The Standard Model

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1 Introduction. From Fermi theory to Higgs mechanism

Currently Standard Model describes to an excellent precision the vast majority of the laboratory experiments. There are several notable exceptions, which indicate that our understanding of the fundamental details of everything is yet far from complete. However, the SM itself does not contain indications of any problems.

This was not always the case. Actually, SM itself emerged as a theory that solved the intrinsic problems of its predecessors. Let us do a small historic[1] review.

When we study a theory at low energy, up to several hundred MeV, we have lot's of weak and rare effects, like nuclear β decay, μ and π decays, neutrino scattering. The phenomenology of these processes is very rich, and all have stunning properties – parity is maximally violated, flavour is violated. All these effects are perfectly grasped by the Fermi theory of 4-fermion interactions. All these effects are due to the interaction between "charged currents"

$$\mathcal{L} = -4\frac{G_f}{\sqrt{2}} J^+_\mu J^-_\mu + \text{h.c.}, \tag{1}$$

where the *Fermi constant*

$$G_F = 1.116 \times 10^{-5}\ \text{GeV}^{-2} \sim \frac{1}{(300\ \text{GeV})^2}.$$

The currents are composed of leptonic and hadronic parts $J^\pm_\mu = J^\pm_{\mu,\text{lepton}} + J^\pm_{\mu,\text{hardon}}$. For the first two generations (not much sense to use Fermi theory for the third generation)

$$J^+_{\mu,\text{lepton}} = \bar{\nu}_L\gamma_\mu e_L + \bar{\nu_\mu}_L\gamma_\mu\mu_L. \tag{2}$$

Here the index 'L' means left fermion $\psi_L \equiv \frac{1-\gamma^5}{2}\psi$. The hadronic part is a bit more complicated, as far as it mixes first and second generation quarks

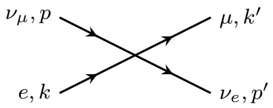
$$J^+_{\mu,\text{hadron}} = \bar{u}_L\gamma_\mu d'_L + \bar{c}_L\gamma_\mu s'_L, \tag{3}$$

[1]Not very historically precise.

where the down quarks are mixed with the Cabibo angle θ_c

$$
\begin{aligned}
d'_L &= \cos\theta_c d_L + \sin\theta_c s_L, \\
s'_L &= -\sin\theta_c d_L + \cos\theta_c s_L.
\end{aligned}
$$

The interaction (1) looks rather innocent (though a careful reader with QFT knowledge may notice the suspicious negative dimension of the coupling constant). Let us see what is the problem. For this, we can calculate the cross-section $\nu_\mu e \to \nu_e \mu$ scattering.[2] This process has only one diagram to compute



$$
= i\mathcal{M} = i\frac{4G_F}{\sqrt{2}}\bar{u}(p')\gamma^\lambda\frac{1-\gamma^5}{2}u(k)\cdot\bar{u}(k')\gamma^\lambda\frac{1-\gamma^5}{2}u(p), \tag{4}
$$

where $u(p)$ are fermionic wave functions for corresponding fermions (neutrinos, electrons, or muons). Squared modulus of the matrix element summed over all spin states, following standard rules, is

$$
\sum_{\text{spins}} |\mathcal{M}|^2 \;=\; 8G_F^2 \operatorname{tr}\left(\gamma_\rho\frac{1-\gamma^5}{2}\not{p}\gamma_\lambda\frac{1-\gamma^5}{2}(\not{k}'+m_\mu)\right)\operatorname{tr}\left(\gamma_\rho\frac{1-\gamma^5}{2}\not{k}\gamma_\lambda\frac{1-\gamma^5}{2}\not{p}'\right), \tag{5}
$$

where we have neglected masses of neutrinos and electron. Actually, the term with muon mass gives zero also, as far as it enters under trace multiplied by an odd number of gamma matrices. Using the standard trace rules, and contracting indexes for the epsilon symbols we get after some tedious algebra

$$
\sum_{\text{spins}} |\mathcal{M}|^2 = 128G_F^2(pk)(p'k') = 128G_F^2\frac{s}{2}\frac{s-m_\mu^2}{2},
$$

where $s \equiv (p+k)^2 = (p'+k')^2$ characterises the collision energy.

As far as the matrix element does not depend on directions of the final particles, we can immediately convert it to the cross-section

$$
\sigma = \frac{1}{8\pi s}\left(\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2\right)\frac{|\mathbf{p}|}{s} = \frac{G_F^2}{\pi}\frac{(s-m_\mu^2)^2}{s}, \tag{6}
$$

where 1/2 is present because we have to average over the initial electron spin states (no averaging for neutrinos as far as the have only one helicity state, and nothing for muon, because we sum over all final states).

This result is rather peculiar. First, it iz zero at threshold, $s = m_\mu^2$. Then it *grows* with energy. Of course, while *sqrts* $\ll G_F^{-2} \sim 300$ GeV it remains small due to the Fermi constant, but for higher energies it becomes large, and grows $\propto s$. At the same time, scattering cross-sections can not be too large. Naive expectation of cross-section being

[2]Probably not the easiest process for experiment, but it requires only one diagram for us to study.

"size" of something just tells us that it should be at most constant, while precise unitarity arguments (basically, requirement that probability should be smaller than one) state that all cross-sections[3] can not grow faster than $\propto (\ln s)^2$. Another related statement tells us that partial amplitudes should be constant, or partial cross-sections should fall as $\propto 1/E^2$, where E is typical energy scale. Thus, the Fermi theory fails to produce physically sensible results at energies above $G_F^{-2} \sim 300$ GeV. Note, that this scale is encoded in the theory, and follows from the low-energy cross-section measurements.

This inconsistency asked for invention of better theories. First, thing that can be done is to replace the point-like 4-fermion interaction by an interaction with the exchange of a vector boson. Then, instead of the Fermi constant one would expect an expression of the form $g^2/(p^2 - m_W^2)$, where g is the coupling constant, and m_W is the boson mass. For momenta much smaller than the mass it would give approximately constant $\sim g^2/M_W^2$, while at high energies the amplitude will become suppressed, or the cross-section (in the limit of $s \gg m_\mu$) is

$$\frac{d\sigma}{d|t|} = \frac{G_F^2}{\pi} \frac{m_W^4}{(|t| + m_W^2)^2},$$

The total cross-section is now

$$\sigma = \int_0^s \frac{d\sigma}{d|t|} d|t| = \frac{G_F^2 m_W^2}{\pi},$$

which is constant! We see, that the previous answer (6) was cut off at energies of the order of the mass of the W boson.

