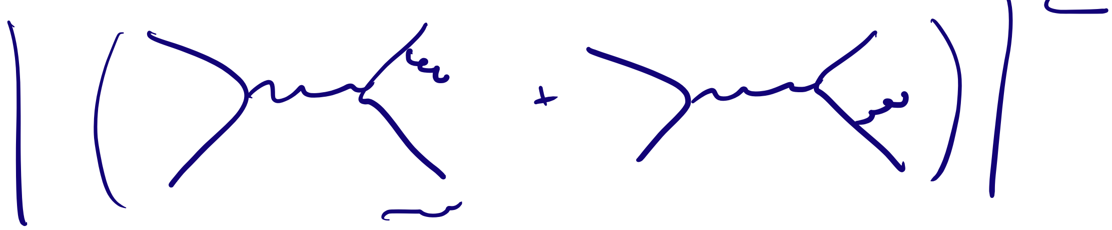
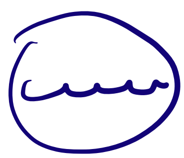
Lecture 3 $e^+e^- \to q\bar{q}g$ (again)

We have: $\sigma^{real}_{NLO} = \dfrac{1}{2s_{ab}} \int dPS_3 \langle |M_{q\bar{q}g}|^2 \rangle$



$= L_{\mu\nu} H^{\mu\nu}_{NLO,\,real}$

Colour:  $= N_c C_F$

Not going to calculate everything: give some steps.

After some work, we get:

$$\langle |M_{q\bar{q}g}|^2 \rangle = 4 e^2 f_q^2 g_s^2 N_c C_F \cdot \left(\frac{s_{a1}^2 + s_{a2}^2 + s_{b1}^2 + s_{b2}^2}{s_{ab}\, s_{13}\, s_{23}} \right)$$

w/ $s_{ij} \equiv (p_i + p_j)^2 = 2(p_i \cdot p_j)$

& $e^+(p_a) + e^-(p_b) \to q(p_1) + \bar{q}(p_2) + g(p_3)$

Often use $s_{ab} \leftrightarrow s$ interchangeably

Next step, cross section:

$$\int dPS_3 = \frac{1}{(2\pi)^5} \int d^4p_1 \, d^4p_2 \, \underline{d^4p_3}$$

$$\underline{\delta(p_1^2)\,\delta(p_2^2)\,\delta(p_3^2)}\;\delta^{(4)}(p_a+p_b-p_1-p_2-p_3)$$

Steps: 1) Eliminate d^4p_3 w/ $\delta^{(4)}(\ldots)$

2) $d^4p_1\,\delta(p_1^2) = \frac{d^3\vec{p}_1}{2E_1} = \frac{E_1\,dE_1}{2}\,\underline{d\cos\theta_1\,d\phi_1}$

& sim for p_2

3) Define energy fraction, via:

$$E_i \equiv x_i \frac{\sqrt{s}}{2} \qquad \left(E_{a,b} = \frac{\sqrt{s}}{2}\right)$$

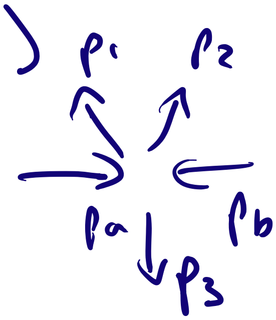
$$\Rightarrow dE_i \; (d|\vec{p}_i|) = \frac{\sqrt{s}}{2}\,\underline{\underline{dx_i}} \leftarrow$$

4) Eliminate / perform angular integrals.

Finally, recall denominator of $|M|^2$

is $\underline{s\,s_{13}\,s_{23}}$

e.g. $s_{13} = 2E_1E_3 - 2(\vec{p}_1 \cdot \vec{p}_3)$ ✓



But : $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$

$$\Rightarrow \vec{p}_2 = -\vec{p}_1 - \vec{p}_3$$

$$\vec{p}_2^{\,2} = E_2^2 = E_1^2 + E_3^2 + 2(\vec{p}_1 \cdot \vec{p}_3)$$

Dust settles: $s_{13} = s(1-x_2)$

So, finally get:

$$\sigma^{q\bar{q}g} = \frac{4\pi\alpha^2}{3s} f_q^2 C_F N_c \frac{\alpha_s}{2\pi} \cdot \int_0^1 dx_1\, dx_2 \frac{(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}$$

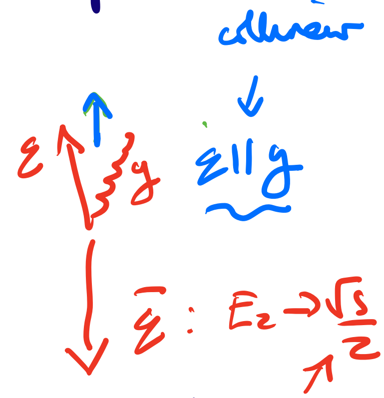
OK, so perform $\int dx_1\, dx_2$ & job done?

