$$E = \frac{\underline{p}^2}{2m} + V(\underline{x})$$

$(\hbar = c = 1)$

$$E \Rightarrow i\frac{\partial}{\partial t} \qquad \underline{p} \Rightarrow -i\underline{\nabla}$$

$$\phi(t,\underline{x}) \propto e^{-i(Et - \underline{p}\cdot\underline{x})} = e^{-ip\cdot x}$$

$$i\partial_t \phi(t,\underline{x}) = \left(-\frac{1}{2m}\nabla^2 + V(x)\right)\phi(t,\underline{x}) = H\phi(t,\underline{x})$$

H: Hamiltonean. This is the Schrödinger eqn.

$$E^2 = m^2 + p^2 \qquad p_\mu = i\partial_\mu$$

$$(\partial_t^2 - \nabla^2 + m^2)\phi(t, \underline{x}) = (\underbrace{\partial_\mu \partial^\mu}_{} + m^2)\phi(x)$$

$$(\Box + m^2)\phi(x) = 0$$

Klein-Gordon equation.

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \partial_\mu \to \partial'_\mu = (\Lambda^{-1})^\rho{}_\mu \partial_\rho$$

ϕ scalar $\phi(x) \to \phi'(x') = \phi'(\Lambda x) = \phi(x)$

$$[\partial'_\mu \partial'^\mu + m^2]\phi'(x') = [(\Lambda^{-1})^\rho{}_\mu \partial_\rho (\Lambda^{-1})^\sigma{}_\nu \partial_\sigma g^{\mu\nu} + m^2]\phi'(\Lambda x)$$

$$= [\partial_\rho \partial_\sigma g^{\rho\sigma} + m^2]\phi(x) = 0$$

Dirac Equation

seek wave equation which is linear in ∂_t, and solves $E^2 = m^2 + p^2$.

$$i\partial_t \Psi(t,\underline{x}) = (-i\underline{\alpha}.\underline{\nabla} + \beta m)\Psi(t,\underline{x}) \quad (*)$$

Must also be sol^n to KG wave eq^n.

"square" $(*)$

$$-\partial_t^2 \Psi(t,\underline{x}) = i\partial_t (-i\underline{\alpha}.\underline{\nabla} + \beta m)\Psi(t,\underline{x})$$

$$= (-i\underline{\alpha}.\underline{\nabla} + \beta m)^2 \Psi(t,\underline{x})$$

$$= (-\alpha^i\alpha^j\partial_i\partial_j - i(\beta\alpha_i + \alpha_i\beta)m\partial_i + \beta^2 m^2]\Psi(t,\underline{x})$$

Require this solves the KG wave eq^n.

Require RHS $[-\nabla^2 + m^2]\Psi(t,\underline{x})$

Therefore $\underline{\alpha}$ and β must satisfy:

$$\alpha^i\alpha^j + \alpha^j\alpha^i = \{\alpha^i, \alpha^j\}_{ab} = 2\delta^{ij}\delta_{ab}$$

$$\beta\alpha^i + \alpha^i\beta = \{\alpha^i, \beta\} = 0$$

$$\beta^2_{ab} = \delta_{ab}$$

To find sol^n for α_i, β, one needs $n\times n$ matrices.

Ψ is then a n-column vector.

Conditions $\mathrm{Tr}(\alpha^i) = 0 = \mathrm{Tr}(\beta)$ \
And eigenvalues ± 1. $\Bigg\} \Rightarrow n$ must be even.

Solutions in 4 dimension $(n=4)$

One such solution:

$$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$$

$\underline{\sigma}$ is a vector of the $\underset{2\times 2}{\text{Pauli}}$ matrices

$\gamma^0 = \beta \quad \underline{\gamma} = \beta \underline{\alpha}$

$\gamma^\mu = (\gamma^0, \underline{\gamma})$ μ is Lorentz

$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$

$\alpha_i' = U \alpha U^{-1} \qquad \beta' = U \beta U^{-1}$, U unitary

$(i\gamma^\mu \partial_\mu - m \mathbb{1}_4)\psi(t, \underline{x}) = (i\cancel{\partial} - m)\psi(x) = 0$

Dirac Equation

$\cancel{a} = \gamma^\mu a_\mu$

$\partial_\mu \to -i p_\mu$

$(\gamma^\mu p_\mu - m \mathbb{1}_4)\psi(p) = 0$

ψ is a column vector

Dirac Spinor

γ^μ operate on these Dirac spinors

$$i\frac{\partial}{\partial t}\phi(t,x) = H\,\phi(t,x)$$

$$i\frac{\partial}{\partial t}\psi(t,x) = (-i\underline{\alpha}\cdot\underline{\nabla} + \beta m)\psi(t,x)$$

$$H \quad (-i\underline{\alpha}\cdot\underline{\nabla} + \beta m)$$

$$\mathrm{Tr}\, H_{\text{Dirac}} = 0$$

<u>Dirac Sea</u>: Accept sol^n $E<0$

Vacuum state is that with all states $E<0$ already occupied.

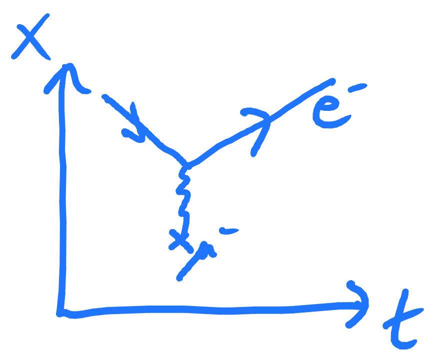
Time-ordered perturbation theory

$E>0$ travel forward in time

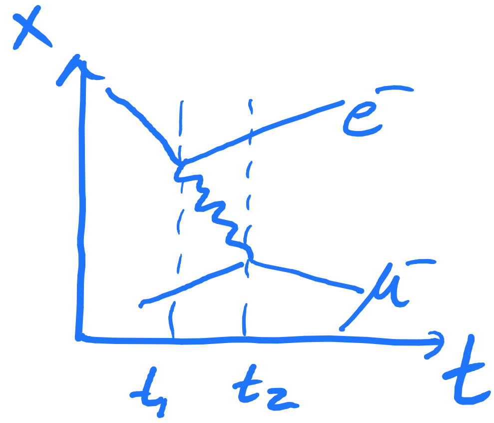
$E<0$ travel backward in time

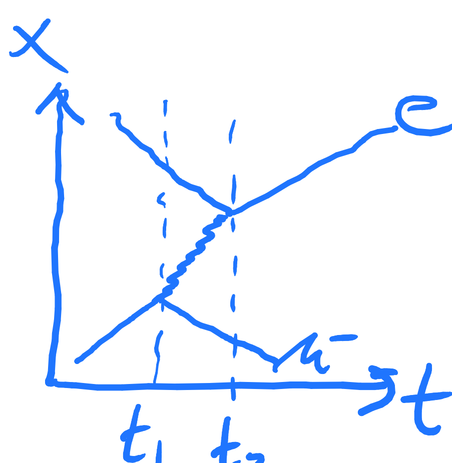
$E<0$ particle with p^μ [travel backwards in time] $\rightarrow$

$E>0$ anti-particle with momentum $-p^\mu$ tr ng forward in time.



$e^-\mu^- \rightarrow e^-\mu^-$





$(\gamma^\mu p_\mu - m\mathbb{1}_4)\psi(p) = (\not{p} - m)\psi(p) = 0$

$\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$

$\gamma^0 = \beta \qquad \gamma^i = \beta\alpha^i$

$\gamma^i = \beta\alpha^i = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

Dirac eqn:

$$\left[p^0 \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} + p_i\gamma^i - m\mathbb{1}_4\right]\psi(p) = 0$$

Plane wave solutions on the form

$$\psi(p) = \begin{pmatrix} \chi(p) \\ \phi(p) \end{pmatrix} e^{-ipx}$$

$$p_i\gamma^i = \begin{pmatrix} 0 & -\sigma^i p^i \\ \sigma^i p^i & 0 \end{pmatrix}$$

$p^0\chi(p) - \underline{\sigma}.\underline{p}\,\phi(p) - m\chi(p) = 0$

$\therefore (\sigma.p)\phi(p) = (p^0 - m)\chi(p)$

$-p^0\phi(p) + \underline{\sigma}.\underline{p}\,\chi(p) - m\phi(p) = 0$

$\therefore (\sigma.p)\chi(p) = (p^0 + m)\phi(p)$

Pauli matrices:

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

$\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}_2$

$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$

$(\sigma.p)^2 = p^2 \quad \{(\sigma.p) = \sigma_i p_i \quad , (\sigma.p)^2 = \underbrace{\sigma_i p_i \sigma_j p_j}_{=p^2}$

$$\begin{aligned}
\underbrace{(\sigma.p)(\sigma.p)}_{p^2}\phi &= (\sigma.p)(p^0 - m)\chi(p) \\
&= (p^0 - m)(\sigma.p)\chi(p) \\
&= (p^0 - m)(p^0 + m)\phi(p) \\
&= [(p^0)^2 - m^2]\phi(p)
\end{aligned}$$

$\therefore (p^0)^2\phi = [\underline{p}^2 + m^2]\phi \quad \checkmark$

$p^0 = \pm\sqrt{\underline{p}^2 + m^2}$

Consider the solutions in their rest frame $\underline{p} = 0$

$p^0\chi = m\chi \qquad p^0\phi = -m\phi$

Positive solutions $\psi_+^{\underline{p}=0}$ must therefore have $\phi = 0$ and the negative energy solutions $\psi_-^{\underline{p}=0}$ must have $\chi = 0$

$\psi_+^{\underline{p}=0} = \begin{pmatrix} \chi \\ 0 \end{pmatrix} e^{-imt}$ and $\psi_-^{\underline{p}=0} = \begin{pmatrix} 0 \\ \phi \end{pmatrix} e^{imt}$

For $\underline{p} \neq 0$

$$\psi_+(x) = \mathcal{N}\begin{pmatrix} \chi_r \\ \frac{\underline{\sigma}.\underline{p}}{E+m}\chi_r \end{pmatrix} e^{-ip.x} = u_r(p)\, e^{-ip.x}$$

$p^0 = E = \sqrt{\underline{p}^2 + m^2}$

The space of solutions is spanned by χ_r, $r = 1,2$ and $\mathcal{N}$ is a normalisation

$u_r^\dagger(p)\, u_s(p) = 2E\delta^{rs} \quad \therefore \mathcal{N} = \sqrt{E+m}$

$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The Negative energy solutions are usually written

$$\psi_{-,r} = \mathcal{N}\begin{pmatrix} \frac{\underline{\sigma}.\underline{p}}{E+m}\phi_r \\ \phi_r \end{pmatrix} e^{ipx} \equiv v_r(p)\, e^{ipx} \qquad p^0 = E$$

$\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The spinors $u(p)$, $v(p)$ represent particle (u) and anti-particle solutions with momentum $\underline{p}$ and energy $E = \sqrt{\underline{p}^2 + m^2}$.

