- Use EFT language from the beginning!
  - Momenta exchanged between particles much smaller than particles' masses

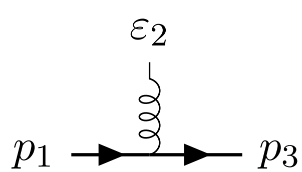
- Goal: construct compact HEFT tree amplitudes with two heavy scalars plus many gluons/gravitons
  - Enter the unitarity cuts, e.g.



  - Similar to heavy-quark effective theory

- Tool: gauge-invariant formulation of the double-copy

Basics of HEFT

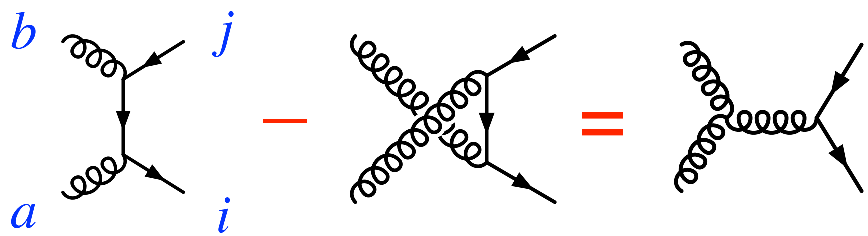
- Momentum of a particle in heavy-mass effective theory:
  - Incoming: $p^\mu = m\, v^\mu$
  - After the interaction with a soft particle: $p^{\mu'} = m\, v^\mu + k^\mu$
  - In QCD, one would take k of order $\Lambda_{\rm QCD} \ll m$
  - For classical gravitational physics: $k^\mu = \hbar\, \hat{k}^\mu$, with $\hat{k}^\mu$ fixed as $\hbar \to 0$
- Three-point amplitudes (with scalars):
  - Yang-Mills: $A_3^{\rm YM-M} =$  $= m\, \epsilon_2 \cdot v$
  - Gravity: $A_3^{\rm GR-M} = (A_3^{\rm YM-M})^2 = (m\, \epsilon_2 \cdot v)^2$ quadratic in m
  - Squaring from KLT relations…or the double copy

- Apply double copy to HEFT
  - Standard double copy studied by Haddad & Helset
  - We propose a novel double-copy based on work developed in YM (Chen, Johansson, Teng & Wang '19)
  - Advantages:
    - manifestly gauge invariant numerators (=term by term)
    - compact expressions, fewer diagrams
    - easier loop integrations

Colour/kinematics duality in one slide

(Bern, Carrasco, Johansson)

- Compton amplitude in gauge theory:



$$(T^a)^i{}_m (T^b)^m{}_j \; - \; (T^b)^i{}_m (T^b)^m{}_j \; = \; i f^{abc} (T^c)^i{}_j$$

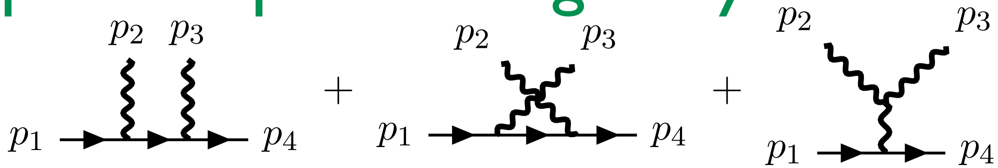
- Relations between triplets of colour factors: $c_I - c_{II} = c_{III}$
  - Jacobi identity, from colour algebra

- Write amplitudes as $A_n = \sum_{i \in \Gamma} \frac{c_i n_i}{d_i}$
  - $\Gamma =$ set of all cubic graphs

- Numerators satisfy $n_I - n_{II} = n_{III}$
  - manifestation of an underlying kinematic algebra ? Colour/Kinematics duality
  - For self-dual YM: area-preserving diffeomorphisms (Monteiro & O' Connell)

Double copy to gravity

- Gauge theory amplitude $A_n^{\mathrm{YM}} = \sum_{i \in \Gamma} \frac{c_i \, n_i}{d_i}$
- Obtain a gravity amplitude as $A_n^{\mathrm{GR}} = \sum_{i \in \Gamma} \frac{n_i \, n_i'}{d_i}$
- Compton amplitude in gravity



$$A_4^{\mathrm{GR-M}} = \frac{N_4(1234)^2}{2p_1 \cdot p_2} + \frac{N_4(1324)^2}{2p_1 \cdot p_3} + \frac{N_4(1[2,3]4)^2}{s_{23}}$$

- Numerators satisfy $N_4(1[2,3]4) = N_4(1234) - N_4(1324)$
  - Drawback: numerators are not gauge invariant, leading to potentially very large expressions
  - Improve on this!

