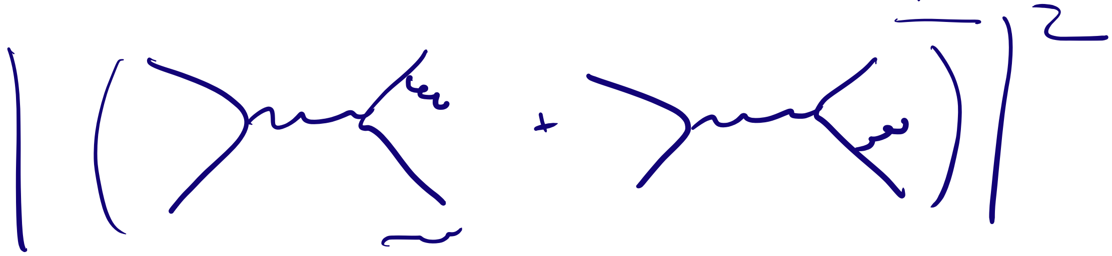
Lecture 3 $e^+e^- \to q\bar{q}g$ (again)

We have: $\sigma^{\text{real}}_{\text{NLO}} = \frac{1}{2 s_{ab}} \int dPS_3 \langle |M_{q\bar{q}g}|^2 \rangle$



$= L_{\mu\nu} H^{\mu\nu}_{\text{NLO, real}}$

Colour: $= N_c C_F$

Not going to calculate everything: give some steps.

After some work, we get:

$$\langle |M_{q\bar{q}g}|^2 \rangle = 4 e^2 f_q^2 g_s^2 N_c C_F \cdot \left(\frac{s_{a1}^2 + s_{a2}^2 + s_{b1}^2 + s_{b2}^2}{s_{ab}\, s_{13}\, s_{23}} \right)$$

w/ $s_{ij} \equiv (p_i + p_j)^2 = 2(p_i \cdot p_j)$

& $e^+(p_a) + e^-(p_b) \to q(p_1) + \bar{q}(p_2) + g(p_3)$

Often use $s_{ab} \leftrightarrow s$ interchangeably

Next step, cross section:

$$\int dPS_3 = \frac{1}{(2\pi)^5} \int d^4p_1\, d^4p_2\, d^4p_3$$

$$\delta(p_1^2)\, \delta(p_2^2)\, \delta(p_3^2)\; \delta^{(4)}(p_a+p_b-p_1-p_2-p_3)$$

Steps: 1) Eliminate d^4p_3 w/ $\delta^{(4)}(\ldots)$

2) $d^4p_1\, \delta(p_1^2) = \frac{d^3\vec{p}_1}{2E_1} = \frac{E_1\, dE_1}{2}\, d\cos\theta_1\, d\phi_1$

& sim for p_2

3) Define energy fraction, via:

$$E_i \equiv x_i \frac{\sqrt{s}}{2} \qquad \left(E_{a,b} = \frac{\sqrt{s}}{2}\right)$$

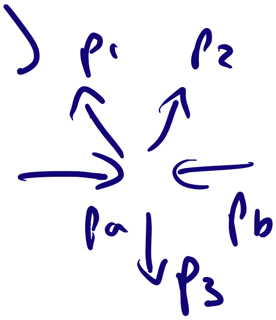
$$\Rightarrow dE_i \; (d|\vec{p}_i|) = \frac{\sqrt{s}}{2}\, \underline{\underline{dx_i}} \leftarrow$$

4) Eliminate / perform angular integrals.

Finally, recall denominator of $|M|^2$

is $\underline{s\, s_{13}\, s_{23}}$

e.g. $s_{13} = 2E_1E_3 - 2(\vec{p}_1\cdot\vec{p}_3)$



But: $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$

$$\Rightarrow \vec{p}_2 = -\vec{p}_1 - \vec{p}_3$$

$$\vec{p}_2^{\,2} = E_2^2 = E_1^2 + E_3^2 + 2(\vec{p}_1\cdot\vec{p}_3)$$

Dust settles: $s_{13} = s(1-x_2)$

So, finally get:

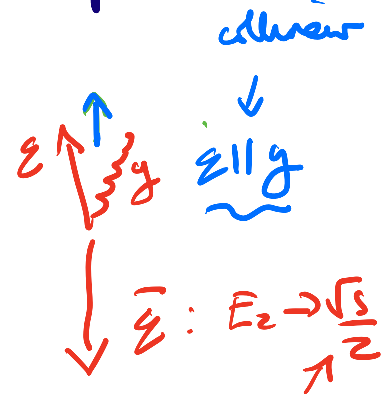
$$\sigma^{q\bar{q}g} = \frac{4\pi\alpha^2}{3s} f_q^2 C_F N_c \frac{\alpha_s}{2\pi} \cdot \int_0^1 dx_1\, dx_2 \frac{(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}$$

OK, so perform $\int dx_1\, dx_2$ & job done? Well NO! Observe that this integral diverges as $x_{1,2} \to 1$

What is going on?

First question: what does $x_{1,2} \to 1$ mean in terms of our phase space?

- $x_2 \to 1 \Rightarrow E_2 \to \frac{\sqrt{s}}{2}$

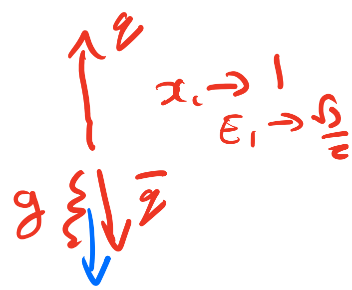


$\Rightarrow$ kinematics '2 body-like'

p_1 & p_3 are going in same direction

i.e. q is collinear to g.

- $x_1 \to 1 \Rightarrow E_1 \to \frac{\sqrt{s}}{2}$

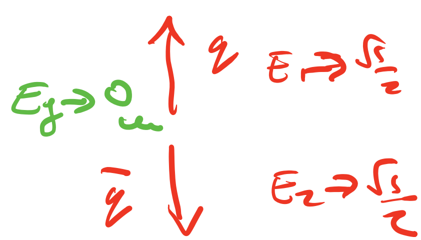


Now $\bar{q}$ & g must be collinear.

- Finally, what if both x_1 & $x_2 \to 1$

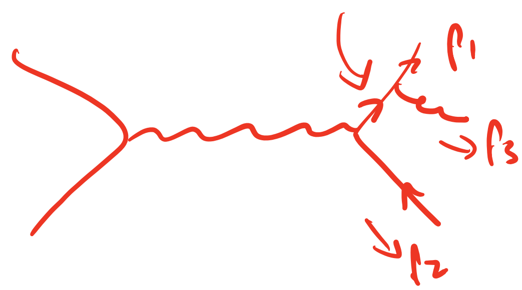
Then q & $\bar{q}$ back-to-back w/ $E = \frac{\sqrt{s}}{2}$

& $E_g \to 0$ soft gluon



In terms of Feynman diagrams, where is this coming from?

∘ e.g $x_2 \to 1$



$$A \sim \frac{1}{(p_1+p_3)^2}$$

Now, $(p_1+p_3)^2 = 2(p_1 p_3)$

$$= 2E_1E_3 - 2\vec{p}_1\cdot\vec{p}_3$$

$$\Rightarrow \quad = 2E_1E_3(1-\cos\theta_{13})$$

So, $p_1 \parallel p_3 \Rightarrow \theta_{13} \to 0$ i.e $\cos\theta_{13} \to 1$

& $s_{13} \to 0$

& $A \to \infty$, diverging!

More precisely: θ_{13} small:

$$\cos\theta_{13} \simeq 1 - \frac{\theta_{13}^2}{2}$$

& $s_{13} \simeq E_1E_3\,\theta_{13}^2$

& $$\frac{1}{(p_1+p_3)^2} \underset{g\parallel q}{\sim} \cancel{\propto} \frac{1}{E_1E_3\theta_{13}^2} \to \infty$$

& sim. for $x_1 \to 1 \Rightarrow \theta_{23} \to 0$

- Also, we can see that $s_{13} \to 0$
 as $E_3 \to 0$ (soft gluon)

How to interpret this?

