1

Do you think that by modifying Newtonian dynamics to something like General Relativity one could explain the anomalies in the orbit of Uranus in a viable way? (without introducing Neptune)

2

A very important result in dynamical systems is virial theorem. Can you reproduce it for Newtonian dynamics? (Show that the time average satisfies $2\langle T\rangle = -\langle V\rangle$ where $T = \sum m_i v_i^2/2$ is the kinetic energy of a collection of particles, $V = \sum_{i<j} V_{ij}(r) = \sum_{i<j} G m_j m_j / r_{ij}$ the potential energy, $\frac{1}{\tau}\int_0^\tau dt ... = \langle ... \rangle$ and we take the limit of large τ)

Hint: Start with the quantity

$$\mathcal{D} = \sum_i \vec{p}_i \cdot \vec{r}_i, \tag{1}$$

where we are summing over number of particles. Take the time derivative, and average over time. Assume that $\mathcal{D}$ does not grow with time in the situation of equilibrium

3

We know from the lectures that DM is almost collisionless. Can you estimate a bound on the cross-section by assuming that the typical clusters do not interact when they collide? (assume the energy density of DM is $\sim \mathrm{GeV/cm^3}$ and recall that the typical size a cluster is $\sim$ few Mpc. Similarly, you can assume that the typical time between collisions should be larger than the crossing time of clusters. Assume this time to be $1\ Gyr$. You can leave the estimate in terms of the velocity in this second case).

Hint: The mean free path in a medium of n number density and given a cross section σ is

$$l_{mfp} \sim 1/(n\sigma) \tag{2}$$

while the typical time between collisions is (we don't use it)

$$l_{mfp}/v. \tag{3}$$

4

You can become an cosmologist for one day. Go to the webpage `https://lambda.gsfc.nasa.gov/toolbox/tb_camb_form.cfm`**. In the first page you can choose different values for $\Omega_b h^2$. Check how if one increases this value (compensate it by reducing the value of $\Omega_c h^2$, which is the value of the DM component such that $\Omega_b + \Omega_c$ is the same as before. If you click the 'Transfer Functions' box you also get the power spectrum. Plot the C_l (data from 'camb_xxxxx_scalcls.dat' shown as LinLog) and compare by eye with** `https://wiki.cosmos.esa.int/planckpla2015/index.php/File:A15_TT.png`**. If you have asked for 'Transfer functions' you can also loglog plot the power spectrum file 'camb_xxxxx_matterpower_z0.dat' and see how it changes.**

5

Find the minimum value of dark matter mass allowed by quantum mechanics for bosonic and fermionic candidates. You need to fit the DM candidate to dwarf spheroidals ($r \sim$kpc, typical velocity $\sim 10^{-4}c$ and mean density $\sim 5\,\mathrm{GeV/cm^3}$)

Hint: As we discussed during the lectures, the idea for baryons is that the uncertainty principle tells us the de Broglie wavelength allowed given a typical momentum (more precisely, the uncertainty in momentum, which shouldn't exceed the momentum that is required for these structures to be bounded)

Hint: For fermions, the idea is that you need to fit the fermions in the free states that live in a halo of certain maximum size and maximum momentum. Assuming a box in phase space of size kpc and $10^{-4}m$, compute the number of degrees of freedom available, and fill them up to accommodate all the mass of the galaxy.

6

What's H_0 in years? $H_0 \sim 0.7km/s/Mpc$

7

Compute the yield for a relativistic and non relativistic species. Estimate the yield that we need in order to reproduce the correct DM relic abundance $\Omega h^2 \approx 0.1$

8

Show that in terms of Y, the equation of evolution reads

$$\frac{dY}{dt} = -s\langle\sigma v\rangle\left(Y^2 - Y_{eq}^2\right). \tag{4}$$

9

Using Boltzmann equation, expressed in terms of the yield $Y = n/s$, which reads

$$\frac{dY}{dx} = -\frac{\lambda\langle\sigma v\rangle}{x^2}\left(Y^2 - Y_{eq}^2\right), \tag{5}$$

define the quantity $\Delta Y = Y - Y_{eq}$ and show that, for non-relativistic particles, the solution can be approximated as (x_f is the time of freezeout at which $\Gamma \sim H$)

$$\begin{aligned} \Delta_Y &= -\frac{\frac{dY_{eq}}{dx}}{Y_{eq}}\frac{x^2}{2\lambda\langle\sigma v\rangle}, \quad 1 < x \ll x_f \\ \Delta_{Y_\infty} = Y_\infty &= \frac{x_f}{\lambda\left(a + \frac{b}{2x_f^2}\right)}, \quad x \gg x_f \text{ (when } Y \gg Y_{eq}) \end{aligned} \tag{6}$$

