

Introduction to QED & QCD

Tutorial Questions 2021

1. Suppose we have a plane-wave solution to the Klein-Gordon equation of the form

$$\phi(\mathbf{x}, t) = A e^{-i(\omega t - \mathbf{k} \cdot \mathbf{x})}.$$

Use the Klein-Gordon equation to find the *dispersion relation*, i.e. find ω in terms of \mathbf{k} . How do you interpret the two solutions?

Show that these solutions are eigenstates of the energy operator, $i\partial_t$, and the 3-momentum operator, $-i\nabla$.

2. Show that the Dirac γ -matrices defined in the lectures:

$$\gamma^0 = \beta, \quad \gamma^k = \beta \alpha^k,$$

obey the hermiticity relation

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0.$$

3. When evaluating cross sections, you will frequently need to manipulate Dirac matrices. Using the anti-commutation relations for the γ -matrices, show that in 4 dimensions:

- (i) $\gamma^\mu \gamma_\mu = 4$,
- (ii) $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$,
- (iii) $\gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\mu = 4g^{\nu\lambda}$,
- (iv) $\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho \gamma_\mu = -2\gamma^\rho \gamma^\lambda \gamma^\nu$.

How do these change in arbitrary dimensions where $g^{\mu\nu} g_{\mu\nu} = \delta_\mu^\mu = d$?

4. Verify the orthonormality and completeness relations for the solutions of the Dirac equation:

$$\bar{u}_r(\mathbf{p}) u_s(\mathbf{p}) = -\bar{v}_r(\mathbf{p}) v_s(\mathbf{p}) = 2m \delta^{rs}, \quad \bar{u}_r(\mathbf{p}) v_s(\mathbf{p}) = \bar{v}_r(\mathbf{p}) u_s(\mathbf{p}) = 0,$$

and

$$\sum_{r=1}^2 u_r(\mathbf{p}) \bar{u}_r(\mathbf{p}) = (\not{p} + m), \quad \sum_{r=1}^2 v_r(\mathbf{p}) \bar{v}_r(\mathbf{p}) = (\not{p} - m).$$

5. Show that the Dirac hamiltonian, $H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m$, commutes with the total angular momentum operator

$$[\mathbf{L} + \mathbf{S}, H] = 0,$$

where $\mathbf{L} = \mathbf{x} \times \mathbf{p}$ is the orbital angular momentum and \mathbf{S} is the spin operator

$$\mathbf{S} = \frac{1}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

6. Using the plane-wave solutions of the Dirac equation given in the lectures, show that for $\mathbf{p} = (0, 0, p_z)$

$$S_z u_1 = \frac{1}{2} u_1, \quad S_z u_2 = -\frac{1}{2} u_2, \quad S_z v_1 = \frac{1}{2} v_1 \quad \text{and} \quad S_z v_2 = -\frac{1}{2} v_2,$$

where S_z is the z -component of the spin operator.

7. Draw all the tree-level diagrams for Bhabha-scattering, $e^+(p) e^-(k) \rightarrow e^+(p') e^-(k')$ and give the expression for the scattering amplitude, $i\mathcal{M}$, in Feynman gauge. What happens in an arbitrary gauge?
8. (a) Show that the process $e^+(k') e^-(k) \rightarrow \mu^+(p') \mu^-(p)$, in the limit $m_e \rightarrow 0$, has a matrix-element-squared given by

$$|\overline{\mathcal{M}}|^2 = \frac{1}{4} \frac{e^4}{(k+k')^4} \text{Tr} [k' \gamma^\mu k \gamma^\nu] \text{Tr} [(\not{p} + M) \gamma_\mu (\not{p}' - M) \gamma_\nu],$$

when summed and averaged over final and initial spins, where M is the mass of the muon.

- (b) Show that

$$\left(\frac{d\sigma}{d\Omega} \right)_{e^+e^- \rightarrow \mu^+\mu^-} = \frac{|\overline{\mathcal{M}}|^2}{64\pi^2 s} \sqrt{1 - \frac{4M^2}{s}},$$

where $s = (k+k')^2$.