Unfortunately, this does not work immediately – the massive vector boson propagator is more complicated

$$\frac{i}{p^2 - m_W^2} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m_W^2} \right).$$

The last term does not become small at large momenta. However, it is possible to get away from this problem, if this term, which has peculiar tensor structure, cancels in the final answer. This is similar to cancellation of gauge dependence in QED, and actually happens if the theory has gauge symmetry. A new problem appears here – normally, massive vector bosons are not gauge invariant. This problem itslef can be solved by introducing a new particle, the Higgs boson.

In the following, we will introduce the Higgs mechanism, and describe it specifically for the case of the electroweak interactions.

2 Global and local symmetries

The Standard Model is the gauge theory with $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group, with spontaneous symmetry breaking. It is known, that the only good (renormalizable)

[3]See C. Itzykson, J.-B. Zuber, Quantum Field Theory, McGraw-Hill: 1980, vol. 1. Strictly speaking, this is true for theories without massless particles. Really, the "total" cross-section of electron-electron scattering is infinite, because massless photons lead to scattering at arbitrary distance, but for infinitely small scattering angle.

quantum field theory that gives rise to the massive non-abelian vector bosons is a gauge theory with spontaneous symmetry breaking, or Higgs mechanism. We will review the construction of the abelian and non-abelian symmetries, and the mechanism of symmetry breaking. Then, we will also describe the interactions with the matter fermions (leptons and quarks).

2.1 Global Abelian $\mathrm{U}(1)$

The simplest example of global symmetry is the Abelian U(1) symmetry. For a complex scalar field it is given by

$$\phi(x) \to \phi'(x) = \mathrm{e}^{i\alpha}\phi(x), \tag{7}$$

and the invariant Lagrangian density (up to the operators of dimension 4) is

$$\mathcal{L}[\phi(x), \partial_\mu\phi(x)] = \partial_\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi - \frac{\lambda}{2}(\phi^*\phi)^2. \tag{8}$$

Let us note the consequences of this symmetry. First, it tells that the masses of the real scalar fields defined as $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ are equal, $m_1 = m_2 = m$. Second, by Noether theorem, there is a conserved current

$$j_\mu = i\phi^*\partial_\mu\phi - i\phi\partial_\mu\phi^*. \tag{9}$$

If the fields satisfy the equations of motion, then $\partial_\mu j_\mu = 0$, and charge $Q \equiv \int d^3x j_0$ does not change with time.

2.2 Gauge (local) $U(1)$

If we would like to have *local* symmetry

$$\phi(x) \to \phi'(x) = \mathrm{e}^{i\alpha(x)}\phi(x), \tag{10}$$

then the kinetic term in (8) stops being invariant. This can be compensated by promotion of the derivative to the covariant derivative

$$D_\mu\phi = (\partial_\mu - ieA_\mu)\phi. \tag{11}$$

Requirement that transformation of $D_\mu\phi$ is covariant (just multiplication by the phase, as for the field ϕ itself)

$$\begin{aligned} D_\mu\phi \to (D_\mu\phi)' &= \mathrm{e}^{i\alpha(x)}(\partial_\mu\phi + i\phi\partial_\mu\alpha - ieA'_\mu\phi) \\ &= \mathrm{e}^{i\alpha(x)}(\partial_\mu\phi - ieA_\mu\phi) = \mathrm{e}^{i\alpha(x)}D_\mu\phi, \end{aligned} \tag{12}$$

defines the transformation of the the gauge field A_μ

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha. \tag{13}$$

The gauge invariant expression constructed out of the gauge field is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{14}$$

Note, that a convenient form to define the gauge invariant field strength tensor is as a commutator of the covariant derivatives (which gives the same result)

$$F_{\mu\nu} \equiv \frac{i}{e}[D_\mu, D_\nu]. \tag{15}$$

This leads to the local $U(1)$ symmetric action (scalar electrodynamics)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - (D_\mu\phi)^*(D_\mu\phi) - V(\phi^*\phi). \tag{16}$$

Note, that we still have two equal mass scalar degrees of freedom, two polarisations for the massless vector field, and conserved charge.

The case of fermionic matter field was discussed in detail in QED lectures.

2.3 Global non-Abelian

The next generalization is to use more complex symmetries for the action. The simplest example is the system with N scalar fields (for concreteness think $N = 3$ for the QCD case, or $N = 2$ for the electroweak case we will be interested mostly)

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \tag{17}$$

and the transformation

$$\phi_i(x) \to \phi'_i(x) = \omega_{ij}\phi_j(x), \tag{18}$$

with $\omega_{ij} \in \mathrm{SU}(N)$ (we assume implicitly sum over the repeated index j). $\mathrm{SU}(N)$ is the group of unitary $N \times N$ matrices $\omega\omega^\dagger = 1$ with $\det\omega = 1$. Note, that the transformation matrix can be conveniently parametrised by $r = N^2 - N$ parameters α^a as

$$\omega = \exp\left(i\sum_{a=1}^{r}\alpha^a t^a\right),$$

where t^a are *generators* of the group.[4] In particular, for $SU(2)$ case the generators are $t^a = \sigma^a/2$ with Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19}$$

[4] There are other groups and other representations which can be used, but all can be expressed in this form

Unitary transformation leaves the scalar product $(\phi, \phi) \equiv \phi_i^* \phi_i$ invariant, so the following Lagrangian is invariant

$$\mathcal{L} = \partial_\mu \phi_i^* \partial_\mu \phi_i - m^2 \phi_i^* \phi_i - \lambda (\phi_i^* \phi_i)^2. \tag{20}$$

In a similar way a symmetric action can be written for a group of fermion fields.

As a consequence of the symmetry all the fields have the same mass, the same coupling constant. Also, in agreement with the Noether theorem there is a set of conserved currents (or charges).

I should mention, that generalization of this construction is possible for more complicated groups or group representations—then in general, instead of using the element $\omega \in G$ of a group G ($\mathrm{SU}(N)$ in our example) we should use a matrix $T(\omega)$ which transforms the fields according to representation T of the group.