Well NO! Observe that this integral diverges as $x_{1,2} \to 1$

What is going on?

First question: what does $x_{1,2} \to 1$ mean in terms of our phase space?

- $x_2 \to 1 \Rightarrow E_2 \to \frac{\sqrt{s}}{2}$

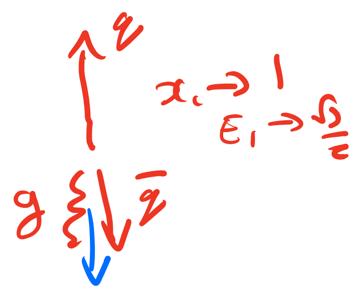


$\Rightarrow$ kinematics '2 body-like'

p_1 & p_3 are going in <u>same</u> direction

i.e. q is collinear to g.

- $x_1 \to 1 \Rightarrow E_1 \to \frac{\sqrt{s}}{2}$

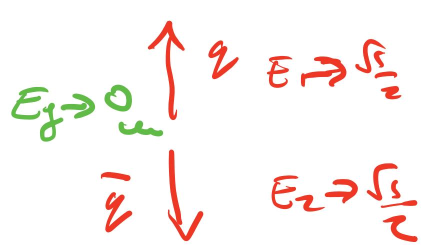


Now $\bar{q}$ & g must be collinear.

- Finally, what if both x_1 & $x_2 \to 1$

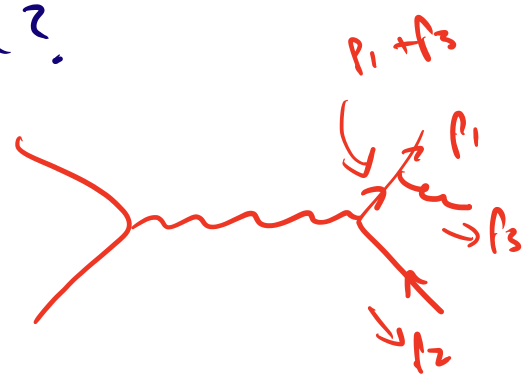
Then q & $\bar{q}$ back-to-back w/ $E = \frac{\sqrt{s}}{2}$

& $E_g \to 0$ soft gluon



In terms of Feynman diagrams, where is this coming from?

◦ e.g $x_2 \to 1$



$A \sim \frac{1}{(p_1+p_3)^2}$

Now, $(p_1+p_3)^2 = 2(p_1 p_3)$

$= 2E_1E_3 - 2\vec{p}_1\cdot\vec{p}_3$

$\Rightarrow = 2E_1E_3(1-\cos\theta_{13})$

So, $p_1 \parallel p_3 \Rightarrow \theta_{13} \to 0$ i.e $\cos\theta_{13} \to 1$

& $s_{13} \to 0$

& $A \to \infty$, diverging!

More precisely: θ_{13} small:

$\cos\theta_{13} \simeq 1 - \frac{\theta_{13}^2}{2}$

& $s_{13} \simeq E_1E_3\theta_{13}^2$

& $\frac{1}{(p_1+p_3)^2} \underset{g\parallel q}{\sim} \propto \frac{1}{E_1E_3\theta_{13}^2} \to \infty$

& sim. for $x_1 \to 1 \Rightarrow \theta_{23} \to 0$

- Also, we can see that $s_{13} \to 0$ as $E_3 \to 0$ (soft gluon)

How to interpret this?

Basic idea:

$e^+e^- \to$ hadrons $\Leftrightarrow e^+e^- \to q\bar{q}$ $(q\bar{q}g)$

Timescale for $\gamma^* \to q\bar{q} \sim \frac{1}{\sqrt{s}}$

$(\Delta E \Delta t \sim \hbar)$

While timescale for hadronisation ($q\bar{q} \to$ hadrons)

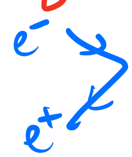
$t_{had} \sim \frac{1}{\Lambda_{QCD}}$

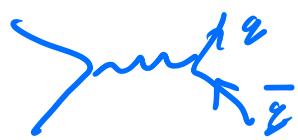
Then provided $\sqrt{s} \gg \Lambda_{QCD}$

Then $t_{\gamma^* \to q\bar{q}} \ll t_{had}$

$\Rightarrow$ basis of calculation is that hadronisation happens <u>much</u> later than $\gamma^* \to q\bar{q}$ & therefore we can ignore it in total cross section.