Basic idea:

$$e^+e^- \to \text{hadrons} \Leftrightarrow e^+e^- \to q\bar{q}\ (q\bar{q}g)$$

Timescale for $\gamma^* \to q\bar{q} \sim \frac{1}{\sqrt{s}}$

($\Delta E \Delta t \sim \hbar$)

While timescale for hadronisation ($q\bar{q} \to$ hadrons)

$$t_{had} \sim \frac{1}{\Lambda_{QCD}}$$

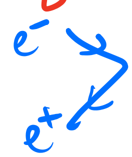
Then provided $\sqrt{s} >> \Lambda_{QCD}$

Then $t_{\gamma^* \to q\bar{q}} << t_{had}$

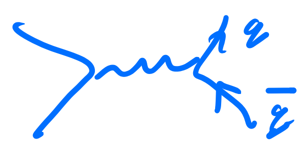
⇒ basis of calculation is that hadronisation happens <u>much</u> later than $\gamma^* \to q\bar{q}$ & therefore we can ignore it in total cross section.

Schematically:

$t = 0$



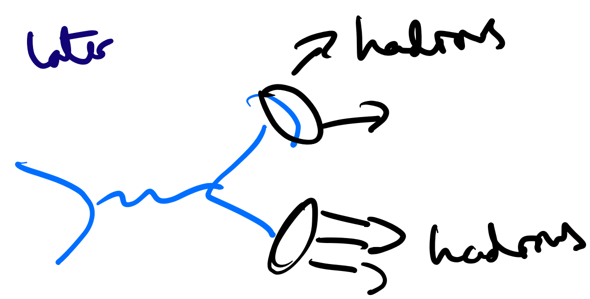
$t \sim \frac{1}{\sqrt{s}}$



...... wait ...

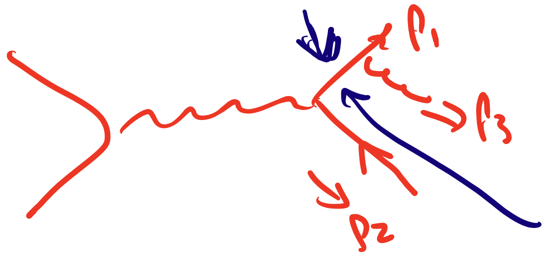
..... much later

$t \sim \frac{1}{\Lambda_{QCD}}$



→ long distance physics of hadronization should be independent of / factorized from our $e^+e^- \to q\bar{q}$ $(q\bar{q}g \ldots)$ calculation.

* What about our $\theta_{13} \to 0$ limit?



$(p_1 \cdot p_3) \sim E_1 E_3 \theta_{13}^2 \sim 0$

$\Rightarrow$ in $\theta_{13} \to 0$ limit our quark propagator is going on-shell, w/ $p_{prop}^2 \sim 0$.

- What does $p^2 = 0$ mean? On-shell particle $\to$ externally propagating state.

- So, as $\theta_{13} \to 0$, propagator starts to go on-shell & quark is propagating over a long-distance / timescale.

$\Rightarrow$ Timescale for gluon emission

$t_g \sim \frac{1}{\sqrt{E_1 E_3}\,\theta_{13}}$ & this can be $\gg \frac{1}{\sqrt{s}}$

$\left(\sim \frac{1}{\Lambda_{QCD}} \right)$

$\Rightarrow$ Precisely entering the $t \sim \frac{1}{\Lambda_{QCD}}$ regime we ~~need~~ need to be insensitive to.

⇒ Indicates a sensitivity to long distance physics → known as an infrared (IR) divergence

- Now, in reality nothing is ∞ here. As $t_g \sim \frac{1}{\Lambda_{QCD}}$ our pQCD theory cannot be applied.

- Disaster?? In fact, not so. What have we missed? Virtual corrections.

$M^V_{q\bar{q}}$ =

$\sim \int \frac{d^4 l}{(2\pi)^4}$

& this also has IR divergences as loop $E_l \to 0$ or $p_l \parallel p_1$, $p_l \parallel p_2$

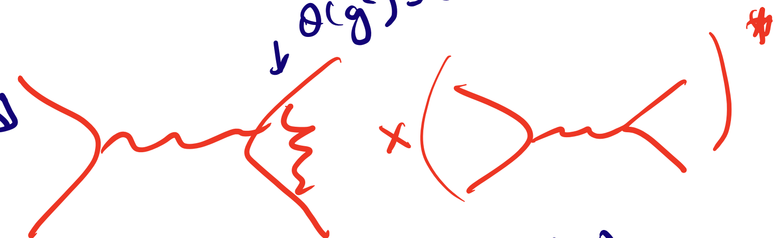
Remember our final state is "everything":

$$M^{NLO} = M^{R} + M^{V}$$

both need to be included:

$O(g^2) = O(\alpha)$

In particular



$= O(\alpha_s)$ (as in real)

Including everything, find that IR divergences in the real emission & the virtual diagrams exactly cancel.