Spin and Helicity

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

These have eigenvalues $\pm 1/2$ of the operator

$$1/2 \begin{pmatrix} \sigma_z & 0 \\ 0 & 0 \end{pmatrix}.$$

$$1/2 \begin{pmatrix} \sigma_z & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1/2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For anti-particles

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Spin operator

$$\underline{S} = 1/2 \begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix}$$

$$S^2 = \underline{S} \cdot \underline{S} = 1/2 \begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix} 1/2 \begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix} \begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} \underline{\sigma}^2 & 0 \\ 0 & \underline{\sigma}^2 \end{pmatrix}$$

$$\left[\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 3 \cdot 1\!\!1_2 \right]$$

$$= \frac{3}{4} \begin{pmatrix} 1\!\!1_2 & 0 \\ 0 & 1\!\!1_2 \end{pmatrix} = \frac{3}{4} 1\!\!1_4 \qquad \psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix}$$

For particles at rest, χ_r describe fermions with $S_z = +1/2$ $\left(\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, and $S_z = -1/2$ $\left(\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$.

For arbitrary $\underline{p}$: consider the spin projection along $\underline{p}/|\underline{p}|$: Helicity operator

$$h(\underline{p}) = \begin{pmatrix} \frac{\underline{\sigma} \cdot \underline{p}}{|\underline{p}|} & 0 \\ 0 & \frac{\underline{\sigma} \cdot \underline{p}}{|\underline{p}|} \end{pmatrix}$$

This operator satisfies $h^2(\underline{p}) = 1$.

$\therefore$ eigenvalues must be ± 1.

Dirac equation

$$(i\gamma^\mu\partial_\mu - m\mathbb{1}_4)\psi(x) = 0 \qquad \partial_\mu \to -ip_\mu$$

$$(i\cancel{\partial} - m)\psi(x) = 0 \qquad i\partial_\mu \to p_\mu$$

$$(\cancel{p} - m)\psi(x) = 0$$

$$(\gamma^i)^+ = -\gamma^i \quad (\gamma^0)^+ = \gamma^0$$

$(i\gamma^\mu\partial_\mu - m)\psi$ conjugate

$$\psi^+(-i(\gamma^\mu)^+\overleftarrow{\partial}_\mu - m) = 0$$

$$\psi^+(-i\gamma^0\overleftarrow{\partial}_0 + i\gamma^i\overleftarrow{\partial}_i - m) = 0 \qquad \{\gamma^0, \gamma^i\} = 0$$

$$\psi^+(-i\gamma^0\overleftarrow{\partial}_0 + i\gamma^i\overleftarrow{\partial}_i - m)\gamma^0 = 0 \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

$$\psi^+\gamma^0(-i\gamma^0\overleftarrow{\partial}_0 - \gamma^i\overleftarrow{\partial}_i - m) = 0 \qquad \gamma^0\gamma^i = -\gamma^i\gamma^0$$

$$\psi^+\gamma^0(-i\gamma^\mu\overleftarrow{\partial}_\mu - m) = 0 \qquad \bar{\psi} = \psi^+\gamma^0$$

$$\bar{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0$$

$$\bar{\psi}(\cancel{p} + m) = 0$$

---

plane wave solutions:

$$\psi_{+,r} = u_r(p)\, e^{-ipx} \qquad u_r(p) = \sqrt{E+m}\begin{pmatrix} \chi_r \\ \frac{\sigma\cdot p}{E+m}\chi_r \end{pmatrix}$$

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\psi_{-,r} = v_r(p)\, e^{ipx} \qquad v_r(p) = \sqrt{E+m}\begin{pmatrix} \frac{\sigma\cdot p}{E+m}\phi_r \\ \phi_r \end{pmatrix}$$

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \phi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$(\cancel{p} - m)u_r(p) = 0 \qquad (\cancel{p} + m)v_r(p) = 0$$

u, v orthonormal:

$$\bar{u}_r(p)u_s(p) = 2m\,\delta^{rs} = -\bar{v}_r(p)v_s(p)$$

$$\bar{u}_r(p)v_s(p) = \alpha - \bar{v}_r(p)u_s(p)$$

outer products

$$\sum_{r=1}^{2} u_r(p)\bar{u}_r(p) = (\cancel{p} + m)$$

$$\sum_{r=1}^{2} v_r(p)\bar{v}_r(p) = (\cancel{p} - m)$$

Lorentz transformations of spinors

$\psi(x) \to \psi'(x') = \psi'(\Lambda x) = S(\Lambda)\psi(x)$

$\bar\psi = \psi^\dagger\gamma^0 \to \bar\psi'(x') = \bar\psi(x)\gamma^0 S^\dagger(\Lambda)\gamma^0$

$S(\Lambda)$ suitable 4x4 matrix operating in spinor space.

$$(i\partial'_\mu\gamma^\mu - m)\psi'(x') = (i(\Lambda^{-1})^\nu{}_\mu\partial_\nu\gamma^\mu - m)S(\Lambda)\psi(x)$$

Guess $\boxed{\gamma^\mu S(\Lambda) = S(\Lambda)\Lambda^\mu{}_\rho\gamma^\rho}$

$$(i\partial'_\mu\gamma^\mu - m)\psi'(x') = S(\Lambda)[i\underbrace{(\Lambda^{-1})^\nu{}_\mu\Lambda^\mu{}_\rho}_{g^\nu{}_\rho}\partial_\nu\gamma^\rho - m]\psi(x)$$

$$= S(\Lambda)(i\partial_\nu\gamma^\nu - m)\psi(x) = 0$$

If ψ is a sol^n to Dirac eq^n in S then $\psi' = S(\Lambda)\psi$ is a sol^n in S'

$\Lambda^\mu{}_\rho$ is the Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\rho x^\rho$

$S^\dagger(\Lambda) = \gamma^0 S^{-1}(\Lambda)\gamma^0$ $\quad\therefore\ \bar\psi'(x') = \overline{(S(\Lambda)\psi(x))}$

$S^{-1}(\Lambda) = \gamma^0 S^\dagger(\Lambda)\gamma^0$ $\quad = \psi'^\dagger\gamma^0 = (S\psi)^\dagger\gamma^0 = \psi^\dagger S^\dagger\gamma^0$

$[(\gamma^0)^2 = 1\!\!1_4]$ $\quad = (\psi^\dagger\gamma^0)\gamma^0 S^\dagger\gamma^0 = \bar\psi S^{-1}(\Lambda)$

Lorentz transformations of Bilinears

Consider constructs of the form

$\bar\psi\Gamma\psi$ $\quad\Gamma$ is 4x4 matrix in spinor space.

Since Γ is 4x4, expect 16 bilinear

$\bar\psi\psi \to \bar\psi S^{-1}(\Lambda)S(\Lambda)\psi = \bar\psi\psi$ invariant, scalar!

$\bar\psi(x)\psi(x) \to \bar\psi'(x')\psi'(x') =$

$\bar\psi\gamma^\mu\psi \to \bar\psi'(x')\gamma^\mu\psi'(x') = \bar\psi(x)S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi(x)$

$= \Lambda^\mu{}_\rho(\bar\psi\gamma^\rho\psi)$ vector!

We can construct another 6 by considering

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Note that $\gamma^\mu\gamma^\nu$ is not linearly independent $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}1\!\!1$.

$\bar\psi\Sigma^{\mu\nu}\psi \to \bar\psi S^{-1}(\Lambda)\frac{i}{4}[\gamma^\mu,\gamma^\nu]S(\Lambda)\psi$ Rank-2

$= \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma(\bar\psi\Sigma^{\rho\sigma}\psi)$ Tensor

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix}$$

which satisfies

$(\gamma^5)^2 = 1\!\!1_4 \qquad \{\gamma^5,\gamma^\mu\} = 0 \qquad \gamma^{5\dagger} = \gamma^5$

$\bar\psi\gamma^5\psi \to \bar\psi S^{-1}(\Lambda)\, i\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma S(\Lambda)\psi$

$= i\varepsilon_{\mu\nu\rho\sigma}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\gamma\Lambda^\sigma{}_\delta(\bar\psi\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\psi)$

$= \det\Lambda\,\bar\psi\underbrace{i\varepsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta}_{\gamma^5}\psi = \det(\Lambda)\bar\psi\gamma^5\psi$ pseudo-scalar!

