1

Do you think that by modifying Newtonian dynamics to something like General Relativity one could explain the anomalies in the orbit of Uranus in a viable way? (without introducing Neptune)

As seen in the lectures, the modification related to GR is valid when the gravitational potential starts to be large. The gravitational potential can be approximated to

$$\phi = \frac{G_N M_{sun}}{r} \approx \frac{3 \text{ km}}{r} \tag{1}$$

Orbital radius of Uranus $\approx 10^9$ km. This is a tiny modification, much smaller than for inner planets: GR is different from Newtonian dynamics at distances closer to the Sun. It can't explain anything in Neptune without spoiling the rest of the Solar System.

2

A very important result in dynamical systems is virial theorem. Can you reproduce it for Newtonian dynamics? (Show that the time average satisfies $2\langle T\rangle = -\langle V\rangle$ where $T = \sum m_i v_i^2/2$ is the kinetic energy of a collection of particles, $V = \sum_{i<j} V_{ij}(r) = \sum_{i<j} G m_j m_j / r_{ij}$ the potential energy, $\frac{1}{\tau}\int_0^\tau dt ... = \langle ...\rangle$ and we take the limit of large τ)

Hint: Start with the quantity

$$\mathcal{D} = \sum_i \vec{p}_i \cdot \vec{r}_i, \tag{2}$$

where we are summing over number of particles. Take the time derivative, and average over time. Assume that $\mathcal{D}$ does not grow with time in the situation of equilibrium

$$\frac{d\mathcal{D}}{dt} = \sum_i \vec{F}_i \cdot \vec{r}_i + 2T. \tag{3}$$

If we average the previous quantity over time τ,

$$\frac{1}{\tau}\int_0^\tau dt ... = \langle ...\rangle, \tag{4}$$

we find

$$\frac{\mathcal{D}(\tau) - \mathcal{D}(0)}{\tau} = 2\langle T\rangle + \langle \vec{F}\vec{r}\rangle. \tag{5}$$

The first object cancels at long times if the situation is almost in equilibrium (in other words, as long as $D(\tau)$ is not growing with time. Which should be the case in a situation in equilibrium). The last term,

$$\sum_{i\neq j} -\vec{\partial} V_{ij} \vec{r}_i = \sum_{i<j} -\vec{\partial} V_{ij} (\vec{r}_i - \vec{r}_j) = \sum_{i<j} V_{ij} \qquad (6)$$

where we used that $V_{ij}(r) = G m_j m_j / r_{ij}$.

3

We know from the lectures that DM is almost collisionless. Can you estimate a bound on the cross-section by assuming that the typical clusters do not interact when they collide? (assume the energy density of DM is $\sim \mathrm{GeV/cm^3}$ and recall that the typical size a cluster is $\sim$ few Mpc. Similarly, you can assume that the typical time between collisions should be larger than the crossing time of clusters. Assume this time to be $1\ Gyr$. You can leave the estimate in terms of the velocity in this second case).

Hint: The mean free path in a medium of n number density and given a cross section σ is

$$l_{mfp} \sim 1/(n\sigma) \qquad (7)$$

while the typical time between collisions is (we don't use it)

$$l_{mfp}/v. \qquad (8)$$

Assume that

$$l_{mfp} > \#Mpc. \qquad (9)$$

This means

$$\sigma/m < cm^3/(GeV)/Mpc = 1/(10^{-24} gr) 10^{-24} cm^2 \approx cm^2/gr \qquad (10)$$

4

You can become an cosmologist for one day. Go to the webpage `https://lambda.gsfc.nasa.gov/toolbox/tb_camb_form.cfm`. In the first page you can choose different values for $\Omega_b h^2$. Check how if one increases this value (compensate it by reducing the value of $\Omega_c h^2$, which is the value of the DM component such that $\Omega_b + \Omega_c$ is the same as before. If you click the 'Transfer Functions' box you also get the power spectrum. Plot the C_l (data from 'camb_xxxxx_scalcls.dat' shown as LinLog) and compare by eye with `https:`