- Gauge-invariant double copy from algebraic numerators for HEFT (Brandhuber, Chen, GT, Wen)
  - Introduce vector and tensor currents representing the generators of the kinematic algebra
  - Construct a fusion rule among them
- Key features:
  - Sum over a subset of cubic diagrams where the two massive particles always connect via a single cubic vertex
  - Diagrams contain only massless propagators
  - Much fewer terms, compact expressions, easy to integrate!

Example

- Five-point amplitude with new double copy

$$A_5^{\mathrm{YM-M}}(12345) = \frac{\mathcal{N}_5([[2,3],4],v)}{s_{234}s_{23}} + \frac{\mathcal{N}_5([2,[3,4]],v)}{s_{234}s_{34}}$$

$$A_5^{\mathrm{GR-M}}(12345) = \frac{\left[\mathcal{N}_5([[2,3],4],v)\right]^2}{s_{234}s_{23}} + \frac{\left[\mathcal{N}_5([[2,4],3],v)\right]^2}{s_{234}s_{24}} + \frac{\left[\mathcal{N}_5([[3,4],2],v)\right]^2}{s_{234}s_{34}}$$

- Particles 1 and 5 are the massive scalars
- $\mathcal{N}_5([[2,3],4],v) = \mathbb{L}(2,3,4) \circ \left[m \frac{v \cdot F_2 \cdot F_3 \cdot V_3 \cdot F_4 \cdot v}{(v \cdot p_3)\,(v \cdot p_4)}\right]$ nested numerator $\quad V_i^{\mu\nu} = v^\mu p_i^\nu$
- $\mathbb{L}(i_1, i_2, \ldots, i_r) := \left[\mathbb{I} - \mathbb{P}_{(i_2 i_1)}\right]\left[\mathbb{I} - \mathbb{P}_{(i_3 i_2 i_1)}\right]\cdots\left[\mathbb{I} - \mathbb{P}_{(i_r \ldots i_2 i_1)}\right]$
- $\mathbb{P}_{(j_1 j_2 j_3 \ldots j_m)}$ denotes the cyclic permutation $j_1 \to j_2 \to j_3 \to \cdots \to j_m \to j_1$

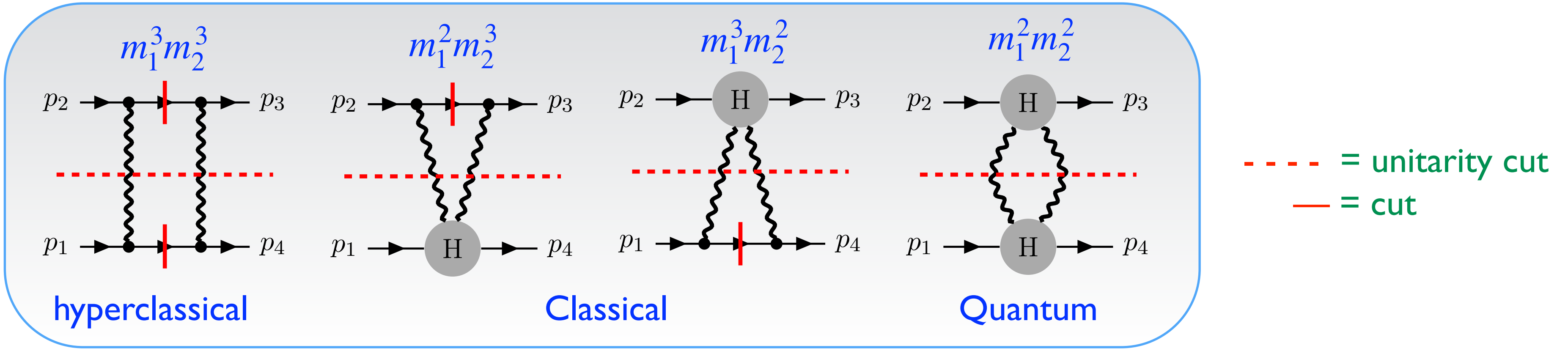
- Term by term gauge invariant!
  - kinematic numerators expressed in terms of field strengths
  - results from standard double copy / Feynman diagrams considerably more complicated!
- Jacobi relations automatically satisfied!
  - Numerators with "nested commutators"
  - $\mathcal{N}_5([[2,3],4], v) := \mathcal{N}_5(234,v) - \mathcal{N}_5(324,v) - \mathcal{N}_5(423,v) + \mathcal{N}_5(432,v)$ and so on
  - We have automatically

$$\mathcal{N}_5([2,[3,4]], v) = \mathcal{N}_5([[2,3],4], v) - \mathcal{N}_5([[2,4],3], v)$$