Schematically:

$$\lim_{E_g \to 0} (\underbrace{R}_{\sim +\frac{1}{\infty}} + \underbrace{V}_{\sim -\frac{1}{\infty}}) = 0$$

$$\lim_{\theta \to 0} (R+V) = 0$$

$\Rightarrow$ Total cross section at NLO (NNLO...)

is not sensitive to long distance physics

& our factorised approach is maintained.

Not a coincidence, but a general result:

Kinoshita - Lee - Nauenberg (KLN) theorem (1962/63): all IR divergences will cancel for suitably defined observable

Key in LHC physics: Not enough that $\alpha_s(M_Z)$ is small.

In practice:

R+V IR finite, but R & V individually not. Could "in principle" work w/ R+V at all times, but in practice need to work w/ them individually.

To do this, we must regulate our

theory. Just as in renormalization:

Dim. reg.: $\sigma_{q\bar{q}g} \sim \left(\frac{4}{\varepsilon^2} + \frac{3}{\varepsilon} + O(\varepsilon)\right)$

$\rightarrow \sigma^{V}_{q\bar{q}} \sim \left(\frac{-4}{\varepsilon^2} - \frac{3}{\varepsilon} + O(\varepsilon)\right)$

Dealing with these IR divergences is one of the key issues in higher order calculations.

- In particular, we want a numerical MC generator to produce a finite cross section.
- This goes under the name of "IR subtraction"

Write:

$$\sigma_{NLO} = \int dPS_{q\bar{q}g}\left(\sigma^{q\bar{q}g} - \sigma^{sub}\right) + \int dPS_{q\bar{q}}\left(\sigma^{V}_{q\bar{q}} + \sigma^{sub}\right)$$

Where $\sigma^{sub} \sim \sigma^{q\bar{q}g}$ in $E_g \to 0$ limits, $\theta \to 0$

- Solved / automated at NLO

- At NNLO more complicated, various methods on market.

IR limits & matrix element factorisation

The fact that $\sigma_{q\bar{q}g} + \sigma_{q\bar{q}}$ give finite results for arbitrary q & $\bar{q}$ phase space points, seems a little surprising & hints at universal behaviour!

We have: $$d\sigma_{q\bar{q}g} \xrightarrow[\substack{q \parallel g \\ \bar{q} \parallel g \\ E_g \to 0}]{} d\sigma_{q\bar{q}} \times F_{\text{universal}}$$

This is indeed the case. Let's show this.

Recall: $$\langle |M_{q\bar{q}g}|^2 \rangle \propto \frac{s_{a1}^2 + s_{a2}^2 + s_{b1}^2 + s_{b2}^2}{s_{ab}\, s_{13}\, s_{23}}$$

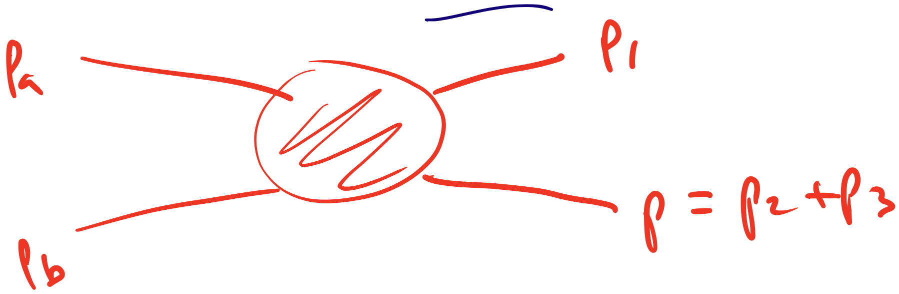
Let's investigate how this looks in $g \parallel \bar{q}$ ($p_3 \parallel p_2$) limit.