$\bar\psi\gamma^5\gamma^\mu\psi \to \det(\Lambda)\Lambda^\mu{}_\nu(\bar\psi\gamma^5\gamma^\nu\psi)$

pseudo-vector

$1\!\!1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu]$

s.t. any bilinear $\bar\psi\Gamma\psi$

γ^5 appears in Feynman Rules since it is used for projections

$P_L = (1-\gamma^5)/2 \qquad P_R = (1+\gamma^5)/2$

Check these are projections

$P^2 = P \quad ; \quad P_L P_R = 0$

$\bar\psi_L = \psi_L^\dagger\gamma^0 = \psi^\dagger P_L\gamma^0 = \psi^\dagger\gamma^0 P_R = \bar\psi P_R$

The QED Lagrangian

Maxwell's equations in relativistic notation

$\partial_\mu F^{\mu\nu} = J^\nu$ $\qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

where J^ν is a conserved current $\partial_\nu J^\nu = 0$

$\frac{\partial}{\partial t}\rho = -\nabla\cdot j$

Maxwells equations can be derived from the Lagrangian

$\mathcal{L} = \underbrace{\mathcal{L}_{em}}_{\text{describes free photon fields}} + \underbrace{\mathcal{L}_{int}}_{\text{describes interactions with sources}}$ $\qquad \mathcal{L}_{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

$\mathcal{L}_{int} = -J^\mu A_\mu$

Euler-Lagrange equations:

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} + J^\nu$$

The Dirac equation for ψ (and $\bar\psi$) can be derived from

$$\mathcal{L}_{Dirac} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$$

$$0 = \partial_\mu \bar\psi i\gamma^\mu + \bar\psi m = \bar\psi(i\gamma^\mu\partial_\mu + m)$$

Derivative w.r.t. $\underset{\bar\psi}{\psi}$: Dirac eqn for $\underset{\psi}{\bar\psi}$,

Starting point for QED Lagrangian: Free fields of γ (photon), e $\quad \mathcal{L} = \mathcal{L}_{em} + \mathcal{L}_{Dirac}$

Interactions:

A^μ ψ : $\mathcal{L}_{int} = -J^\mu A_\mu$ $\quad J^\mu$ is a conserved vector current.

$J^\mu = \bar\psi\gamma^\mu\psi$ $\quad \psi$ is a solⁿ to Dirac eqn then this J^μ is conserved:

$$\partial_\mu J^\mu = \bar\psi\overleftarrow{\not\partial}_\mu\psi + \bar\psi\not\partial_\mu\psi = (-m\bar\psi\psi) + \bar\psi(m\psi) = 0$$

$J^\mu = -e\bar\psi\gamma^\mu\psi$

$[J^\mu = (\rho c, j)]$ $\qquad \nabla\cdot E = \rho/\varepsilon_0$

$$\mathcal{L} = \mathcal{L}_{em} + \mathcal{L}_{Dirac} + \mathcal{L}_{int}$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\underset{\not D}{\not\partial} - m)\psi + e\bar\psi\gamma^\mu\psi A_\mu$$

Notice that this Lagrangian is invariant under local gauge transformations

$$\psi(x) \to \psi'(x) = e^{-ie\alpha(x)}\psi(x)$$

$\alpha(x)$ depends on x!

true for $\mathcal{L}_{Dirac}$ when $\alpha(x)=\alpha$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) \pm \partial_\mu\alpha(x)$$

true also for free $\mathcal{L}_{em}$ when $\alpha(x)=\alpha$

Actually, the interaction term can be written as [included] in the $\mathcal{L}_{Dirac}$ by replacing

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\gamma^\mu(\partial_\mu + ieA_\mu))\psi$$

Fixing gauge:

Coulomb gauge $\nabla\cdot A = 0$

Lorenz $\partial_\mu A^\mu = 0$

$\Box\alpha = -\partial_\mu A^\mu$ $\qquad \Box = \partial_\mu\partial^\mu$

Maxwell equations $\Box A^\nu = 0$

$A_\mu \to A_\mu' = A_\mu + \partial_\mu\chi$, $\Box\chi = 0$

$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ and pick ξ

Feynman gauge $\xi = 1$

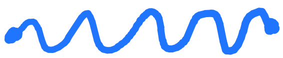
$(\xi - 1)$

$\xi = 0$

Feynman Rules for QED for constructing $i\mathcal{M}$ (in momentum space)

$ie\gamma^\mu$

$$\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

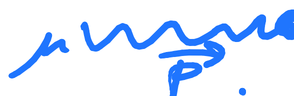


$$\frac{-i}{p^2+i\epsilon}\left(g^{\mu\nu}-(1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$$

incoming:

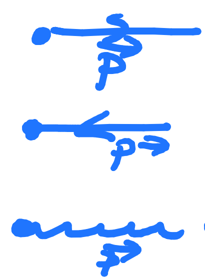
$u(p)$

$\bar{v}(p)$



$\epsilon_\mu(p)$

outgoing:



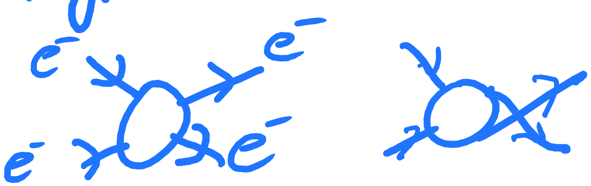
$\bar{u}(p)$ $v(p)$ $\epsilon^*_\mu(p)$

Additional factors of (-1):

ⓐ anti-fermion line connecting initial and final state

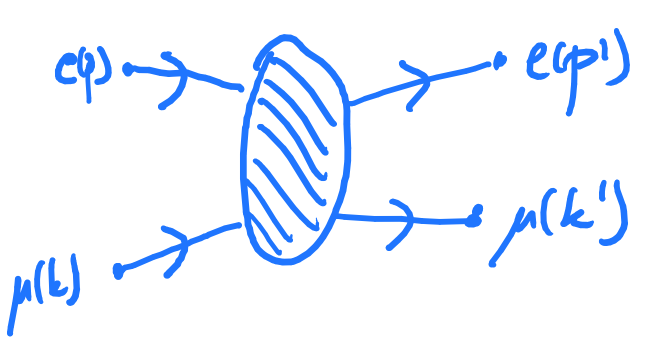
ⓑ every closed fermion loop

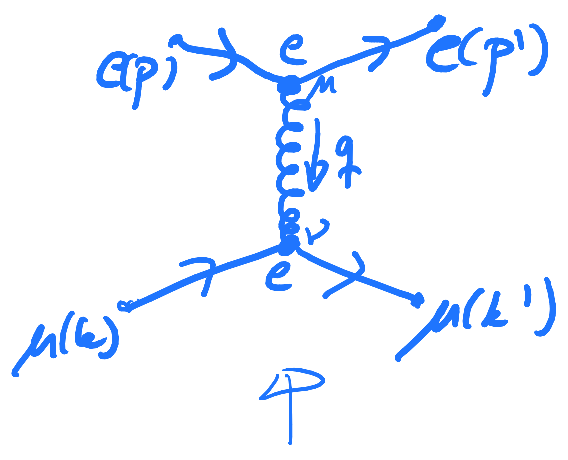
ⓒ between diagrams with identical fermions in the final state



Coulomb scattering

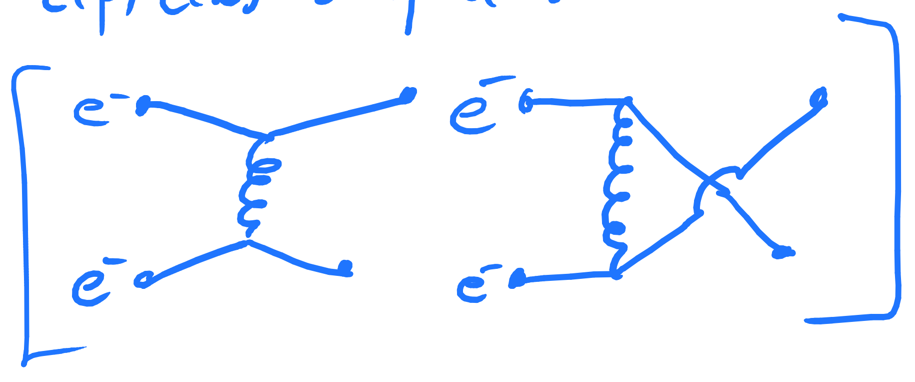
$e(p)\mu(k) \to e(p')\mu(k')$





$\propto e^2$

$e^-(p)\, e^-(k) \to e^-(p')e^-(k')$



electron line: $\bar{u}(p')\ (ie\gamma^\mu)\ u(p)$

muon line: $\bar{u}(k')(ie\gamma^\nu)u(k)$ $\qquad q = p'-p = k-k'$

photon propagator: $\dfrac{-ig_{\mu\nu}}{q^2+i\epsilon}$

The amplitude is then

$$i\mathcal{M} = ie^2\underbrace{[\bar{u}_r(p')\gamma^\mu u_r(p)]}_{\text{for each } \mu\text{, this is a number, Lorentz vectors}}\frac{g_{\mu\nu}}{q^2}\underbrace{[\bar{u}_s(k')\gamma^\nu u_s(k)]}_{\text{for each } \nu\text{, this is a number, Lorent vector}}$$

$\gamma^\mu g_{\mu\nu}\gamma^\nu = \gamma^\mu\gamma_\mu$

$\mathcal{M}$ is just a complex number, so could just square $|\mathcal{M}|^2$ $\ \mathcal{M}\mathcal{M}^*$