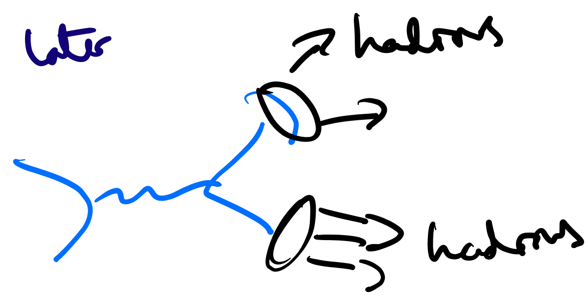
Schematically:

$t = 0$ 

$t \sim \frac{1}{\sqrt{s}}$ 

.... wait ...

..... much later

$t \sim \frac{1}{\Lambda_{QCD}}$ 

$\rightarrow$ long distance physics of hadronization should be independent of / factorized from our $e^+e^- \to q\bar{q}$ ($q\bar{q}g \ldots$) calculation.

* What about our $\theta_{13} \to 0$ limit?

$(p_1 \cdot p_3) \sim E_1 E_3 \theta_{13}^2 \sim 0$

$\Rightarrow$ in $\theta_{13} \to 0$ limit our quark propagator is going on-shell, w/ $p_{prop}^2 \sim 0$.

- What does $p^2 = 0$ mean? On-shell particle $\to$ externally propagating state.
- So, as $\theta_{13} \to 0$, propagator starts to go on-shell & quark is propagating over a long-distance / timescale.

$\Rightarrow$ Timescale for gluon emission

$$t_g \sim \frac{1}{\sqrt{E_1 E_3}\,\theta_{13}} \quad \text{\& this can be} \gg \frac{1}{\sqrt{s}}$$

$$\left(\sim \frac{1}{\Lambda_{QCD}} \right)$$

$\Rightarrow$ Precisely entering the $t \sim \frac{1}{\Lambda_{QCD}}$ regime we ~~~~ need to be insensitive to.

⇒ Indicates a sensitivity to long distance physics → known as an infrared (IR) divergence

- Now, in reality nothing is ∞ here. As $t_q \sim \frac{1}{\Lambda_{QCD}}$ our pQCD theory cannot be applied.

- Disaster?? In fact, not so. What have we missed? Virtual corrections.

$M^V_{q\bar{q}} =$

$\sim \int \frac{d^4 l}{(2\pi)^4}$

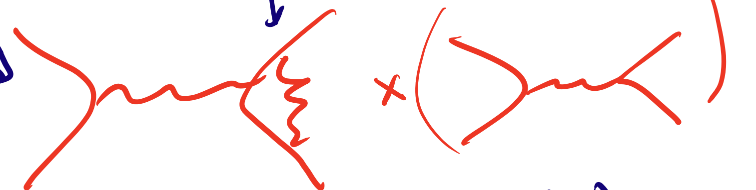
& this also has IR divergences as loop $E_l \to 0$ or $p_l \parallel p_1$, $p_l \parallel p_2$

Remember our final state is "everything":

$$M^{NLO} = M^R + M^V$$

both need to be included:

In particular $O(g^2) = O(\alpha)$



$= O(\alpha_s)$ (as in real)

Including everything, find that IR divergences in the real emission & the virtual diagrams __exactly__ cancel.

Schematically:

$$\lim_{E_g \to 0} (\underbrace{R}_{\sim +\frac{1}{\alpha}} + \underbrace{V}_{\sim -\frac{1}{\alpha}}) = 0$$

$$\lim_{\theta \to 0} (R+V) = 0$$

$\Rightarrow$ Total cross section at NLO (NNLO...)

is not sensitive to long distance physics

& our factorised approach is maintained.

Not a coincidence, but a general result:

Kinoshita - Lee - Nauenberg (KLN) theorem (1962/63): all IR divergencies will cancel for suitably defined observable

* Key in LHC physics: Not enough that $\alpha_s(M_Z)$ is small.

In practice:

* R+V IR finite, but R & V individually not. Could "in principle" work w/ R+V at all times, but in practice need to work w/ them individually.

* To do this, we must regulate our

theory. Just as in renormalization:

Dim. reg.: $\sigma_{q\bar{q}g} \sim \left(\frac{4}{\varepsilon^2} + \frac{3}{\varepsilon} + \mathcal{O}(\varepsilon) \right)$

$\rightarrow \sigma^{V}_{q\bar{q}} \sim \left(\frac{-4}{\varepsilon^2} - \frac{3}{\varepsilon} + \mathcal{O}(\varepsilon) \right)$

* Dealing with these IR divergences is one of the key issues in higher order calculations.

- In particular, we want a numerical MC generator to produce a finite cross section.

- This goes under the name of "IR subtraction"

Write:

$$\sigma_{NLO} = \int dPS_{q\bar{q}g} \left(\sigma^{q\bar{q}g} - \sigma^{sub} \right)$$
$$+ \int dPS_{q\bar{q}} \left(\sigma^{V}_{q\bar{q}} + \sigma^{sub} \right)$$

Where $\sigma^{sub} \sim \sigma^{q\bar{q}g}$ in $E_g \to 0$ limits
$\theta \to 0$

- solved / automated at NLO

- At NNLO more complicated, various methods on market.