(c) The traces evaluate to (check if you have time!)

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{s^2} [(pk)^2 + (pk')^2 + M^2(kk')].$$

Move to the centre-of-mass frame and let the scattering angle be θ . Show that

$$\left(\frac{d\sigma}{d\Omega}\right)_{e^+e^- \rightarrow \mu^+\mu^-} = \frac{e^4}{64\pi^2 s} \sqrt{1 - \frac{4M^2}{s}} \left[1 + \left(1 - \frac{4M^2}{s}\right) \cos^2 \theta + \frac{4M^2}{s}\right].$$

(d) Find an expression for the total cross section in the high-energy limit where the mass of the muon can be neglected.

9. Write the amplitude for Compton scattering $e(p)\gamma(k) \rightarrow e(p')\gamma(k')$ in the form $i\mathcal{M} = M_{\mu\nu}\varepsilon^{*\mu}(k')\varepsilon^\nu(k)$. Verify that this is gauge-invariant.

10. In the lectures, we found the matrix element squared for unpolarised Compton scattering was

$$\overline{|\mathcal{M}|^2} = 2e^4 \left(\frac{pk}{pk'} + \frac{pk'}{pk} + 2m^2 \left(\frac{1}{pk} - \frac{1}{pk'} \right) + m^4 \left(\frac{1}{pk} - \frac{1}{pk'} \right)^2 \right).$$

Working in the centre-of-mass system, in the limit where the electron mass m can be neglected, show that the matrix element squared is dominated by backward scattering, $\theta \simeq \pi$, where θ is the scattering angle of the photon.

11. Use the matrix-element squared for Compton scattering to obtain the matrix-element squared for the annihilation process $e^+e^- \rightarrow \gamma\gamma$. Again work in the centre-of-mass frame and show that, in the high-energy limit $E \gg m$,

$$\overline{|\mathcal{M}|^2} \simeq 4e^4 \frac{1 + \cos^2 \theta}{\sin^2 \theta}.$$

12. Consider the diagrams in figure 1. Show that the colour factors are given by

$$(a) t^c t^a t^b \delta^{bc} = -\frac{1}{2N_c} t^a, \quad \text{and} \quad (b) i f^{abc} t^b t^c = -\frac{1}{2} C_A t^a$$

respectively.

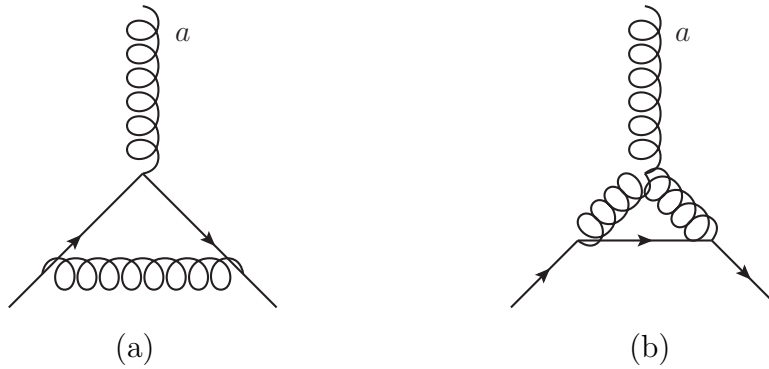


Figure 1: One-loop corrections to a $q\bar{q}g$ -vertex.

13. Calculate the summed and averaged matrix-element squared, $|\overline{\mathcal{M}}|^2$, for the quark-scattering process $ud \rightarrow ud$.
14. Solve the one-loop β -functions for QCD and QED:

$$\mu^2 \frac{d\alpha_s}{d\mu^2} = -\frac{11C_A - 2n_f}{12\pi} \alpha_s^2, \quad \text{and} \quad \mu^2 \frac{d\alpha}{d\mu^2} = \frac{1}{3\pi} \alpha^2,$$

using as initial condition the value of the couplings at the Z mass. Sketch the solutions as a function of μ^2 .