//wiki.cosmos.esa.int/planckpla2015/index.php/File:A15_TT.png. **If you have asked for 'Transfer functions' you can also loglog plot the power spectrum file 'camb_xxxxx_matterpower_z0.dat' and see how it changes.**

5

Find the minimum value of dark matter mass allowed by quantum mechanics for bosonic and fermionic candidates. You need to fit the DM candidate to dwarf spheroidals ($r \sim$kpc, typical velocity $\sim 10^{-4}c$ and mean density $\sim 5\,\mathrm{GeV/cm^3}$)

Hint: As we discussed during the lectures, the idea for baryons is that the uncertainty principle tells us the de Broglie wavelength allowed given a typical momentum (more precisely, the uncertainty in momentum, which shouldn't exceed the momentum that is required for these structures to be bounded)

$$\Delta x \Delta p \geq 2\pi\hbar \tag{11}$$

The uncertainty in position is kpc, thus

$$kpc \cdot m 10^{-4} \geq 2\pi, \tag{12}$$

meaning

$$m \gtrsim 10^{-22}\,\mathrm{eV}. \tag{13}$$

Hint: For fermions, the idea is that you need to fit the fermions in the free states that live in a halo of certain maximum size and maximum momentum. Assuming a box in phase space of size kpc and $10^{-4}m$, compute the number of degrees of freedom available, and fill them up to accommodate all the mass of the galaxy.

$$N = \frac{kpc^3 (10^{-4}m)^3}{(\Delta x \Delta p)^3} \sim 10^{64} \left(\frac{m}{eV}\right)^3. \tag{14}$$

These states should fill up the total mass, so

$$\rho (kpc)^3 \sim M_{galaxy} = Nm \tag{15}$$

I find

$$m \gtrsim 300\ eV. \tag{16}$$

6

What's H_0 in years? $H_0 \sim 0.7 km/s/Mpc$

The result is ~ 14 Gyr.

7

Compute the yield for a relativistic and non relativistic species. Estimate the yield that we need in order to reproduce the correct DM relic abundance $\Omega h^2 \approx 0.1$

$$s = \frac{2\pi^2}{45} g_{*s} T^3 \tag{17}$$

Also , for relativistic particles

$$n_\gamma = \zeta(3) \frac{g_{eff}}{\pi^2} T^3. \tag{18}$$

Hence

$$Y_{eq} = \frac{45}{2\pi^4} \zeta(3) \frac{g_{eff}}{g_{*s}} \approx 0.278 \frac{g_{eff}}{g_{*s}} \tag{19}$$

For NR,

$$n = g_{eff} \left(\frac{mT}{2\pi} \right)^{3/2} e^{-m/T} \tag{20}$$

and hence

$$Y_{eq} = \frac{45}{2\pi^4} \left(\frac{\pi}{8} \right)^{1/2} \frac{g_{eff}}{g_{*s}} \left(\frac{m}{T} \right)^{3/2} e^{-m/T}. \tag{21}$$

Then

$$\Omega h^2 = \frac{\rho_\chi}{\rho_c} h^2 = \frac{m_\chi n \chi h^2}{\rho_c} = \frac{m_\chi Y_\infty s_0 h^2}{\rho_c} \tag{22}$$

The yield stays constant since freeze-out,

$$\Omega h^2 = \frac{m_\chi Y_f s_0 h^2}{\rho_c} \tag{23}$$

One gets

$$Y_f \approx 3.55 \times 10^{-10} \left(\frac{1\text{GeV}}{m_\chi} \right). \tag{24}$$

8

Show that in terms of Y, the equation of evolution reads

$$\frac{dY}{dt} = -s\langle \sigma v \rangle \left(Y^2 - Y_{eq}^2 \right). \tag{25}$$

$$\frac{dY}{dt} = \frac{d}{dt}\left(\frac{n}{s}\right) = \frac{d}{dt}\left(\frac{a^3 n}{a^3 s}\right) = \frac{1}{a^3 s}\left(3a^2 \dot{a} n + a^3 \frac{dn}{dt}\right) = \frac{1}{s}\left(3Hn + \frac{dn}{dt}\right) \tag{26}$$

Here we have used that the expansion of the Universe is iso-entropic and thus $a^3 s$ remains constant. Also we use the definition of the Hubble parameter. Using (37) of the notes we get the final result.