2.4 Gauge (local) non-Abelian

The Lagrangian (20) is no longer invariant under *coordinate dependent* transformations

$$\phi(x) \to \phi'(x) = \omega(x)\phi(x) \tag{21}$$

$$\omega(x) \in \mathrm{SU}(N), \tag{22}$$

the terms with derivatives of $\omega(x)$ emerge from the kinetic term

$$\partial_\mu \phi'(x) = \omega(x)\partial_\mu \phi(x) + \partial_\mu \omega(x) \cdot \phi(x). \tag{23}$$

The remedy is the same as for the abelian case—introduce the gauge fields and upgrade all derivatives to covariant ones,

$$D_\mu \phi = \partial_\mu \phi - i A_\mu \phi,$$

with some matrix A_μ which transform in a covariant way under the gauge transformations

$$(D_\mu \phi)' = \omega D_\mu \phi.. \tag{24}$$

Here Comparing the left and right parts of this equation gives us the required transformation rule for the gauge fields

$$D_\mu \phi' = \partial_\mu \phi' - i A'_\mu \phi' = \omega \partial_\mu \phi + \partial_\mu \omega \phi - i A'_\mu \omega \phi$$

$$A_\mu \to A'_\mu = \omega A_\mu \omega^{-1} + i \omega \partial_\mu \omega^{-1}. \tag{25}$$

The theory of continuous group guarantees that the matrix A_μ is an element of the Lie algebra of the symmetry group and can be written as

$$A_\mu = g \sum_a A_\mu^a t^a,$$

where t^a are generators of the group (see (19) for the $SU(2)$ case), and A^a_μ are $N^2 - N$ independent fields. Thus

$$D_\mu \phi = \partial_\mu \phi \phi - ig \sum_a A^a_\mu t^a$$

Thus, the Lagrangian

$$\mathcal{L} = (D_\mu \phi)^\dagger D_\mu \phi - m^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2,$$

is invariant under gauge transformations (21,25). The kinetic term for the gauge bosons is a generalization of the Abelian case. However, now the definition (15) leads to the field strength

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig A^a_\mu A^b_\nu f^{abc} t^c, \end{aligned}$$

which is no longer gauge invariant, but transforms in a covariant way

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = \omega(x) F_{\mu\nu} \omega^{-1}(x). \tag{26}$$

Then, the kinetic term takes the form

$$\mathcal{L}_A = -\frac{1}{2g^2} \operatorname{Tr} F_{\mu\nu} F_{\mu\nu} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}. \tag{27}$$

where again matrix $F_{\mu\nu} \equiv g^2 \sum F^a_{\mu\nu} t^a$.

The resulting theory has a set of conserved currents, and a lot (equal to the number of generators of the Lie algebra) massless vector bosons. Due to the commutator in the definition of the filed strength tensor these gauge bosons interact between each other, making the dynamics quite different from the Abelian (QED) case.

3 Spontaneous symmetry breaking

3.1 Global symmetry—Goldstone theorem

For now, we are still far from the interesting goal—getting a theory of massive vector bosons. The next attempt is to take a Lagrangian with some internal symmetry, and break it spontaneously. That is, find a Lagrangian which *is* invariant, but which has vacuum state which is *not* invariant under the symmetry transformation. The simplest example is to take the U(1) symmetric theory (8) (written for better visualisation using two real fields $\phi = \frac{\varphi_1 + \varphi_2}{\sqrt{2}}$, instead of one complex field), but select a rather peculiar potential

$$V(\varphi_1, \varphi_2) = -\frac{\mu^2}{2}(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2.$$

The minimum of this potential is not at zero, but at any point of the circle $\varphi_1^2 + \varphi_2^2 = v^2 \equiv \mu^2/\lambda$. Note, that symmetry transformation (rotation in the plane φ_1, φ_2) takes a point on this circle and moves it to *another* point—so any given vacuum is not invariant under the symmetry. To study the theory we should first select one of the vacuums (not really important, which one), for example

$$\varphi_1 = v, \quad \varphi_2 = 0.$$

Then, we can expand the fields on top of this vacuum

$$\begin{aligned} \varphi_1(x) &= v + \chi(x) \\ \varphi_2(x) &= \theta(x). \end{aligned} \tag{28}$$

and study the dynamics of the small perturbations $\chi(x)$, $\theta(x)$. The expansion of the kinetic part is trivial while the potential is more complicated

$$V = -\frac{\mu^2}{2}[(v+\chi)^2 + \theta^2] + \frac{\lambda}{4}[(v+\chi)^2 + \theta^2]^2 + \frac{\mu^4}{4\lambda}.$$

Up to the quadratic order we get

$$\mathcal{L}^{(2)}_{\chi,\theta} = \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2}(\partial_\mu \theta)^2 - \mu^2 \chi^2.$$

So, we have got one massive field χ, and one *massless* field θ. Also, we have got a number of higher order terms, leading to various interactions between χ and θ. This massless field is called *Goldstone boson.* It is a generic consequence of spontaneous breaking of the global symmetry. The Goldstone theorem tells, that for each broken generator of a global symmetry one massless boson appears in the spectrum of the perturbations.

There is an approximate physical example – pions are approximate Goldstone bosons of global $SU(2)$ symmetry in QCD with massless quarks, which is broken by non-perturbative quark condensate. They are not exactly massless, because of small quark masses present in the fundamental theory, but a much lighter than all other states in the theory.