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\frac{1}{2}\sum_{r=1}^{2}\sum_{s=1}^{2}\sum_{r'=1}^{2}\sum_{s'=1}^{2}|\mathcal{M}|^2$$

$$= \frac{1}{4}\frac{(e)^4}{(q^2)^2}\sum_{r,r'}[\bar{u}_{r'}(p')\gamma^\mu u_r(p)][\bar{u}_{r'}(p')\gamma^\rho u_r(p)]^*$$

$$\sum_{s,s'}[\bar{u}_{s'}(k')\gamma_\mu u_s(k)][\bar{u}_{s'}(k')\gamma_\rho u_s(k)]^*$$

Since $[\bar{u}\gamma^\rho u]^*$ is a complex number for each ρ

$[\bar{u}_{r'}(p')\gamma^\rho u_r(p)]^* = u_r^\dagger(p)\gamma^{\rho\dagger}\gamma^{0\dagger}u_{r'}(p') = u_r^\dagger(p)\gamma^0\gamma^\rho u_{r'}(p')$

$(\gamma^0\gamma^\rho)^\dagger = \gamma^{\rho\dagger}\gamma^{0\dagger} = \gamma^\rho\gamma^0 \qquad = \bar{u}_r(p)\gamma^\rho u_{r'}(p')$

$$\sum_{r,r'}[\bar{u}_{r'}(p')\gamma^\mu \underline{u_r(p)}][\underline{\bar{u}_r(p)}\gamma^\rho u_{r'}(p')] \qquad \not{p}+m$$

$$= \sum_{r'}\bar{u}_{r'}(p')\gamma^\mu(\not{p}+m)\gamma^\rho u_{r'}(p').$$

introduce spinor indicies

$\sum_{r'}\bar{u}_{r'i}\Gamma_{ij}u_{r'j}$ $\qquad \Gamma$ represents the string of γ matrices

$$\sum_{r'}\Gamma_{ij}u_{r'j}\bar{u}_{r'i} = \Gamma_{ij}(\not{p}'+m)_{ji} = \mathrm{Tr}(\gamma^\mu(\not{p}+m)\gamma^\rho(\not{p}'+m))$$

$\mathrm{Tr}(\text{odd \# of }\gamma) = 0$ $\qquad p_\nu p'_\sigma \ \gamma^\nu \ \gamma^\sigma$

$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$

$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$

$(*) = 4p_\nu p'_\sigma(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) + 4m^2g^{\mu\rho}$

The same steps for the muon line gives

$$4k^\alpha k'^\beta(g_{\mu\alpha}g_{\rho\beta} - g_{\mu\rho}g_{\alpha\beta} + g_{\mu\beta}g_{\alpha\rho}) + 4M^2g_{\mu\rho}$$

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{(q^2)^2}\big((p.k)(p'.k') + (p.k')(p'.k) + 2m^2M^2 - M^2(pp')$$
$$- m^2(k.k')\big).$$

Rewrite in terms of s and $t = q^2$

$s = (p+k)^2$

$2(p.k) = (p+k)^2 - m^2 - M^2 = s - m^2 - M^2$

$2(p.p') = -(p-p')^2 + 2m^2 = -q^2 + 2m^2$

$2(p.k') = 2p.(p+k-p') = s + q^2 - m^2 - M^2$

$2(p'k') = s - m^2 - M^2$

$2(k.k') = -q^2 + 2M^2$

$2(p'.k) = s + q^2 - m^2 - M^2$

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{(q^2)^2}\left((s-m^2-M^2)^2 + (s+q^2-m^2-M^2) + 2q^2(m^2+M^2)\right).$$

$s, t, u\ ;\ u = (p-k')^2 = (p'-k)^2$

$$\overline{|\mathcal{M}|^2} = \frac{2e^4}{t^2}\left((s-m^2-M^2)^2 + (u-m^2-M^2)^2 + 2t(m^2+M^2)\right)$$

Massless: $\dfrac{2e^4}{t^2}(s^2+u^2)$

Phase Space Integrals & cross sections

onshell condition $p_f^2 = m_f^2$ Lorentz Integral

$$\prod_f \left(\int \frac{d^4 p_f}{(2\pi)^{4}} (2\pi) \delta(p_f^2 - m^2) \Theta^0(p_f^0) \right) \quad dp_f^0 \quad \frac{1}{2E}$$

$$= \prod_f \int \frac{d^3 p_f}{(2\pi)^3 (2E_f)} \quad , E_f = \sqrt{p_f^2 + m^2} .$$

Flux factor $\quad p_a p_b \to \ldots$

$$\mathcal{F} = \frac{1}{4 E_a E_b \, |\underline{v}_a - \underline{v}_b|}$$

$$= \frac{1}{4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}}$$

$$\xrightarrow[m_a, m_b \to 0]{} \frac{1}{2s} \quad \text{in the massless limit}$$

$$\sigma = \mathcal{F} \left(\prod_f \int \frac{d^3 p_f}{(2\pi)^3 (2E_f)} \right) \overline{|\mathcal{M}|^2} (2\pi)^4 \delta\left(\Sigma p_f - p_a - p_b\right) .$$

For decay widths, $\mathcal{F} = \frac{1}{2M}$

$$\Gamma = \frac{1}{2M} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3 (2E_f)} \right) \overline{|\mathcal{M}|^2} (2\pi)^4 \left(\Sigma p_f - p_M \right) .$$

Return to Coulomb scattering $e\mu \to e\mu$
$p \; k \quad p' \; k'$

$$\sigma = \mathcal{F} \int \frac{d^3 p'}{(2\pi)^3 (2E_p')} \frac{d^3 k'}{(2\pi)^3 (2E_k')} \overline{|\mathcal{M}|^2} (2\pi)^4 \delta^4(p'+k'-p-k)$$

choose eq. frame where $\mathbf{p} = -\mathbf{k}$

$$\sigma = \mathcal{F} \int \frac{d^3 \mathbf{p}'}{(2\pi)^3 (2E_p')} \frac{1}{2E_k'} (2\pi) \delta(E_p' + E_k' - E_p - E_k)$$

$$d^3 \mathbf{p}' = |\mathbf{p}'|^2 d|\mathbf{p}'| \, d\Omega \quad , \quad d\Omega = \sin\theta d\theta d\phi$$

$$\sigma = \frac{\mathcal{F}}{(2\pi)^2} \int d\Omega \, d|\mathbf{p}'| \frac{|\mathbf{p}'|^2}{4E_p' E_k'} \overline{|\mathcal{M}|^2} \delta(E_{p'} + E_k' - E_p - E_k)$$

We want the answer expressed in terms of $\sqrt{s} = E_{p'} + E_{k'}$

$|\mathbf{p}'| \to E = E_{p'} + E_{k'}$

$$\frac{\partial E}{\partial |\mathbf{p}'|} = \frac{E |\mathbf{p}'|}{E_{p'} E_{k'}}$$

$$\sigma = \frac{\mathcal{F}}{(2\pi)^2} \int d\Omega \, dE \frac{|\mathbf{p}'|}{4E} \overline{|\mathcal{M}|^2} \delta(E - \sqrt{s})$$

$$= \frac{\mathcal{F}}{(2\pi)^2} \int d\Omega \frac{|\mathbf{p}'|}{4\sqrt{s}} \overline{|\mathcal{M}|^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{\mathcal{F}}{16\pi^2} \frac{|\mathbf{p}'|}{\sqrt{s}} \overline{|\mathcal{M}|^2}$$

In the high energy limit $s \gg m_e^2, m_\mu^2$

$s,t,u \quad s = 4\mathbf{p}^2 = (p_a + p_b)^2$

$$t = -4\mathbf{p}^2 \sin^2(\theta/2)$$

$$u = -4\mathbf{p}^2 \cos^2(\theta/2)$$

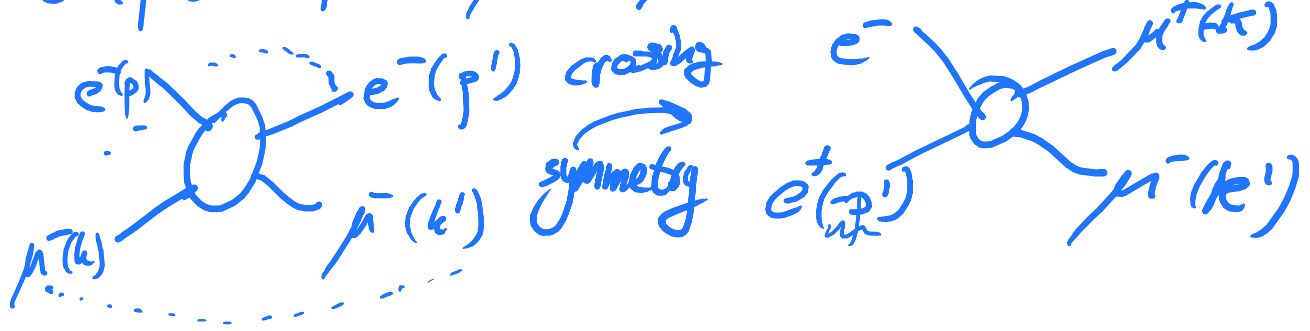
$$\overline{|\mathcal{M}|^2} = 2e^4 \frac{s^2 + u^2}{t^2} = \frac{2e^4}{\sin^4(\theta/2)} \left(1 + \cos^4 \theta/2\right)$$

$$\alpha^2 = e^2/4\pi$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s} \frac{1 + \cos^4(\theta/2)}{\sin^4(\theta/2)} .$$

e^+e^- annihilation

$e^+(p')\, e^-(p) \to \mu^+(k)\, \mu^-(k')$



Take approximation $m_e = 0$

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{s^2}\left[(p\cdot k)^2 + (p\cdot k')^2 + m_\mu^2 (k\,k')^2\right]$$