9

Using Boltzmann equation, expressed in terms of the yield $Y = n/s$, which reads

$$\frac{dY}{dx} = -\frac{\lambda\langle\sigma v\rangle}{x^2}\left(Y^2 - Y_{eq}^2\right), \tag{27}$$

define the quantity $\Delta Y = Y - Y_{eq}$ and show that, for non-relativistic particles, the solution can be approximated as (x_f is the time of freezeout at which $\Gamma \sim H$)

$$\begin{aligned} \Delta_Y &= -\frac{\frac{dY_{eq}}{dx}}{Y_{eq}}\frac{x^2}{2\lambda\langle\sigma v\rangle}, \quad 1 < x \ll x_f \\ \Delta_{Y_\infty} = Y_\infty &= \frac{x_f}{\lambda\left(a + \frac{b}{2x_f^2}\right)}, \quad x \gg x_f \text{ (when } Y \gg Y_{eq}) \end{aligned} \tag{28}$$

For the second part assume that the thermally averaged annihilation cross section can be expanded in powers of $1/x$ as $\langle\sigma v\rangle = a + \frac{b}{x}$,

Hint: For early times, $1 < x \ll x_f$, the yield follows closely its equilibrium, $Y \approx Y_{eq}$ and we can assume that $d\Delta Y/dx = 0$, and just follow the algebra

We find

$$\begin{aligned} \Delta_Y &= -\frac{\frac{dY_{eq}}{dx}}{Y_{eq}}\frac{x^2}{2\lambda\langle\sigma v\rangle} \\ \Delta_{Y_f} &\approx \frac{x_f^2}{2\lambda\langle\sigma v\rangle} \end{aligned} \tag{29}$$

where in the last line we have used that for large enough x, using

$$Y_{eq} = \frac{45}{2\pi^4}\left(\frac{\pi}{8}\right)^{1/2}\frac{g_{eff}}{g_{*s}}\left(\frac{m}{T}\right)^{3/2}e^{-m/T} \tag{30}$$

implies $dY_{eq}/dx \approx -Y_{eq}$ at leading order in x.

Hint: For late times, $x \gg x_f$, we can assume that $Y \gg Y_{eq}$, and thus $\Delta Y_\infty \approx Y_\infty$. You need to integrate from x_f to x_∞. You can neglect the Y_f int he final formula

This leads to the following expression,

$$\frac{d\Delta_Y}{dx} \approx -\frac{\lambda\langle\sigma v\rangle}{x^2}\Delta_Y^2 \tag{31}$$

This is a separable equation that we integrate from the freeze-out time up to nowadays. In doing so, it is customary to expand the thermally averaged annihilation cross section in powers of $1/x$ as $\langle\sigma v\rangle = a + \frac{b}{x}$,

$$\int_{\Delta_{Y_f}}^{\Delta_{Y\infty}} \frac{d\Delta_Y}{\Delta_Y^2} = -\int_{x_f}^{x_\infty} \frac{\lambda\langle\sigma v\rangle}{x^2} dx \tag{32}$$

Taking into account that $x_\infty \gg x_f$, this leads to

$$\frac{1}{\Delta_{Y_\infty}} = \frac{1}{\Delta_{Y_f}} + \frac{\lambda}{x_f}\left(a + \frac{b}{2x_f}\right) \tag{33}$$