For the second part assume that the thermally averaged annihilation cross section can be expanded in powers of $1/x$ as $\langle\sigma v\rangle = a + \frac{b}{x}$,

Hint: For early times, $1 < x \ll x_f$, the yield follows closely its equilibrium, $Y \approx Y_{eq}$ and we can assume that $d\Delta Y/dx = 0$, and just follow the algebra

Hint: For late times, $x \gg x_f$, we can assume that $Y \gg Y_{eq}$, and thus $\Delta Y_\infty \approx Y_\infty$. You need to integrate from x_f to x_∞. You can neglect the Y_f int he final formula

10

IIn the Early Universe, neutrinos remain in equilibrium through the process $e^+ + e^- \longleftrightarrow \nu_e + \bar{\nu}_e$. Using that both the electron-positron and neutrino populations are relativistic and therefore their number density scales as $n \sim T^3$, the decoupling temperature of neutrinos can be roughly estimated by equating the annihilation rate $\Gamma = n\langle\sigma v\rangle$ and the Hubble expansion rate $H = \sqrt{8\pi G\rho/3}$. The energy density of the Universe scales as $\rho \sim T$. Show that neutrinos decouple at approximately $T \sim 1$ MeV.

Hint: Neutrinos keep in thermal equilibrium through interactions with electrons through the processes $e^- + e^+ \longleftrightarrow \nu_e + \bar{\nu}_e$ and $e^- + \nu_e \longleftrightarrow e^- + \nu_e$. When neutrinos decouple there are in the thermal bath electrons, positrons, photons and the three neutrinos and antineutrinos, $g_* = 10.75$ (check it).

Using dimensional arguments, the cross section of these processes at a temperature T (which defines the c.o.m. energy) is approximately $\sigma = G_F^2 T^2$, where $G_F = 1.17 \times 10^{-5}$ GeV .

11

From the question above, we know that when neutrinos decouple, they are still relativistic. The other relativistic species in the thermal bath are electrons, positrons, photons and the three neutrinos and antineutrinos. With this information the relic density of neutrinos in the Universe today can be estimated as a function of the neutrino mass. .

Hint: To do that, follow eq (56) of the notes, and substitute g_{eff} by the corresponding value for two helicities.

12

What's the relic density of a species of mass m that is kept in equilibrium with SM particles through $3_{DM} \to 2_{SM}$ processes assuming it decouples at $T \sim m$? (follow the same steps as 3.3.2 of the notes). Which value of the mass generates the observed DM abundance? (assume $\langle \sigma v \rangle \sim \alpha^3/m^5$, where α is a dimensionless coupling)

Hint: The rate of interaction is should now be proportional to the flux squared $\sim n^2$. So we expect

$$\Gamma \sim n^2 \langle \sigma v \rangle \sim m^2/M_{Pl} \tag{7}$$

13

Direct detection. What is the minimum velocity needed v_m for a WIMP with mass m_χ to produce a 10 keV recoil in a nucleus of mass m_N Which cross-sections will generate one event/day in a 1 T detector of targets of 100 GeV?

14

Boost the DM distribution to the Solar System frame and compute $\bar{v}$

15

Consider two massive bosonic fields coupled with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 - m_1^2 \phi_1^2 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - m_2^2 \phi_1^2 + g\phi_1^2 \phi_2 \tag{8}$$

Assume that ϕ has a background value $\bar{\phi}_1$. Show that the fluctuations over this background satisfy (in Fourier space)

$$(\omega^2 - k^2 - m_1^2)\delta\phi_1 + g\bar{\phi}_1 \delta\phi_2 = 0, \qquad (\omega^2 - k^2 - m_2^2)\delta\phi_2 + g\bar{\phi}_1 \delta\phi_1 = 0. \tag{9}$$

If the system starts with initial conditions $\delta\phi_1 = \phi_0$ and $\delta\dot{\phi}_1 = \delta\dot{\phi}_2 = \delta\phi_2 = 0$, compute the value of $\delta\phi_2$ as the wave propagates in the limit where $m_1 = m_2$ (you can do it in the limit of small g).

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