Centre of mass:

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^-\to\mu^+\mu^-} = \frac{\alpha^2}{4s}\sqrt{1-\frac{4m_\mu^2}{s}}\left(1+\frac{4m_\mu^2}{s}+\left(1-\frac{4m_\mu^2}{s}\right)\cos^2\theta\right)$$

$s \gg m_\mu^2$

$$\to \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right)$$

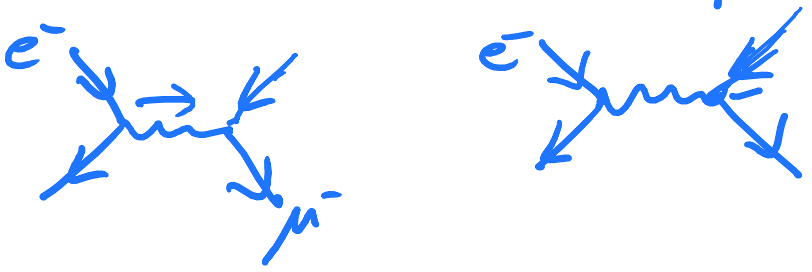
$d\Omega = \sin\theta\, d\theta\, d\phi$ polar $\theta \in [0,\pi[$ azimuthal $\phi \in [0, 2\pi[$

$= \underline{-d\cos\theta}\, d\phi$

$$-\int_1^{-1}(1+\cos^2\theta)\,d(\cos\theta) = -\int_1^{-1}(1+x^2)\,dx = \int_{-1}^{1}(1+x^2)\,dx$$

$$= 2 + \frac{1}{3}\left[x^3\right]_{-1}^{1} = 2 + \frac{2}{3} = \frac{8}{3}$$

$$\sigma(e^+e^-\to\mu^+\mu^-) = \frac{8}{3}\cdot 2\pi \cdot \frac{\alpha^2}{4s} = \frac{4\pi}{3}\frac{\alpha^2}{s}.$$



3 QCD colours

$$\sigma(e^+e^-\to \text{hadrons}) = \frac{4\pi\alpha^2}{3s} N_c \sum_{i=1}^{n_f} Q_i^2 \qquad \frac{2}{3} \quad -\frac{1}{3}$$

+ higher orders

$$R = \frac{\sigma(e^+e^-\to\text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)} = \underset{4}{N_c}\sum_{i=1}^{n_f} Q_i^2$$

Scattering with Photons

$e\gamma \to e\gamma$

Photon polarisation

$A^\mu(k) = \varepsilon^\mu(k) e^{-ikx}$

$\varepsilon^\mu(k)$ polarisation vector

In Lorenz gauge $\partial_\mu A^\mu = 0$

$\to \Box A^\mu = 0 \qquad \Box = \partial_\nu \partial^\nu$

$k^2 = 0$

$\partial_\mu A^\mu = 0 \therefore k_\mu \varepsilon^\mu = 0$

$\varepsilon'^\mu = \varepsilon^\mu + \lambda k^\mu$ if ε^μ is a sol^n then so is ε'^μ

This gives us freedom to require

$\varepsilon^0 = 0 \quad \therefore \underline{k} \cdot \underline{\varepsilon} = 0$

That means that $\underline{\varepsilon}$ is $\perp$ $\underline{k}$

$\varepsilon^\alpha, \quad \alpha = 1,2$

$$\sum_{\alpha=1}^{2} \varepsilon^{\alpha i}_{(k)} \varepsilon^{*\alpha j}_{(k)} = \delta^{ij} - \hat{k}^i \hat{k}^j, \quad \hat{k}^i = \frac{k^i}{|k|} = \frac{k^i}{k^0}$$

Feynman rule for incoming photon

$\mu \overset{k}{\rightsquigarrow}\bullet \qquad \varepsilon^\mu(k)$

$\overset{k}{\rightsquigarrow}\mu \qquad \varepsilon^{*\mu}(k)$

$i\mathcal{M} = \mathcal{M}^\mu \varepsilon_\mu(k)$

gauge inv.

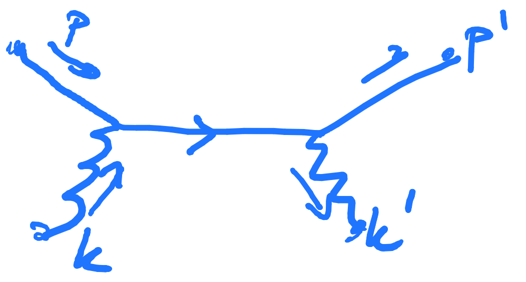
$\varepsilon_\mu \to \varepsilon'_\mu = \varepsilon_\mu + \lambda k_\mu$

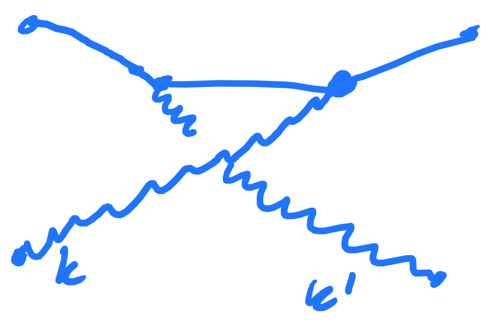
$i\mathcal{M} \to i\mathcal{M} = \mathcal{M}^\mu(\varepsilon_\mu + \lambda k_\mu)$

$\mathcal{M}^\mu k_\mu = 0$ when $\mathcal{M}^\mu$ is gauge invariant.

Ward Identity $k_\mu \mathcal{M}^\mu = 0 \qquad i\mathcal{M} = \mathcal{M}^\mu \varepsilon_\mu$

$_p e \gamma \to e \gamma_{p'}$





Squaring

$$\sum_{\alpha=1}^{2} |\mathcal{M}^\mu \varepsilon^\alpha_\mu(k)|^2 = \sum_{\alpha=1}^{2} \mathcal{M}^\mu \mathcal{M}^{*\nu} \varepsilon^\alpha_\mu(k) \varepsilon^*_\nu(k)$$

$$= A^i A^{*j}(\delta^{ij} - \hat{k}^i \hat{k}^j)$$

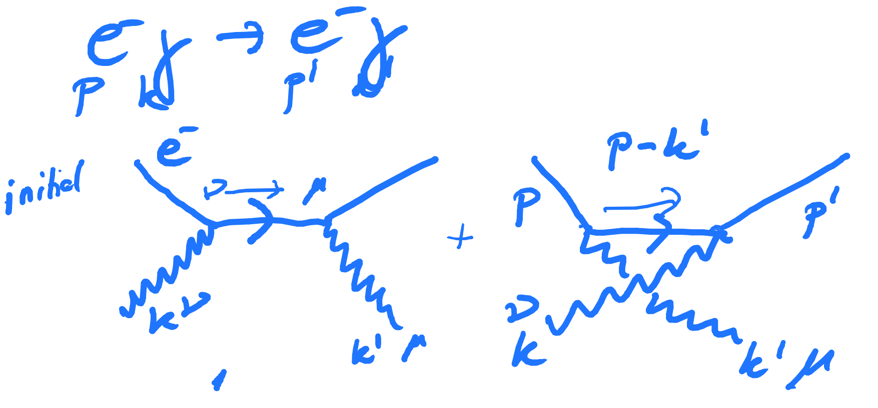
$\mathcal{M}^\mu k_\mu = 0 \quad \therefore A^i \hat{k}^i = A^0$

$$\sum_{\alpha=1}^{2} |\mathcal{M}^\mu \varepsilon^\alpha_\mu(k)|^2 = A^i A^{*i} - A^0 A^{*0} = -A^\mu A^{*\nu} g_{\mu\nu}$$

$$\sum_{\alpha=1}^{2} \varepsilon^\alpha_\mu \varepsilon^{*\alpha}_\nu \to -g_{\mu\nu}$$

Compton Scattering

$$\underset{p}{e^-}\,\underset{k}{\gamma} \to \underset{p'}{e^-}\,\gamma$$



$$i\mathcal{M} = \bar{u}(p')(-ie\gamma^\mu)\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}(-ie\gamma^\nu)u(p)\,\varepsilon_\nu(k)\,\varepsilon^*_\mu(k')$$

$$+\bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2}(-ie\gamma^\mu)u(p)\,\varepsilon_\nu(k)\,\varepsilon^*_\mu(k')$$

$$= -ie^2\varepsilon^*_\mu(k')\varepsilon_\nu(k)$$

$$\times\,\bar{u}(p')\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2} + \frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u(p)$$

$p^2 = m^2,\ k^2 = 0$ $\qquad (p+k)^2 - m^2 = 2p\cdot k$

$(p+k)^2 = p^2+k^2+2p\cdot k$ $\qquad (p-k')^2 - m^2 = -2p\cdot k'$

$$(\not{p}+m)\gamma^\nu u(p) = (2p^\nu - \gamma^\nu\not{p} + \gamma^\nu m)u(p)$$

$\not{p} = p^\mu\gamma^\mu$

$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$= 2p^\nu u(p) - \gamma^\nu(\not{p}-m)u(p)$$
$$= 2p^\nu u(p)$$

$$i\mathcal{M} = -ie^2\varepsilon^*_\mu(k')\varepsilon_\nu(k)$$

$$\bar{u}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p)$$

$\sum_r u_r(k)u^+_r(k) = \not{k}+m$ $\qquad$ for $\mathcal{M}$ use μ for outgoing γ k', ν incoming k

$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^2}{4}g_{\mu\rho}g_{\nu\sigma}\mathrm{Tr}\{\cdots\cdots\}$ $\qquad$ $\mathcal{M}^*$ ρ outgoing γ k', σ incoming γ k