- Just an example, can obtain higher-point amplitudes
  - e.g. six-point amplitudes for three-loop potential calculation

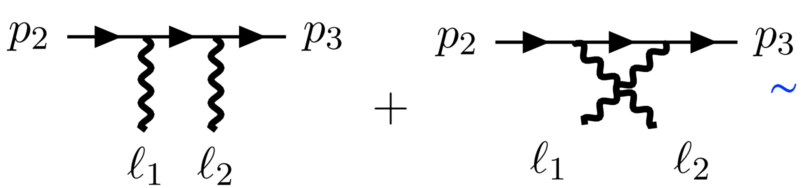
One loop

- Expansion of the one-loop amplitude in the masses:



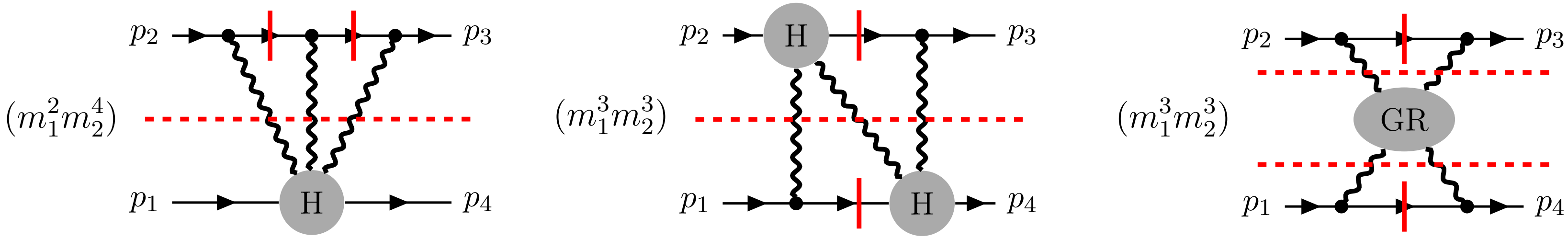
  - Power of mass is power of $1/\hbar$
  - Nice diagrammatic decomposition! (possibly reminiscent of the non-abelian exponentiation theorem for the Wilson loop…)
  - Hyperclassical term exponentiates tree level in impact parameter space

- To leading order in $q = p_3 - p_2$: (Kabat & Ortiz; Akhouri, Saotome & Sterman)

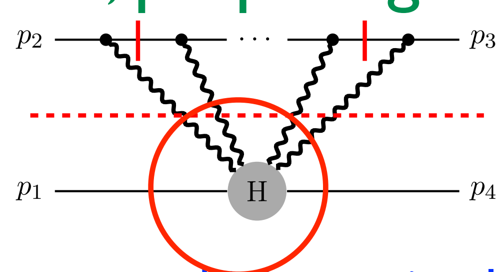


$$\sim \frac{i}{2m_2(v_2\cdot\ell_1)+i\varepsilon} + \frac{i}{-2m_2(v_2\cdot\ell_1)+i\varepsilon} = 2\pi\,\delta\big(2m_2(v_2\cdot\ell_1)\big) \sim \frac{1}{m_2}$$

Two loops



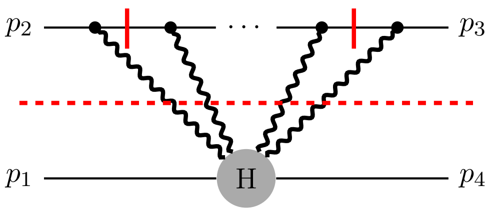
- Done already a number of times
  - First computation by Bern, Cheung, Roiban, Shen, Solon, Zeng (2019)
  - More recently: Cheung & Solon (2020), Bjerrum-Bohr et al (2021)
- Our goal: simplify the calculation, preparing the way to higher loops
  - Compactness of our expressions
  - HEFT amplitudes contain linear propagators which are simple(r) to integrate!



- One-loop done

- Two loops: almost completed
  - Integrations performed with Henn's differential equation method
  - Boundary conditions easy to impose

- Conjecture for the probe limit at any loops



Conclusions & open problems

- On-shell methods applied to problems in classical GR
  - Newton potential, particle deflection…
  - powerful applications of amplitude methods
- General relativity regarded as an effective theory
  - We focused on cubic corrections to curvature
- Heavy-mass effective theory and its double copy
  - Compact trees for better loop integrations
- (Some) open issues
  - computation of higher PM terms in the potential
  - Radiation, and connection to wave forms for gravitational waves!

and many more…