3.2 Gauge symmetry—Higgs mechanism

More interesting situation is realized if an attempt is made to spontaneously break gauge symmetry. Let us see how this happens on the simplest example of Abelian Higgs model. We will take the model from the previous section (we switch again to the complex field notations which are more natural for the charged fields) with the addition of the gauge fields

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^* D_\mu\phi - [-\mu^2\phi^*\phi + \lambda(\phi^*\phi)^2], \tag{29}$$

where, as before, the transformation rules are

$$\begin{aligned} A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \\ \phi(x) &\to \phi'(x) = \mathrm{e}^{i\alpha(x)}\phi(x). \end{aligned}$$

We can select the vacuum (minimum of the energy of the system) by setting the gauge field to zero and using the results of the previous section for the scalar field

$$A_\mu^{(v)} = 0, \ \phi^{(v)} = \frac{1}{\sqrt{2}} v. \tag{30}$$

Following standard rules the fields should be expanded into small perturbations on top of the vacuum

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \chi(x) + i\theta(x)). \tag{31}$$

The covariant derivative in terms of A_μ, χ, θ looks like

$$D_\mu \phi = \frac{1}{\sqrt{2}}(\partial_\mu \chi + i\partial_\mu \theta - iev A_\mu).$$

Substituting this expression into the action (29) and expanding to the quadratic order we get

$$\mathcal{L}^{(2)} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2 \chi^2 + \frac{e^2 v^2}{2}\left(A_\mu - \frac{1}{ev}\partial_\mu \theta\right)^2. \tag{32}$$

Note, that θ and A_μ enter only in the combination

$$B_\mu = A_\mu - \frac{1}{ev}\partial_\mu \theta,$$

so we can rewrite the Lagrangian for small perturbations using only B_μ and χ

$$\mathcal{L}^{(2)} = -\frac{1}{4}B_{\mu\nu}^2 + \frac{e^2 v^2}{2}B_\mu B_\mu + \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2 \chi^2, \tag{33}$$

where $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is the field strength tensor. This is the theory for two massive fields with masses

$$m_V = ev = \frac{e}{\sqrt{\lambda}}\mu$$

$$m_\chi = \sqrt{2\lambda}\mu.$$

No trace of the field θ left here. No massless Goldstone bosons appeared, contrary to the SSB of a global theory.

The absence of one field deserves a closer analysis. Let us count the number of propagating degrees of freedom in the theory. If the potential does not lead to the SSB ($-\mu^2 > 0$, see section 2.2) we have two massive scalar fields, and one massless vector field. The latter has only two polarizations (remember the electrodynamics), overall 4 d.o.f. If the symmetry is broken ($\mu^2 > 0$) we have only one massive scalar field, and a *massive* vector field, which has 3 degrees of freedom (we can no longer use gauge symmetry to remove the longitudinal polarization). Overall, in both cases the system has 4 d.o.f. This is the essence of the Higgs mechanism—the Goldstone boson that would have been associated with the symmetry breaking is getting “eaten” by the gauge boson. The result is exactly what we ere looking for—a theory with a massive vector bosons. Moreover, the theory still has gauge theory as the origin, which leads to good UV properties of the theory. At low energies the remains of the gauge symmetry can be seen in the relations between interaction constants and masses of the theory.

Note, that we could use gauge invariance for a simpler way to obtain the interaction Lagrangian of the physical degrees of freedom. As far as all configurations connected by gauge transformation are completely equivalent, we could define a gauge

$$\begin{aligned} \mathrm{Im}\,\phi &= 0 \\ \mathrm{Re}\,\phi &= \frac{1}{\sqrt{2}}(v+\chi). \end{aligned}$$

This gauge is called *unitary* and has no trace of the unphysical degree of freedom. Writing the action in this gauge immediately gives the spectrum and interaction vertices for all the physical particles of the theory. However, analysis of the quantum radiative corrections in this gauge is rather involved, because the original gauge structure is very well hidden.

The generalization of the Higgs mechanism to non-abelian symmetries we can study directly on the real world example—Standard Model.

4 Standard Model $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$

4.1 Gauge and Higgs bosons

Let us start form the construction of the real full bosonic sector of the Standard Model. We want two gauge symmetries, weak $\mathrm{SU}(2)_L$ ('L' stands for left, which reflects its relation to the fermionic degrees of freedom), and U(1) which is called hypercharge. The corresponding fields will be F^a_μ, with $a = 1, 2, 3$, corresponding to three SU(2) generators, and B_μ. We will also add one complex doublet

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

The gauge invariant Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{4}B_{\mu\nu}B_{\mu\nu} + (D_\mu\Phi)^\dagger D_\mu\Phi - \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2,$$

where the field strength tensors are

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\mu A^a_\mu + g\varepsilon^{abc}A^b_\mu A^c_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \end{aligned}$$

and the covariant derivative is

$$D_\mu\Phi = \partial_\mu\Phi - i\frac{g}{2}\tau^a A^a_\mu\phi - i\frac{g'}{2}B_\mu\Phi, \tag{34}$$

Here g and g' are coupling constants for the two gauge groups. Note, that the scalar transforms in fundamental representation of $SU(2)_L$, and is also charged with the charge $g'/2$ under the $U(1)$.

The vacuum configuration can be chosen as

$$\begin{aligned} A^a_\mu &= B_\mu = 0 \\ \Phi &= \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \equiv \Phi^{(v)}. \end{aligned} \tag{35}$$

Note, that this configuration is still invariant under some specific gauge transformations. Really, the group transformations corresponding to the generator

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

would not change the vacuum

$$Q\Phi^{(v)} = 0 \tag{36}$$

(if the generator annihilates the vacuum, then the group transformation of the form $\exp(i\alpha Q)$ would leave the vacuum intact). This specific generator is is a combination of the generators of the U(1) hypercharge rotation and the third $\mathrm{SU}(2)_L$ generator

$$Q = T^3 + Y/2, . \tag{37}$$

where Y is the hypercharge operator, which is proportional to the unit matrix for all fields, but may have different value for different fields. Therefore, we expect that the corresponding U(1) group will remain unbroken, leading to the electromagnetic interaction with massless photon.

The simplest way to analyse the spectrum of the theory is to fix the unitary gauge

$$\Phi = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} + \frac{\chi}{\sqrt{2}} \end{pmatrix}. \tag{38}$$

Really, it can be easily verified that arbitrary (non-zero) doublet can be transformed by an $SU(2)$ transformation to this form. The covariant derivative is

$$\begin{aligned} D_\mu\Phi = \; & \partial_\mu\Phi + \left[-\frac{ig}{2}A^1_\mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{ig}{2}A^2_\mu \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right. \\ & \left. -\frac{ig}{2}A^3_\mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{ig'}{2}B_\mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \Phi. \end{aligned}$$

In the unitary gauge we get

$$D_\mu\Phi = \begin{pmatrix} -\frac{ig}{2\sqrt{2}}(A^1_\mu - iA^2_\mu)(v+\chi) \\ -\frac{i}{2\sqrt{2}}(g'B_\mu - gA^3_\mu)(v+\chi) + \frac{1}{\sqrt{2}}\partial_\mu\chi \end{pmatrix}. \tag{39}$$