The term $1/\Delta Y_f$ is generally ignored (if we are only aiming at a precision up to a few per cent) . We can check that this is a good approximation using the previously derived (29) for $x_f \approx 20$ (which, is the value for which the equilibrium Yield has the right value). This leads to

$$\Delta_{Y_\infty} = Y_\infty = \frac{x_f}{\lambda\left(a + \frac{b}{2x_f}\right)} \tag{34}$$

The relic density can now be expressed in terms of this result as follows

$$\begin{aligned} \Omega h^2 &= \frac{m_\chi Y_\infty s_0 h^2}{\rho_c} \\ &\approx \frac{10^{-10}\mathrm{GeV}^{-2}}{a + \frac{b}{40}} \\ &\approx \frac{3\times 10^{-27}\mathrm{cm}^3 s^{-1}}{a + \frac{b}{40}} \end{aligned} \tag{35}$$

This expression explicitly shows that for larger values of the annihilation cross section, smaller values of the relic density are obtained.

10

IIn the Early Universe, neutrinos remain in equilibrium through the process $e^+ + e^- \longleftrightarrow \nu_e + \bar{\nu}_e$. Using that both the electron-positron and neutrino populations are relativistic and therefore their number density scales as $n \sim T^3$, the decoupling temperature of neutrinos can be roughly estimated by equating the annihilation rate $\Gamma = n\langle\sigma v\rangle$ and the Hubble expansion rate $H = \sqrt{8\pi G\rho/3}$. The energy density of the Universe scales as $\rho \sim T$. Show that neutrinos decouple at approximately $T \sim 1$ MeV.

Hint: Neutrinos keep in thermal equilibrium through interactions with electrons through the processes $e^- + e^+ \longleftrightarrow \nu_e + \bar{\nu}_e$ and $e^- + \nu_e \longleftrightarrow e^- + \nu_e$. When neutrinos decouple *there are in the thermal bath electrons, positrons, photons and the three neutrinos and antineutrinos,* $g_* = 10.75$ *(check it).*

Using dimensional arguments, the cross section of these processes at a temperature T *(which defines the c.o.m. energy) is approximately* $\sigma = G_F^2 T^2$*, where* $G_F = 1.17 \times 10^{-5}$ *GeV .*

Given that both neutrinos and electrons are fermions, their number density can be written as

$$n_{e,\nu_e} = \frac{g_{eff}}{\pi^2}\zeta(3)T^3 \approx 0.1 g_{e,\nu_e} T^3 \tag{36}$$

where $g_e = 2$ and $g_\nu = 1$. The interaction rate of these processes therefore reads

$$\Gamma = n_e \langle \sigma v \rangle \approx 0.1\left(g_e + g_{\nu_e}\right) T^3 G_F^2 T^2 \approx 0.3 G_F^2 T^5 \tag{37}$$

where we have considered that both species are relativistic and therefore $v \sim c = 1$. For a radiation dominated Universe the Hubble parameter reads

$$H = \frac{\pi}{\sqrt{90}} g_*^{1/2} \frac{T^2}{M_P} \tag{38}$$

In order to see when neutrinos decouple, we need to compare their annihilation rate with the Hubble parameter

$$\frac{\Gamma}{H} = M_P \frac{0.3 G_F^2 T^5}{0.3 g_*^{1/2} T^2} = \frac{M_P G_F^2 T^3}{g_*^{1/2}} \sim \left(\frac{T}{2\text{MeV}}\right)^3 \tag{39}$$

the last expression we have used that the number of relativistic degrees of freedom when neutrinos decouple is $g_* = 10.75$. This approximation suggests that neutrinos decouple from the thermal bath at $T \sim 2$ MeV . A full numerical solution of Boltzmann equation yields $T \sim 1$ MeV, so this is still a good approximation.