In $\mathcal{M}\mathcal{M}^*$: $\varepsilon^*_\mu(k')\varepsilon_\nu(k)\,\varepsilon^*_\sigma(k)\varepsilon_\rho(k')$

$\sum_{\text{pol}}\varepsilon^*_\mu(k')\varepsilon_\rho(k') \to -g_{\mu\rho}$

$\sum_{\text{pol}}\varepsilon^*_\sigma(k)\varepsilon_\nu(k) \to -g_{\nu\sigma}$

$$\mathrm{Tr}\{\bar{u}(p')\cdots\cdots\underbrace{u(p)\bar{u}(p)}_{\not{p}+m}\cdots\cdots u(p')\}$$

$$\mathrm{Tr}\{\underbrace{u(p')\bar{u}(p')}_{(\not{p}'+m)}\cdots\cdots(\not{p}+m)\cdots\cdots\}$$

$$\mathrm{Tr}\Big\{(\not{p}'+m)\left[\frac{\gamma^\mu\not{k}\gamma^\nu+2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu\not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\right](\not{p}+m)$$
$$\left[\frac{\gamma^\sigma\not{k}\gamma^\rho + 2\gamma^\rho p^\sigma}{2p\cdot k} + \frac{\gamma^\rho\not{k}'\gamma^\sigma - 2\gamma^\sigma p^\rho}{2p\cdot k'}\right]\Big\}$$

$$\equiv\left[\frac{\mathrm{I}}{(2p\cdot k)^2} + \frac{\mathrm{II}}{(2p\cdot k)(2p\cdot k')} + \frac{\mathrm{III}}{(2p\cdot k')(2p\cdot k)} + \frac{\mathrm{IV}}{(2p\cdot k')^2}\right]$$

I...IV : complex traces

IV is the same exp. as I with $k\to -k'$

II = III

$$\mathrm{I} = \mathrm{tr}\left[(\not{p}'+m)(\gamma^\mu\not{k}\gamma^\nu + 2\gamma^\mu p^\nu)(\not{p}+m)(\gamma_\nu\not{k}\gamma_\mu + 2\gamma_\mu p_\nu)\right]$$

remember $\mathrm{Tr}\{\underbrace{\gamma^\mu\gamma^\rho\gamma^\sigma}_{\text{odd}}\} = 0$ $\qquad \gamma^\mu\not{p}'\gamma_\mu = (-2\not{p}')$

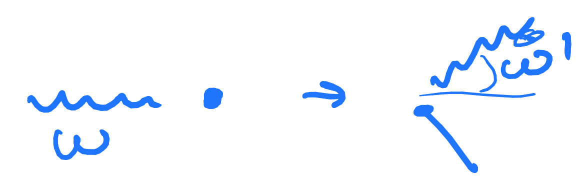
$$\mathrm{tr}[\not{p}'\gamma^\mu\not{k}\gamma^\nu\not{p}\gamma_\nu\not{k}\gamma_\mu] = \mathrm{tr}[(-2\not{p}')\not{k}(-2\not{p})\not{k}]$$

$$= \mathrm{tr}[4\not{p}'\not{k}(2p\cdot k - \not{k}\not{p})]$$ $\qquad \{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$, $\not{k}\not{k} = k^2 = 0$

$$= 8p\cdot k\ \mathrm{tr}[\not{p}'\not{k}]$$

$$= 32(p\cdot k)(p'\cdot k)$$

$$\overline{|\mathcal{M}|^2} = 2e^4\left(\frac{(p\cdot k)}{(p\cdot k')} + \frac{(p\cdot k')}{(p\cdot k)} + 2m^2\left(\frac{1}{(p\cdot k)} - \frac{1}{(p\cdot k')}\right) + m^4\left(\frac{1}{(p\cdot k)} - \frac{1}{p\cdot k'}\right)^2\right)$$



$$p + k = p' + k'$$
$$p' = p + k - p'$$

$$m^2 = (p')^2 = (p+k-k')^2 = p^2 + 2p(k-k') - 2k\cdot k'$$

$$= m^2 + 2m(\omega-\omega') - 2\omega\omega'(1-\cos\theta)$$

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1-\cos\theta)$$

$$\omega' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}$$

$p\cdot k = m\omega$, $p\cdot k' = m\omega'$

$$\overline{|\mathcal{M}|^2} = 2e^4\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right).$$

$$\alpha = \frac{e^2}{4\pi}$$

$\omega \ll m$: $\omega \sim \omega'$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}(1+\cos^2\theta)$$

$\omega \gg m$ $\qquad \omega' \simeq \dfrac{m}{1-\cos\theta}$

$$\frac{d\sigma}{d\Omega} \simeq \frac{\alpha^2}{2m\omega}\frac{1}{1-\cos\theta}.$$

QCD Lagrangian

$$\frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3 \sum_{f=1}^{N_c} q_f^2$$

$$\mathcal{L} \in \bar{\psi}(\not{\partial} - m)\psi$$

$i = 1,..,3$

$$\mathcal{L}_{QCD} = \bar{\psi}_i (i\not{\partial}_{ij} - m)\psi_j$$

ψ_i : i in colour space

Require gauge invariance under local gauge transf

$$\psi \to \psi' = e^{-i\theta_a(x) t^a}\psi \qquad t^a : 3\times 3 \text{ matrix}$$

inf $\psi \to (1 - i\theta_a t^a)\psi$

Gluon (vector) field $A^\mu = A^\mu_a t^a$ $\qquad a = 1, \dots, 8$

$$D_{ij} = \partial^\mu \delta_{ij} + i g_s t_{ij} A^\mu_a \qquad i,j = 1,\dots,3$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu$$

f^{abc} : structure constants

$$[t^a, t^b] = i f^{abc} t^c \qquad (SU(2): [J_i, J_j] = i\epsilon_{ijk} J_k)$$

For $SU(3)$ $t^a = \frac{1}{2}\lambda^a$, λ^a Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$Tr(t^a) = 0 \qquad Tr(t^a t^b) = \frac{1}{2}\delta^{ab} \qquad \sum_a t^a_{ij} t^a_{jk} = C_F \delta_{ik}$$

$$\sum_{a,b} f^{abc} f^{abd} = C_A \delta^{cd} \qquad C_F = \frac{4}{3} \qquad C_A = N = 3$$

$\delta^{aa} = N_c^2 - 1 = 8$ (# of gluons)

$\delta_{ii} = N_c = 3$ (# of colour charges for quarks)

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \bar{\psi}_i (i\not{D}_{ij} - m\delta_{ij})\psi_j$$

invariant

$$\psi_i(x) \to (\delta_{ij} - i g_s \theta^a t^a_{ij})\psi(x)$$

$$A^a_\mu(x) \to A^a_\mu(x) + D^{ab}_\mu \theta^b(x)$$

$$D^{ab}_\mu = \partial_\mu \delta^{ab} + i g_s A^c_\mu (T^c)^{ab}$$

$$(T^c)^{ab} = i f^{acb} = -i f^{abc}$$

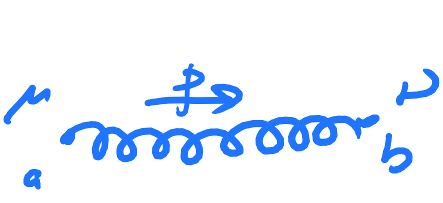
$$[T^a, T^b] = i f^{abc} T^c$$

Jacobi $f^{abc} f^{cde} + f^{bdc} f^{cae} + f^{dac} f^{cbe} = 0$

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A^\mu_a)^2$$

Feynman Rules:

$$: \delta_{kj} \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}$$

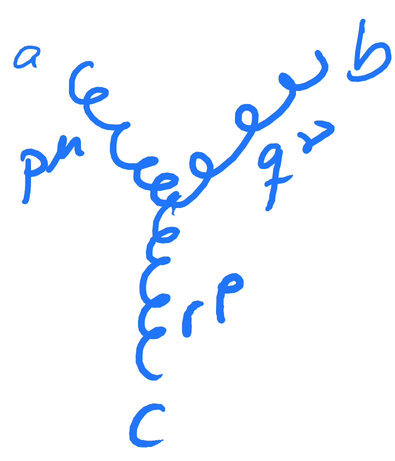


$$: \delta^{ab} \frac{-i g^{\mu\nu}}{p^2 + i\varepsilon}$$

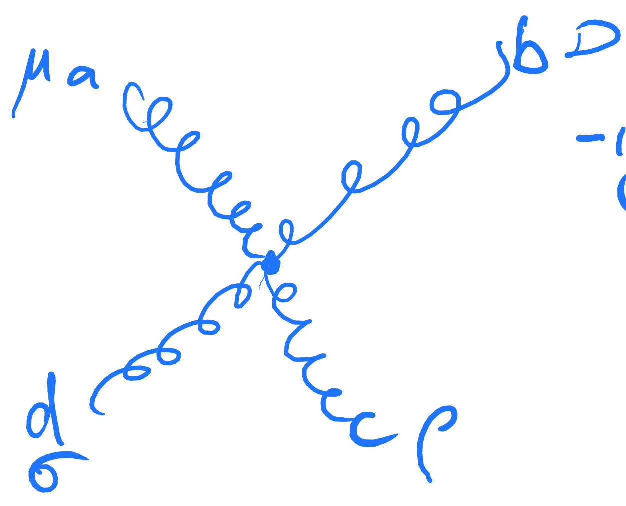
$$-i g_s \gamma^\mu t^a_{kj}$$

a ← gluon

kj : quark colour charges



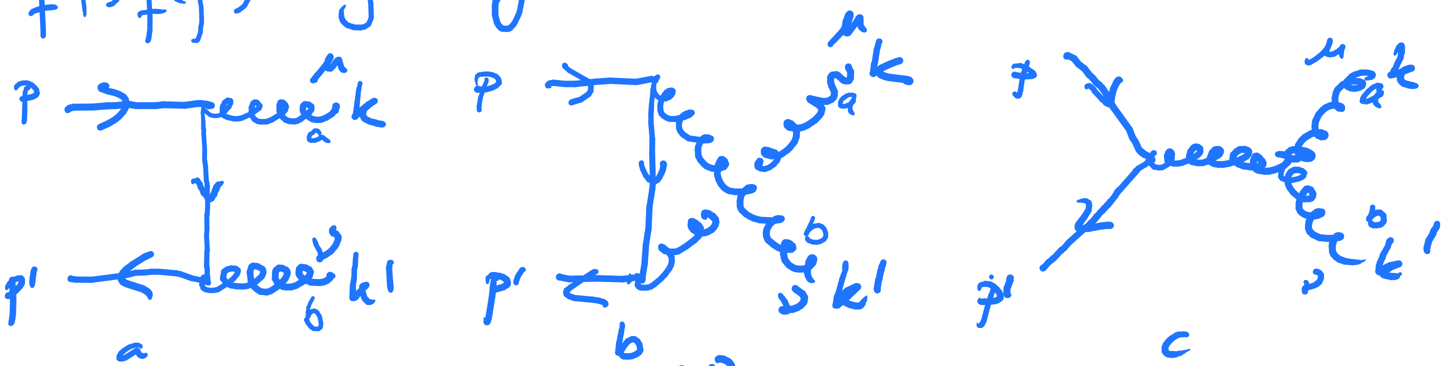
$$g_s f^{abc} \left(g^{\mu\nu}(p-q)^\rho + g^{\nu\rho}(q-r)^\mu + g^{\rho\mu}(r-p)^\nu\right)$$



$$-i g_s^2 \big(f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho})$$
$$+ f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho})$$
$$+ f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \big)$$

Gauge Invariance in QCD

$q(p)\bar q(p') \to g(k)\, g(k')$ to order α_s^2



$$\mathcal{M}^{a+b} = \mathcal{M}^{a+b}_{\mu\nu}\,\varepsilon^{*\mu}(k)\,\varepsilon^{*\nu}(k').$$

$\not{v}\not{v} = v^2$

$\not k - \not p = -(\not p - \not k)$ $\qquad$ $\not k - \not p'$

$$\mathcal{M}^{a+b}_{\mu\nu} = -ig_s^2\,\bar v(p')\left(\gamma_\nu t^b \frac{\not p - \not k}{(p-k)^2}\gamma_\mu t^a + \gamma_\mu t^a \frac{\not p - \not k'}{(p-k')^2}\gamma_\nu t^b\right)u(p)$$

$\varepsilon^\mu \to \varepsilon^\mu + \lambda k^\mu$, QED Ward identity $\mathcal{M}_{\mu\nu}k^\mu = 0$.

since $\not p\, u(p) = 0$; $\bar v(p')\not p' = 0$

First term in bracket is $-\gamma_\nu t^b t^a$

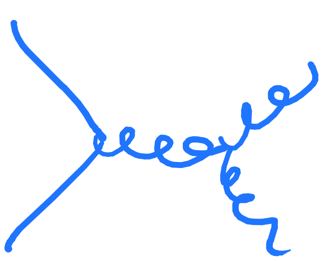
$\not p - \not k' = \not k - \not p'$ $\qquad (p + p' = k + k')$

$t^a t^b \gamma_\nu$

$$\mathcal{M}^{a+b\,\mu}_{\mu\nu}k = -ig_s^2\,\bar v(p')\left(\gamma_\nu\left(t^a t^b - t^b t^a\right)\right)u(p)$$

$$= -ig_s^2\,\bar v(p')\gamma_\nu\left([t^a, t^b]\right)u(p)$$

$$= -ig_s^2\,[t^a, t^b]\,\bar v(p')\gamma_\nu u(p) \neq 0$$

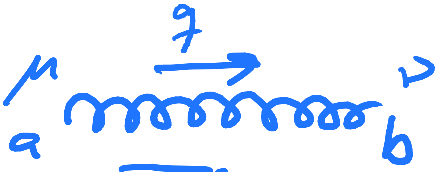


Adding diagram c would still give 0 for contraction with k^μ, once $\varepsilon^{*\nu}(k')$ is contracted.

Physical Gauge.

Gauge condition $A^{a\mu} n_\mu = 0$

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}\left(A^a_\mu n^\mu\right)^2$$



$$\delta^{ab}\frac{i}{q^2+i\varepsilon}\left(-g^{\mu\nu} + \frac{q^\mu n^\nu + q^\nu n^\mu}{(q\cdot n)} - n^2\frac{q^\mu q^\nu}{(n\cdot q)^2}\right)$$

$|\mathcal{M}|^2$ would contain:

$$\sum_{\alpha=1}^{2} \varepsilon^\alpha_\mu(q)\,\varepsilon^{*\alpha}_\nu(q)$$

$$\rightarrow -g_{\mu\nu} + \frac{q^\mu n^\nu + q^\nu n^\mu}{(n\cdot q)} - n^2\frac{q^\mu q^\nu}{(n\cdot q)}$$

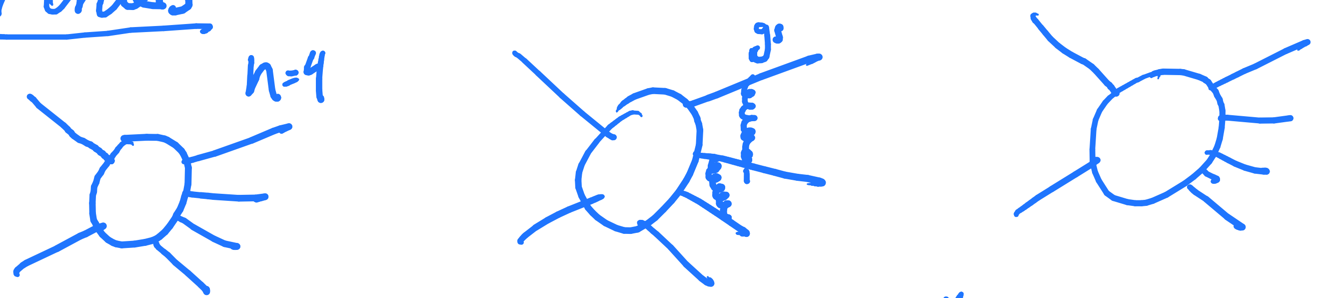
often choose $n^2 = 0$ (light-cone gauge)

$$\underline{\varepsilon^{\pm}_\mu(\underset{4}{k},\underset{4}{q})} = \pm\frac{\langle q^\mp|\gamma_\mu|k^\mp\rangle}{\sqrt{2}\langle q^\mp|k^\pm\rangle}$$

← vector

where $\langle q^\mp|$ is $\bar{\Psi}_\mp(q)$ $\quad\gamma_\mu$ is $\quad|k^\mp\rangle$ is $\Psi_\mp(k)$

Higher orders



$$\mathcal{M}_n = g_s^n M_n^n + g_s^{n+2} M_n^{n+2} + g_s^{n+4} M_n^{n+4} + \cdots$$

$$\mathcal{M}_{n+1} = g_s^{n+1} M_{n+1}^{n+1} + g_s^{n+3} M_{n+1}^{n+3} + \cdots$$

$$\mathcal{M}_{n+2} = g_s^{n+2} M_{n+2}^{n+2} + g_s^{n+4} M_{n+2}^{n+4} + \cdots$$

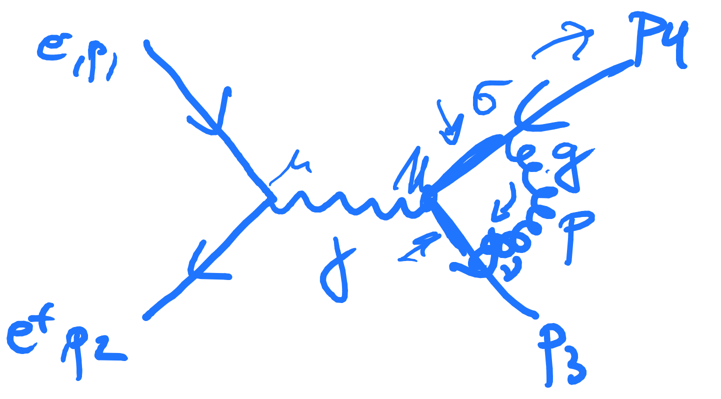
$$|\mathcal{M}_n|^2 = M_n \cdot M_n^* = \left(g_s^n M_n^n + g_s^{n+2} M_n^{n+2} + g_s^{n+4} M_n^{n+4} + \cdots\right)$$
$$\times \left(g_s^n M_n^{n*} + g_s^{n+2} M_n^{*n+2} + g_s^{n+4} M_n^{*n+4} + \cdots\right)$$

$$= g_s^{2n} |M_n^n|^2 + g_s^{2n+2}\, 2\mathrm{Re}\left(M_n^n M_n^{*n+2}\right)$$
$$+ g_s^{2n+4}\left(2\mathrm{Re}\left(M_n^n M_n^{*n+4}\right) + |M_n^{n+2}|^2\right) + \cdots$$

$$|\mathcal{M}_{n+1}|^2 = g_s^{2n+2} |M_{n+1}^{n+1}|^2 + g_s^{2n+4}\, 2\mathrm{Re}\left(M_{n+1}^{n+1} M_{n+1}^{*n+3}\right) + \cdots$$

$$|M_{n+2}|^2 = g_s^{2n+4} |M_{n+2}^{n+2}|^2 + \cdots$$

$e^+e^- \to$ hadrons



$$\frac{\bar{u}(p_3)\gamma^\nu(\not{p}_3 - \not{p})\gamma^\mu(-\not{p}_4 - \not{p})\gamma_\nu v(p_4)}{(p_3-p)^2(p_4+p)^2 p^2} \qquad v(p_2)\gamma_\mu u(p_1)$$

$$\sigma = \frac{1}{8s}\int \Pi\, F_V$$

$$F_V = \sum_i (Q_i)^2 \frac{e^4}{q^4} \mathrm{Tr}\left[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu\right] \mathrm{Tr}\left[\not{p}_3 \Lambda_\mu \not{p}_4 \gamma_\nu\right] + \text{c.c.}$$

$$\Lambda_\mu = g_s^2 C_F \int \frac{d^4p}{(2\pi)^4 i}\, \frac{1}{p^2}\, \gamma_\nu \frac{\not{p}_3 - \not{p}}{(p_3-p)^2} \gamma_\mu \frac{\not{p}_4 + \not{p}}{(p_4+p)^2} \gamma^\nu$$

Divergent

Problem 1: Divergence in soft & collinear regions.
Infrared divergences

Problem 2: The integrand doesn't fall fast enough at large p for the whole ~~al~~ to be finite.