It is convenient to introduce instead of $A^{1,2}$ complex fields

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp i A_\mu^2)$$

so that $(W_\mu^-)^* = W_\mu^+$. Let us also introduce two vector fields

$$Z_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gA_\mu^3 - g'B_\mu) \tag{40}$$

$$A_\mu = \frac{1}{\sqrt{g^2+g'^2}}(gB_\mu + g'A_\mu^3). \tag{41}$$

Fields Z_μ and A_μ are chosen in a way, so that the covariant derivative (39) contains only Z_μ, and the following normalization condition is satisfied

$$Z_\mu^2 + (A_\mu)^2 = (A_\mu^3)^2 + B_\mu^2. \tag{42}$$

Finally, the covariant derivative (39) takes the form

$$D_\mu\phi = \begin{pmatrix} -i\frac{gv}{2}W_\mu^+ \\ \frac{1}{\sqrt{2}}\partial_\mu\chi + \frac{i\sqrt{g^2+g'^2}}{2\sqrt{2}}vZ_\mu \end{pmatrix} + \begin{pmatrix} -i\frac{g}{2}W_\mu^+\chi \\ \frac{i\sqrt{g^2+g'^2}}{2\sqrt{2}}Z_\mu\chi \end{pmatrix}, \tag{43}$$

Here the first column is linear in excitations (fields $W_\mu^\pm$, χ, Z_μ), and the second is of the second order. The contribution of the covariant derivative to the quadratic part of the Lagrangian is then given by

$$[(D_\mu\Phi)^\dagger D_\mu\Phi]^{(2)} = \frac{1}{2}(\partial_\mu\chi)^2 + \frac{g^2v^2}{4}W_\mu^+W_\mu^- + \frac{1}{2}\left(\frac{(g^2+g'^2)v^2}{4}\right)Z_\mu^2. \tag{44}$$

The quadratic parts of the kinetic terms for the gauge fields is expressed in the usual form in terms of the new fields W, Z, A because the redefinitions were orthogonal. Finally, collecting all terms quadratic in the small perturbations gives

$$\begin{aligned} \mathcal{L}^{(2)} = & -\frac{1}{2}\mathcal{W}_{\mu\nu}^+\mathcal{W}_{\mu\nu}^- + m_W^2 W_\mu^+ W_\mu^- \\ & -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} \\ & -\frac{1}{4}\mathcal{Z}_{\mu\nu}\mathcal{Z}_{\mu\nu} + \frac{m_Z^2}{2}Z_\mu Z_\mu \\ & +\frac{1}{2}(\partial_\mu\chi)^2 - \frac{m_\chi^2}{2}\chi^2, \end{aligned}$$

where the field strengths $\mathcal{W}_{\mu\nu}^\pm$, $F_{\mu\nu}$, and $\mathcal{Z}_{\mu\nu}$ are obtained in the standard way from $W_\mu^\pm$, A_μ, Z_μ, and the masses are

$$\begin{aligned} m_W &= \frac{gv}{2} \\ m_Z &= \frac{\sqrt{g^2+g'^2}v}{2} \\ m_\chi &= \sqrt{2\lambda}v. \end{aligned}$$

fields\ groups	SU(3)$_c$	SU(2)$_L$	U(1)$_Y$	U(1)$_{EM}$
$\chi_L \equiv \binom{\nu_L}{e_L}$	1	2	-1	$\binom{0}{-1}$
e_R	1	1	-2	-1
$Q_L = \binom{U_L}{D_L}$	3	2	+1/3	$\binom{+2/3}{-1/3}$
U_R	3	1	+4/3	+2/3
D_R	3	1	-2/3	-1/3

The Lagrangian describes one massive scalar field with the mass m_χ, massless photon A_μ, two massive (complex) vector bosons $W_\mu^\pm$ with mass m_W, and massive vector boson Z_μ with the mass m_Z.

It is convenient also to introduce the Weinberg mixing angle as

$$\begin{aligned} \cos\theta_W &= \frac{g}{\sqrt{g^2+g'^2}} \\ \sin\theta_W &= \frac{g'}{\sqrt{g^2+g'^2}}. \end{aligned}$$

Then the relations (40), (41) take the explicit form of a rotation

$$\begin{aligned} Z_\mu &= \cos\theta_W A_\mu^3 - \sin\theta_W B_\mu \\ A_\mu &= \cos\theta_W B_\mu + \sin\theta_W A_\mu^3 \end{aligned} \qquad (45)$$

and also we get the important relation between the weak vector boson masses

$$m_Z = \frac{m_W}{\cos\theta_W}. \qquad (46)$$

Experimental value of $\sin\theta_W$ is $\sin^2\theta_W = 0,23$, and it can be measured independently of the vector boson masses in the interactions of photons and W and Z bosons with quarks and leptons. The equation (46) is satisfied with very good precision and provides a test of the Standard Model.

Expansion of the Lagrangian beyond the quadratic level gives the interactions of the bosons, with all the vertices expressed from the three coupling constants g, g', λ and vacuume expectation value v.

For completeness, we should mention the QCD gauge kinetic term

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2}\sum_{a=1}^{8} G^a_{\mu\nu} G_{\mu\nu^a},$$

which was described in detail in the corresponding lectures. The $SU(3)$ group is not broken, the corresponding gauge bosons—gluons—are massles, but are self-interacting.

4.2 Leptons

In the SM there were only left handed $(V-A)$ charged currents. Thus, left and right components of the spinors enter differently in the SM action. We can extract the left and right components out of the usual four spinor using the projectors

$$\psi_L \equiv P_L\psi \equiv \frac{1-\gamma^5}{2}\psi, \quad \psi_R \equiv P_R\psi \equiv \frac{1+\gamma^5}{2}\psi. \tag{47}$$

Note, that $\overline{\psi_L} = (\psi_L)^\dagger\gamma^0 = \psi^\dagger\frac{1-\gamma^5}{2}\gamma^0 = \overline{\psi}\frac{1+\gamma^5}{2} = \overline{\psi}P_R$. In the Weyl basis for gamma matrices γ^5 is diagonal, and the meaning of these definitions boils down to selecting upper two or lower two components of the 4-spinor for the left or right part. In Dirac basis (which is often used in non-relativistic limit) the meaning of (47) is less obvious.