$p^2 = 0$

Parametrise that limit: $p_2 = z p \quad p_3 = (1-z)p$

i.e. $p \equiv p_2 + p_3$ (collinear)

We have: $p_a + p_b = p_1 + p_2 + p_3 \overset{\text{ellg}}{\equiv} p_1 + p$



Consider $(p_a + p_b)^2 = (p_1 + p)^2 = 2(p_1 \cdot p)$

$\Rightarrow s_{1p} = s_{ab} \quad (\equiv s)$

$(p_a - p)^2 = (p_1 - p_b)^2 \Rightarrow s_{ap} = s_{b1}$

(& also get $s_{bp} = s_{a1}$)

$s_{a2} = 2(p_a \cdot p_2) = 2z\, p_a \cdot p = z\, s_{ap} = \underline{z\, s_{b1}}$

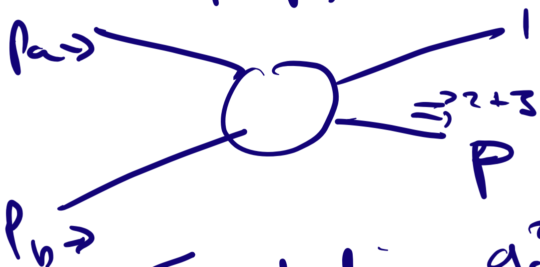
& sim $\underline{s_{b2} = z\, s_{a1}}$

Finally: $\underline{s_{13} = (1-z)\, s_{1p} = (1-z)s}$

$$\Rightarrow \frac{s_{a1}^2 + s_{a2}^2 + s_{b1}^2 + s_{b2}^2}{s\, s_{13}\, s_{23}} \rightarrow \frac{s_{a1}^2 + z^2 s_{b1}^2 + s_{b1}^2 + z^2 s_{a1}^2}{s\,(1-z)s\, s_{23}}$$

$$= \frac{(1+z^2)}{(1-z)} \frac{1}{s_{23}} \frac{(s_{a1}^2 + s_{b1}^2)}{s^2}$$

$s_{a1} = (p_a - p_1)^2,\ s_{b1} = (p_b - p_1)^2$



$$\frac{u^2 + t^2}{s^2} \propto \langle |M_{q\bar{q}}|^2 \rangle$$

Including g_s^2 & colour:

$$\langle |M_{q\bar{q}g}|^2 \rangle \xrightarrow{q \parallel g} \underbrace{2 g_s^2 C_F \frac{(1+z^2)}{(1-z)} \frac{1}{s_{23}}}\langle |M_{q\bar{q}}|^2 \rangle$$

We define: $P_{q\to qg}(z) = C_F \dfrac{(1+z^2)}{(1-z)}$

(P_{gq})

- This is the $q \to qg$ <u>splitting function</u>
- Universal object that relates to $q \parallel g$
  ~ probability of emitting collinear radiation.

- Have derived for specific $q\bar{q}g$ case, but can show that this is universal:

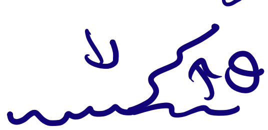
$$\langle |M_{n+g}|^2 \rangle \xrightarrow[g \parallel q]{} \frac{8\pi\alpha_s^2}{s_{qg}} P_{q\to qg} \langle |M_n|^2 \rangle$$

M_n : arbitrary QCD process

* Not quite what we need as also a PS integral to deal with. But it also find this also factorises in $q \parallel g$ limit.

<u>End up with</u>: $$\sigma_{n+g} \underset{q\parallel g}{=} \sigma_n \int dz \, \frac{d\theta_{ij}^2}{\theta_{ij}^2} \cdot \frac{\alpha_s}{2\pi} P_{q\to qg}(z)$$

Similar factorisation for other splittings:

$g \parallel g$ 

: $P_{g\to gg}(z)$

$q \parallel g$ 

: $P_{g\to q\bar{q}}(z)$

Only difference is in the splitting functions, P.

- Similar behaviour for when gluon is soft: $\langle |M_{q\bar{q}g}|^2 \rangle \xrightarrow{E_g \to 0} \langle |M_{q\bar{q}}|^2 \rangle \dfrac{2 s_{12}}{s_{13} s_{23}}$

"Eikonal factor"

Summary

QCD has simple & universal factorisation properties in soft & collinear limits.

- Required by cancellation of IR divergences.
- Splitting functions ~ prob of emitting collinear radiation. Connect low energy non-pert physics to high energy scattering.