For large p: integrand behaves as p^{-4}

$$\int \frac{d^4k}{k^4} = \int_0^\infty \frac{k^{3}\,dk}{k^4}\int d\Omega = 2\pi^2 \int_0^\infty \frac{dk}{k^{1-2\varepsilon}}$$

log. divergent in $k \to \infty$

reduce dimensions to $d = 4 - 2\varepsilon$, $\varepsilon > 0$

$$\frac{\Gamma(1+\varepsilon)}{\varepsilon}(4\pi)^\varepsilon = \frac{1}{\varepsilon} + \ln(4\pi) - \gamma_E + O(\varepsilon)$$

Add counterterms to $\mathcal{L}$ which cancel these poles.

"Only" need finite number of counterterms

Multiplicative renormalisation

$$\psi_B = Z_\psi \psi \qquad A^a_{\mu B} = Z_A^a A^a_\mu$$

Multiplicative UV Renormalisation

$\int d^{4-2\epsilon}x\, \mathcal{L}$ $\qquad \mathcal{L} \ni \bar\psi(\not{D}-m)\psi$

$\int d^4x\, \underbrace{\bar\psi\psi m}_{4}$ $\qquad [\bar\psi\psi]=3 \quad \therefore [\psi]=3/2$

$\boxed{[\underset{1}{g}\underset{3/2}{\bar\psi}\gamma^\mu\underset{3/2}{\psi}\underset{1}{A_\mu}]} \therefore [g]=0$ $\qquad g \to g\mu^{-2\epsilon}$

μ is arbitrary; not fixed by QCD

$\psi_B = Z_\psi \psi \quad ; \quad A^a_{\mu B} = Z_{A^a} A^a_\mu$

$Z_A^{n_A} Z_q^{n_f} \Gamma_B^{n_A,n_f}(g_i,q_i,g) = \Gamma^{n_A,n_f}(g_i,q_i,g(\mu))$

μ-indep

Apply $\mu\frac{d}{d\mu}\Gamma^{n_A,n_f} = \left(\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g}\right)\Gamma^{n_A,n_f}(g_i,q_i,g(\mu))$.

$Z_A^{n_A} = e^{n_A \ln Z_A}$

$\to \left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - n_A\gamma_A(g) - n_f\gamma_F(g)\right)\Gamma^{n_A,n_f}(g_i,q_i,g(\mu))=0$

$\beta(g)=\mu\frac{\partial g}{\partial\mu} \qquad \gamma_A(g)=\mu\frac{\partial}{\partial\mu}Z_A \qquad \gamma_F(g)=\mu\frac{\partial}{\partial\mu}Z_q$.

if D is the mass dimension of the Green's function.

$\Gamma^{n_A,n_f}(tg_i,tq_i,g(t\mu),t\mu) = t^D\,\Gamma^{n_A,n_f}(g_i,q_i,g(\mu),\mu)$.

$\therefore t\frac{d}{dt}\Gamma^{n_A,n_f}(tg_i,tq_i,g(t\mu),t\mu) = D\,\Gamma^{n_A,n_f}(tg_i,tq_i,g(t\mu),t\mu)$.

Expanding $\left(t\frac{d}{dt}\right) = \left(t\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial\mu}\right)$ we find

$\left(t\frac{\partial}{\partial t}+\mu\frac{\partial}{\partial\mu}-D\right)\Gamma^{n_A,n_f}(tg_i,tq_i,g(\mu),\mu)=0$

Eliminate derivative wrt μ:

$\left(-t\frac{\partial}{\partial t}+\beta(g)\frac{\partial}{\partial g}-n_A\gamma_A(g)-n_f\gamma_F(g)-D\right)$
$\Gamma^{n_A,n_f}(tg_i,tq_i,g(\mu),\mu)=0$

characteristic eqn:
$t\frac{\partial\bar g}{\partial t}=\beta(\bar g(t))$. $\quad \bar g(t=1)=g$

$\alpha_s = \frac{g_s^2}{4\pi}$

$gt\frac{\partial g}{\partial t} = \frac{1}{2}\, t\frac{\partial g^2}{\partial t} = 2\pi\frac{\partial\alpha_s}{\partial\ln t} = 2\pi\beta(\alpha_s)$

$\frac{\partial\alpha_s}{\partial\ln q^2} = -\beta_0\alpha_s^2-\beta_1\alpha_s^3-\beta_2\alpha_s^4-\beta_3\alpha_s^5-\dots$

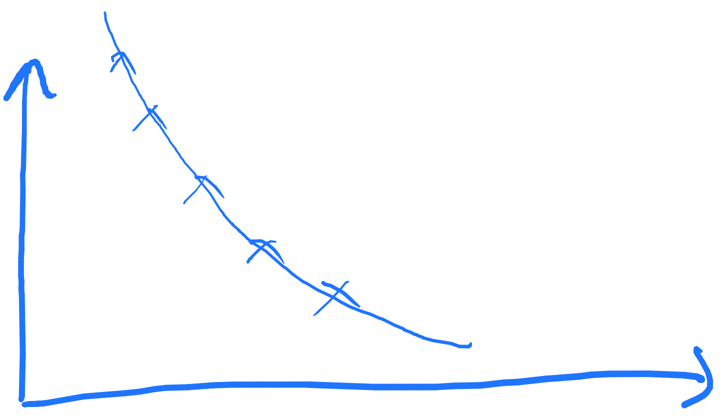
$\beta_0=\frac{1}{4\pi}\left(11-\frac{2}{3}N_f\right) \qquad (C_A=3)$

$\beta_1=\frac{1}{16\pi^2}\left(102-\frac{38}{3}N_f\right)$

$\overline{|\mathcal{M}|^2}_{qq\to qq} \quad \alpha_s^2\,\frac{s^2+u^2}{t^2} \qquad s=(p_a+p_b)^2$

t

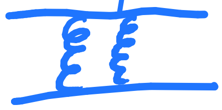
α_s



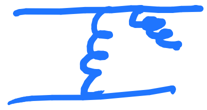
p_t

IR poles

1-loop

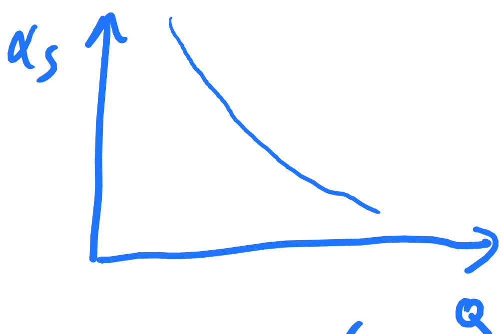


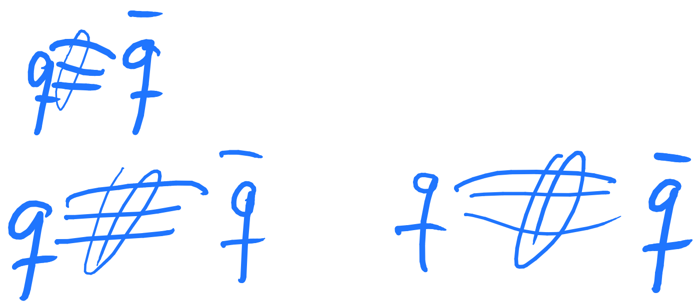
Born+1

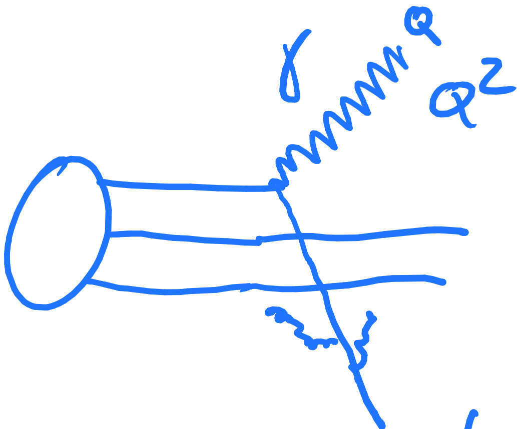


$D = 4 - 2\varepsilon\,,\ \varepsilon < 0$

Confiment







For sufficiently large Q^2 (such that $\alpha_s(Q^2)$ is small (perturbative)) the struck q will not have time to interact with the prot to equalise the Q before it has left / gluon field around the remnant.

The idea of pdf also works for

$pp \to W$

factorisation

onshell scattering amplitude