The left electron e_L and neutrino ν_L form a doublet under gauge transformation $U = \exp\left(-i\frac{\tau^a}{2}\Delta^a(x)\right) \in SU(2)_L$

$$\chi_L \equiv \begin{pmatrix}\nu_L\\ e_L\end{pmatrix}, \quad \chi_L \to \chi_L' = U\chi_L.$$

The Dirac conjugate spinors should transform in a conjugate way, that is $\overline{\chi_L} = (\overline{\nu_L}, \overline{e_L})$, $\overline{\chi_L} \to \overline{\chi_L}U^\dagger = \overline{\chi_L}U^{-1}$.

They also transform under hypercharge with charge $Y_L = -1$:

$$\chi_L \to \exp(iY_L\alpha(x))\chi_L.$$

For the normal massive electron we need also the right handed component. It does not transform at all under $SU(2)_L$, and has hypercharge $Y_R = -2$

$$e_R \to \exp(iY_R\alpha(x))e_R.$$

According to the formula for the electric charge $Q = T^3 + Y/2$ we get $Q = -1$ for both electron components, and 0 for neutrino, as needed.

We can now readily write the kinetic terms for the lepton Lagrangian.

$$\mathcal{L}_{\text{kin}} = \overline{\chi_L}i\gamma^\mu D_\mu\chi_L + \overline{e_R}i\gamma^\mu D_\mu e_R, \tag{48}$$

where the covariant derivatives are

$$D_\mu\chi_L = (\partial_\mu - igA_\mu^aT^a - i\frac{g'}{2}Y_LB_\mu)\chi_L, \tag{49}$$

$$D_\mu e_R = (\partial_\mu - i\frac{g'}{2}Y_RB_\mu)e_R. \tag{50}$$

The only terms that are allowed by the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetries and contains only fermions of the form $\bar{\psi}\psi$ and not more than one scalar field are

$$\mathcal{L}_{\text{Yukawa}} = -G_e\overline{\chi}_L^a\Phi^a e_R - G_e\overline{e}_R\Phi^{\dagger a}\chi_L^a, \tag{51}$$

where repeated $SU(2)$ index a implies summation. Note, that the terms are trivially $SU(2)_L$ invariant. The sum of hypercharges for this term is zero, so it is $U(1)_Y$ invariant. The requirement $-Y_L + Y_\Phi + Y_R = 0$ constraints possible hypercharges that can be assigned to the fields. Note also, that terms of the form $\bar{\chi}_L \chi_L$ are trivially zero because of the chirality projectors.

The kinetic part (48) gives rise to the normal kinetic terms for the fermions, and to the interactions with all the gauge bosons—charge, neutral and electric currents. Writing explicitly in components (48) using (49) and (50) we get for the charged current

$$\mathcal{L}_{CC} = -\frac{g}{\sqrt{2}}(J^+_\mu W^+_\mu + J^-_\mu W^-_\mu), \tag{52}$$
$$J^+_\mu = \bar{\nu}_L \gamma_\mu e_L.$$

To get the neutral and electromagnetic current we also have to express A^3 and B via Z-boson and photon fields using (45), which gives

$$\mathcal{L}_{EC} = eA_\mu \bar{e}\gamma_\mu e, \tag{53}$$

$$\mathcal{L}_{NC} = \frac{g}{2\cos\theta_W} Z_\mu \left(-\bar{\nu}_L\gamma_\mu\nu_L - \bar{e}_L\gamma_\mu e_L + 2\sin^2\theta_W \bar{e}\gamma_\mu e\right), \tag{54}$$

where the electric charge is

$$e \equiv g\sin\theta_W. \tag{55}$$

The Yukawa interaction part in the Lagrangian (51) after symmetry breaking gives both mass term for electrons, and electron interaction with the Higgs boson. It is easiest to see this in the unitary gauge (38), which leads to

$$\mathcal{L}_{\text{Yukawa}} = -\frac{1}{\sqrt{2}}G_e(v+H)\bar{e}_L e_R - \frac{1}{\sqrt{2}}G_e(v+H)\bar{e}_R e_L \equiv -\frac{1}{\sqrt{2}}G_e(v+H)\bar{e}e, \tag{56}$$

leading to the electron mass $m_e = G_e v/\sqrt{2}$, and electron Higgs interaction with the strength $G_e/\sqrt{2} = m_e/v$. Note, that the interaction strength is directly proportional to the mass of the particle, so that this interaction is practically absent for the electron, but is sizeable for τ-lepton.

4.3 Quarks

The left-handed components of the quarks also form a doublet under $\mathrm{SU}(2)_L$, however, as far as the electric quark charges are different from leptons, the choice of the hypercharge is different

$$Q_L = \begin{pmatrix} U_L \\ D_L \end{pmatrix}, \quad Y_Q = 1/3. \tag{57}$$

The right handed components are still $SU(2)_L$ singlets, but as far as all the quarks are Dirac massive fermions, we need right-handed counterparts to both up and down components of the left-handed doublet, U_R and D_R with hypercharges $Y_U = 4/3$, and $Y_D = -2/3$. Also, all of the quark fields, Q_L, U_R, and D_R have SU(3) colour index $i = 1, 2, 3$, which we don't write explicitly here (see QCD lectures for details).

The kinetic quark terms in the Lagrangian are, as usual

$$\mathcal{L}_{\text{kin}} = \bar{Q}_L i\gamma_\mu D^\mu Q_L \bar{U}_R i\gamma_\mu D^\mu U_R \bar{D}_R i\gamma_\mu D^\mu D_R, \tag{58}$$
$$D_\mu Q_L = (\partial_\mu - igA^a_\mu T^a - ig' Y_Q B_\mu - ig_s G_\mu) Q_L, \tag{59}$$
$$D_\mu U_R = (ig' Y_U B_\mu - ig_s G_\mu) U_R, \tag{60}$$
$$D_\mu D_R = (ig' Y_D B_\mu - ig_s G_\mu) D_R. \tag{61}$$

Similarly to the lepton sector, this term generates the normal quark kinetic terms, and gauge boson interactions. The SU(3) strong force interactions are described in QCD lectures, while the electroweak currents in the interaction terms (52), (53), and (54) get the following contributions

$$J^+_\mu = \bar{U}_L \gamma_\mu D_L, \tag{62}$$
$$J^{EM}_\mu = Q_U \bar{U}\gamma_\mu U + Q_D \bar{D}\gamma_\mu D, \tag{63}$$
$$J^{NC}_\mu = \sum_{\text{fermions}} \bar{\psi}\gamma_\mu((1-\gamma^5)T^3 - 2Q\sin^2\theta_W)\psi, \tag{64}$$

where $T^3 = 1/2$, $Q = 2/3$ for the up quarks, $T^3 = -1/2$, $Q = -1/3$ for the down quarks. Note, that this is a generic formula, that can be used for leptons also.