11

From the question above, we know that when neutrinos decouple, they are still relativistic. The other relativistic species in the thermal bath are electrons, positrons, photons and the three neutrinos and antineutrinos. With this information the relic density of neutrinos in the Universe today can be estimated as a function of the neutrino mass. .

Hint: To do that, follow eq (56) of the notes, and substitute g_{eff} *by the corresponding value for two helicities.*

Since neutrinos decouple while they are still relativistic, their yield reads

$$Y_{eq} \approx 0.278 \frac{g_{eff}}{g_{*s}} \tag{40}$$

Neutrinos decouple at a few MeV, when the species that were still relativistic are $e^{\pm}, \gamma, \nu$ and $\bar{\nu}$. Thus, the number of relativistic degrees of freedom is$g_* = g_{*s} = 10.75$.. For one neutrino family, the effective number of degrees of freedom is $g_{eff} = 3g/4 = 3/2$. Using these values, the relic density today an be written as

$$\begin{aligned}\Omega h^2 &= \frac{\sum_i m_{\nu_i} Y_\infty s_0 h^2}{\rho_c} \\ &\approx \frac{\sum_i m_{\nu_i}}{91\text{eV}} h^2\end{aligned} \tag{41}$$

Notice that in order for neutrinos to be the bulk of dark matter, we would need $m \approx 9$ eV, which is much bigger than current upper limits (for example, obtained from cosmological observations). Notice, indeed, that if we consider the current bound $\sum m_{\nu_i} \leq 0.3$ eV we can quantify the contribution of neutrinos to the total amount of dark matter, resulting in $\Omega h^2 \leq 0.003$. This is less than a 3% of the total dark matter density.

12

What's the relic density of a species of mass m that is kept in equilibrium with SM particles through $3_{DM} \to 2_{SM}$ processes assuming it decouples at $T \sim m$? (follow the same steps as 3.3.2 of the notes). Which value of the mass generates the observed DM abundance? (assume $\langle \sigma v \rangle \sim \alpha^3/m^5$, where α is a dimensionless coupling)

Hint: The rate of interaction is should now be proportional to the flux squared $\sim n^2$. So we expect

$$\Gamma \sim n^2 \langle \sigma v \rangle \sim m^2/M_{Pl} \tag{42}$$

The 'cross-section' has now different dimensions, since the total number of events is proportional to the flux squared. A reference where things are done in detail is `https://arxiv.org/pdf/1706.05381.pdf` (see also `https://arxiv.org/abs/1411.3727`). For the moment we keep it as it is. Then

$$\rho_f = mn \sim \frac{m^2}{(M_{Pl}\langle \sigma v \rangle)^{1/2}} \tag{43}$$

After freeze-out, this evolves until the moment of matter-radiation equality, that we use to fix the abundance:

$$\rho_{RME} = \rho_f \left(\frac{T_{RME}}{T_f}\right)^3 = \rho_f \left(\frac{T_{RME}}{m}\right)^3 \sim T_{RME}^4. \tag{44}$$

So,

$$\frac{1}{m^2 M_{Pl} T^2} \sim \langle \sigma v \rangle \tag{45}$$

So, dimensionally, $\langle \sigma v \rangle \sim \alpha^3/m^5$. I introduced the factor α^3 assuming that the process requires three vertices, but this is not important, as long as it is below 1. So the final result is

$$m \sim \alpha (T^2 M_{Pl})^{1/3} \sim 100\alpha \text{ keV} \tag{46}$$

13

Direct detection. What is the minimum velocity needed v_m for a WIMP with mass m_χ to produce a 10 keV recoil in a nucleus of mass m_N Which cross-sections will generate one event/day in a 1 T detector of targets of 100 GeV?

One can simply use the possibility where all the kinetic energy is absorbed by the recoil $10\text{keV} = E_{k,DM} = m\gamma - m$. In the NR limit, $E_{k,DM} = \frac{1}{2}mv^2$, from where you find v.