The Yukawa term is more complicated for quarks. The term similar to (51) gives mass to the down quark, while to generate mass for the up component the Higgs doublet should have non-zero value in the upper component, instead of the lower component. There are two ways to achieve this. One is to introduce another Higgs doublet $\Phi^{(2)}$, which in the vacuum is equal to $\binom{v/\sqrt{2}}{0}$. This is actually the situation that is realized in SUSY, which imposes significant constraints on the possible Lagrangians. Without SUSY, in plain SM, the mass of the up quarks can be generated without addition of new scalar fields. This is possible because[5] the doublet $\Phi^c_a \equiv \epsilon_{ab}\Phi^*_b$, where ϵ_{ab} is the antisymmetric tensor, and we have explicitly written the $SU(2)$ indexes, transforms under the fundamental representation of $SU(2)$ (to insure the gauge invariance we had to use the complex conjugate field in this expression, which is impossible according to the SUSY rules). Then, the Yukawa Lagrangian for quarks can be written as

$$\mathcal{L}_{\text{Yukawa}} = -G_D \bar{Q}_L \Phi D_R - G_U \bar{Q}_L \Phi^c U_R, \tag{65}$$

and using the unitary gauge expressions (38) and $\Phi^c = \frac{1}{\sqrt{2}}\binom{v+H(x)}{0}$ we get masses both for up and down quarks, and proportional interactions with the Higgs boson.

One more complication arises when the construction is generalized to several generations of matter. In general, all the Yukawa constants G_e, G_D, and G_U can become arbitrary matrices in flavour. Thus, the mass matrices $m_{u,ij} = G_{u,ij}v/\sqrt{2}$, $m_{d,ij} = G_{d,ij}v/\sqrt{2}$, that are generated after symmetry breaking are not diagonal. They can be diagonalized

[5]Mathematically this corresponds to the observation that fundamental and conjugate representations are isomorphic in case of $SU(2)$.

by separate orthogonal redefinitions of the left and right up and down quarks,

$$u_{L,i} = \mathcal{V}_{u_L,ij}\tilde{u}_{L,j}, \quad u_{R,i} = \mathcal{V}_{u_R,ij}\tilde{u}_{R,j},$$
$$d_{L,i} = \mathcal{V}_{d_L,ij}\tilde{d}_{L,j}, \quad d_{R,i} = \mathcal{V}_{d_R,ij}\tilde{d}_{R,j},$$

where $\mathcal{V}_{\ldots}$ are orthogonal matrices. With this transformation the masses can be made diagonal, the kinetic terms, electromagnetic and neutral currents stay diagonal, but the charged current changes

$$J^+_\mu = \bar{u}^f_L \gamma_\mu V_{fg} d^f_L, \tag{66}$$

where $f, g = 1, 2, 3$ are flavour indexes, and $V_{fg} \equiv \mathcal{V}^{-1}_{u_L}\mathcal{V}_{d_L}$ is the unitary CKM (Cabibbo-Kobayashi-Maskawa) matrix. In particular, it has quite strong mixing between the first two generations for the charged current

$$V_{fg} \simeq \begin{pmatrix} \cos\theta_c & \sin\theta_c & 0 \\ -\sin\theta_c & \cos\theta_c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{67}$$

where the Cabibbo angle $\theta_c \simeq 19°$, and other off-diagonal elements are of the order 10^{-2}.

A general 3×3 unitary matrix is parametrized by 9 real parameters. However, constant phase rotation of quark fields makes 5 of them unobservable, so there are only 4 physical parameters in the CKM matrix—three mixing angles and one phase.

$$V_{fg} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}, \tag{68}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$, and θ_{ij}, $i, j = 1, 2, 3$ are mixing angles, and δ_{13}—CP-violating phase.

4.4 Parameter count

Overall SM has 19 parameters. The electroweak scalar sector is parametrized by 4 parameters—v, λ, g, g' are the parameters in the Lagrangian, which map to equivalent observables with more physical sense

$$M_Z = 91.2 \text{ GeV}, \quad M_H = 125 \text{ GeV}, \tag{69}$$
$$\alpha \equiv \frac{e^2}{4\pi} = \frac{1}{137}, \quad \sin^2\theta_W = 0.231. \tag{70}$$

Note, that other parameters, like M_W can be expressed from these 4. One more parameter $\alpha_S \equiv \frac{g_3^2}{4\pi}$ defines strong interactions.

For the fermions we need 9 masses for 3 charged leptons and 6 quarks. The quark sector has CKM mixing matrix with 3 angles and one CP phase.

There is one more parameter, QCD phase $\theta_{\rm QCD}$, which we left in our discussion. It appears in the action in the form of an additional gauge invariant term

$$\mathcal{L}_{\text{QCD phase}} = \int d^4x \frac{\theta_{\rm QCD}}{16\pi} \epsilon^{\mu\nu\lambda\rho} \operatorname{Tr} G_{\mu\nu} G_{\lambda\rho}.$$

The term is actually a full derivative, so it does not change the equations of motion. However, there are non-perturbative effects where it contributes. Via these effects this term leads to potentially observable CP-violating effects, like neutron dipole moment.

4.5 Full SM after SSB

Let us just summarize the full SM. The whole action is rather short before spontaneous symmetry breaking

$$\begin{aligned}
\mathcal{L} = & -\frac{1}{2}\operatorname{Tr} G_{\mu\nu}G_{\mu\nu} - \frac{1}{2}\operatorname{Tr} W_{\mu\nu}W_{\mu\nu} - \frac{1}{2}\operatorname{Tr} B_{\mu\nu}B_{\mu\nu} \\
& + \sum_{\psi=\chi_L,e_R,Q_L,U_R,D_R} i\bar{\psi}\gamma^\mu D_\mu \psi + \sum_{f,g}\left[-G_e^{fg}\bar{\chi}_L^f \Phi e_R^g - G_D^{fg}\bar{Q}_L^f \Phi D_R^g - G_U^{fg}\bar{Q}_L^f \Phi^c U_R^g\right] \\
& + (D_\mu\Phi)^\dagger D_\mu \Phi - \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2.
\end{aligned}$$

The Lagrangian after the SSB is much longer, and the fact that it originated from a gauge symmetric action is well hidden. However, all the particle interactions can be directly read from it. The SM Lagrangian after symmetry breaking contains the following parts:

$$\mathcal{L} = \mathcal{L}_{\rm QCD} + \mathcal{L}_{\rm lept} + \mathcal{L}_{\rm f,EM} + \mathcal{L}_{\rm f,weak} + \mathcal{L}_Y + \mathcal{L}_V + \mathcal{L}_H + \mathcal{L}_{VH}.$$

The QCD part is

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \sum_{\rm quarks} \bar{q}(i\gamma^\mu\partial_\mu - m_q - g_s\frac{\lambda^a}{2}\gamma^\mu G^a_\mu)q,$$

where we sum over all quarks. The part describing the free leptons is

$$\mathcal{L}_{\rm lept} = \sum_{g=e,\mu,\tau} \bar{e}_g(i\gamma^\mu\partial_\mu - m_{e_g})e_g + \sum_g \bar{\nu}_g i\gamma^\mu\partial_\mu P_L \nu_g,$$

where the sum is over three generations, and only left neutrino components enter (in pure SM neutrino are massless). The electromagnetic part sums over all charged fermions f

$$\mathcal{L}_{\rm f,EM} = eA_\mu \sum_f Q_f \bar{f}\gamma^\mu f,$$

where Q_f is the electric charge of the fermion in units of e. The weak interactions of the leptons are described by

$$\begin{aligned}\mathcal{L}_{\rm weak} =&\frac{g}{2\sqrt{2}}W_\mu \sum_g \bar{\nu}_g\gamma^\mu(1-\gamma^5)e_g + \text{h.c.}\\ &+\frac{g}{2\sqrt{2}}W_\mu \sum_{f,g} \bar{u}_f\gamma^\mu(1-\gamma^5)V_{fg}d_g + \text{h.c.}\\ &+\frac{g}{2\cos\theta_W}Z_\mu \sum_{fermions} \bar{f}\gamma^\mu(T_3^f(1-\gamma^5)-2Q_f\sin^2\theta_W)f,\end{aligned}$$

where T_3^f is $1/2$ for neutrinos and upper quarks, and $-1/2$ for charged leptons and down quarks. The fermion Higgs interactions are proportional to masses

$$\mathcal{L}_Y = \sum_{\text{fermions}} \frac{m_f}{v}\bar{f}f\cdot H.$$

Careful expansion of the kinetic terms of the gauge fields gives

$$\begin{aligned}\mathcal{L}_V =&-\frac{1}{4}(F_{\mu\nu})^2-\frac{1}{4}(Z_{\mu\nu})^2+\frac{m_Z^2}{2}Z_\mu Z_\mu\\ &-\frac{1}{2}W^-_{\mu\nu}W^+_{\mu\nu}+m_W^2W^-_\mu W^+_\mu+\frac{g^2}{4}(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)^2\\ &-\frac{ig}{2}(F^{\mu\nu}\sin\theta_W+Z^{\mu\nu}\cos\theta_W)(W^-_\mu W^+_\nu - W^+_\mu W^-_\nu),\end{aligned}$$

where

$$W^-_{\mu\nu}\equiv(\partial_\mu - ieA_\mu - ig\cos\theta_W Z_\mu)W^-_\nu - (\mu\leftrightarrow\nu).$$

The Higgs is described the scalar theory with cubic quartic interactions

$$\mathcal{L}_H = \frac{1}{2}(\partial_\mu H)^2 - \frac{m_H^2}{2}H^2 - \lambda v H^3 - \frac{\lambda}{4}H^4.$$

Note, that the vev and self-coupling define both mass and cubic interaction terms. Finally, there are interactions between the Higgs and vector bosons

$$\mathcal{L}_{HV} = \frac{g^4}{4}vHW^-_\mu W^+_\mu + \frac{g^2+g'^2}{4}vHZ_\mu Z_\mu + \frac{g^2}{4}H^2|W^-_\mu|^2 + \frac{g^2+g'^2}{8}H^2Z_\mu Z_\mu.$$

4.6 Symmetries of the SM

The main starting point for the formulation of the SM action was the invariance under gauge symmetries. At the same time, SM has more (global) symmetries.

4.6.1 Baryon number

Simultaneous phase rotation of all quarks (and opposite rotation of antiquarks)

$$q \to \mathrm{e}^{i\beta/3} q, \qquad \bar{q} \to \mathrm{e}^{-i\beta/3} \bar{q}$$

leaves all the terms in the action invariant. By Noether theorem this leads to the conserved current

$$j_\mu^B = \frac{1}{3} \sum_q \bar{q} \gamma_\mu q$$

and, respectively, the conserved *baryon number*

$$B = \int d^3\mathbf{x} j_0 = \frac{1}{3}(N_q - N_{\bar{q}}) = N_{\text{baryons}} - N_{\text{anti-baryons}}.$$

The immediate consequence is that the lightest baryon–proton–should be stable. Therefore, evidence of proton decay would show deviations form the SM.

4.6.2 Lepton numbers

In a similar way lepton numbers can be defined. Moreover, in the absence of neutrino masses there are no mixing between the lepton generations, and lepton numbers can be defined individually for each generation:

$$L_e = N_e + N_{\nu_e} - (N_{\bar{e}} + N_{\bar{\nu}_e}), \tag{71}$$

$$L_\mu = N_\mu + N_{\nu_\mu} - (N_{\bar{\mu}} + N_{\bar{\nu}_\mu}), \tag{72}$$

$$L_\tau = N_e + N_{\nu_\tau} - (N_{\bar{\tau}} + N_{\bar{\nu}_\tau}). \tag{73}$$

Similarly, searches for violation of lepton number, like $\mu \to e\gamma$ decays, would indicate deviations from SM.