For the number of events, see (95) of notes

$$N = nv\sigma t N_T = 10v \frac{\sigma}{(300)^4 GeV^2} \frac{GeV}{m} \text{ events/day.} \tag{47}$$

14

Boost the DM distribution to the Solar System frame and compute $\bar{v}$

In the rest frame of the DM, the distribution is given by

$$f_v = \frac{1}{N_{esc}} \left(\frac{3}{2\pi\sigma_0^2} \right)^{3/2} e^{-\frac{3}{2}\frac{v^2}{2\sigma_0^2}}. \tag{48}$$

The Solar system moves with velocity $\vec{v}_\odot$ with respect to this frame, which implies that for an observer at the Sun, $\vec{v} = \vec{v}_{DM} + \vec{v}_\odot$. The average velocity of a DM particle in this frame is hence

$$\int d^3v \vec{v} f(\vec{v}_{DM} + \vec{v}_\odot) = \int d^3\tilde{v} (\vec{\tilde{v}} - \vec{v}_\odot) f(\vec{\tilde{v}}) = -\vec{v}_\odot \tag{49}$$

where I changed the variable of intergration. Here I'm ignoring the limits related to the escape velocity.

15

Consider two massive bosonic fields coupled with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 - m_1^2 \phi_1^2 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - m_2^2 \phi_1^2 + g\phi_1^2 \phi_2 \tag{50}$$

Assume that ϕ has a background value $\bar{\phi}_1$. Show that the fluctuations over this background satisfy (in Fourier space)

$$(\omega^2 - k^2 - m_1^2)\delta\phi_1 + g\bar{\phi}_1 \delta\phi_2 = 0, \qquad (\omega^2 - k^2 - m_2^2)\delta\phi_2 + g\bar{\phi}_1 \delta\phi_1 = 0. \tag{51}$$

If the system starts with initial conditions $\delta\phi_1 = \phi_0$ and $\delta\dot{\phi}_1 = \delta\dot{\phi}_2 = \delta\phi_2 = 0$, compute the value of $\delta\phi_2$ as the wave propagates in the limit where $m_1 = m_2$ (you can do it in the limit of small g).

The equations of motion follow from expanding $\phi_1 = \bar{\phi} + \delta\phi_1$, and retaining the quadratic order in perturbation in the Lagrangian. Then one finds

$$(-\Box - m_1^2)\delta\phi_1 + g\bar{\phi}_1 \delta\phi_2 = 0, \qquad (-\Box - m_2^2)\delta\phi_2 + g\bar{\phi}_1 \delta\phi_1 = 0. \tag{52}$$

The idea is to find the modes that propagate without mixing. In the case of equal masses, these are simply

$$\phi_+ = \delta\phi_1 + \delta\phi_2, \text{ and } \phi_- = \delta\phi_1 - \delta\phi_2. \tag{53}$$

They propagate as

$$(-\Box - m^2)\phi_\pm \pm g\bar{\phi}_1 \phi_\pm = 0. \tag{54}$$

Now there are different was to proceed. One of them is to use the Green function for the previous linear equation. Let's do something a bit less sophisticated. The modes propagate with dispersion relation

$$\omega^2 = k^2 + m^2 \mp g\bar{\phi}. \tag{55}$$

The group velocity is (in the limit of small mass and coupling)

$$v_g = \frac{d\omega}{dk} = \frac{k}{(k^2 + m^2 \mp g\bar{\phi})^{1/2}} = 1 - \frac{1}{2}\frac{m^2 \mp g\bar{\phi}}{k^2} \tag{56}$$

One mode is a bit faster than the other, the $-$ mode. So we focus on this one. This mode starts with initial conditions

$$\phi_- = \phi_0. \tag{57}$$

It then propagates with velocity v_g. Since

$$\phi_2 = \frac{1}{2}(\phi_+ - \phi_-) \tag{58}$$

One sees that ϕ_2 is generated as the beam propagates. This result is valid at leading order in g, so it is not valid for oscillations.