IR limits & matrix element factorisation

The fact that $\sigma_{q\bar{q}g} + \sigma_{q\bar{q}}$ give finite results for arbitrary q & $\bar{q}$ phase space points, seems a little surprising & hints at <u>universal</u> behaviour!

We have:

$$d\sigma_{q\bar{q}g} \xrightarrow[\substack{q\parallel g \\ \bar{q}\parallel g \\ E_g\to 0}]{} d\sigma_{q\bar{q}} \times \underline{F_{\text{universal}}}$$

This is indeed the case. Let's show this.

<u>Recall</u>: $\langle |M_{q\bar{q}g}|^2 \rangle \propto \dfrac{s_{a1}^2 + s_{a2}^2 + s_{b1}^2 + s_{b2}^2}{s_{ab}\, s_{13}\, s_{23}}$

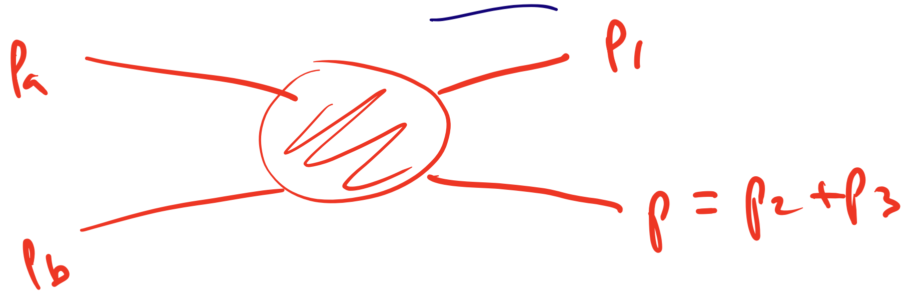
Let's investigate how this looks in $g \parallel \bar{q}$ ($p_3 \parallel p_2$) limit.

$p^2 = 0$

Parametrise that limit: $p_2 = z p$ $\quad p_3 = (1-z) p$

i.e. $p \equiv p_2 + p_3$ (collinear)

We have: $\underline{p_a + p_b} = p_1 + p_2 + p_3 \equiv \underset{\approx}{\overset{\text{ellg}}{p_1 + p}}$



Consider $(p_a + p_b)^2 = (p_1 + p)^2 = 2(p_1 \cdot p)$

$\Rightarrow s_{1p} = s_{ab}$ $(\equiv s)$

$(p_a - p)^2 = (p_1 - p_b)^2 \Rightarrow s_{ap} = s_{b1}$

(& also get $s_{bp} = s_{a1}$)

$s_{a2} = 2(p_a \cdot p_2) = 2z\, p_a \cdot p = z\, s_{ap} = \underline{z\, s_{b1}}$

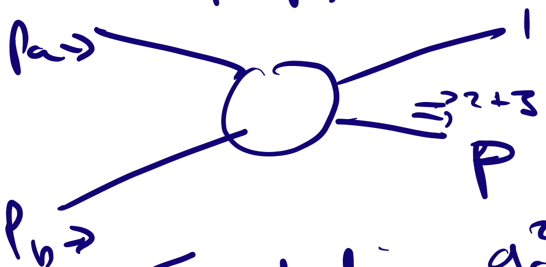
& sim $\underline{s_{b2} = z\, s_{a1}}$

Finally: $\underline{s_{13} = (1-z)\, s_{1p} = (1-z) s}$

$$\Rightarrow \frac{s_{a1}^2 + s_{a2}^2 + s_{b1}^2 + s_{b2}^2}{s\, s_{13}\, s_{23}} \to \frac{s_{a1}^2 + z^2 s_{b1}^2 + s_{b1}^2 + z^2 s_{a1}^2}{s\,(1-z)s\; s_{23}}$$

$$= \frac{(1+z^2)}{(1-z)} \frac{1}{s_{23}} \frac{(s_{a1}^2 + s_{b1}^2)}{s^2}$$

$s_{a1} = (p_a - p_1)^2;\ s_{b1} = (p_b - p_1)^2$



$$\frac{u^2 + t^2}{s^2} \propto \langle |M_{q\bar{q}}|^2 \rangle$$

Including g_s^2 & colour:

$$\langle |M_{q\bar{q}g}|^2 \rangle \xrightarrow{q \| g} \underbrace{2 g_s^2 C_F \frac{(1+z^2)}{(1-z)} \frac{1}{s_{23}}}\langle |M_{q\bar{q}}|^2 \rangle$$

We define: $P_{q\to qg}(z) = C_F \dfrac{(1+z^2)}{(1-z)}$

(P_{gq})

- This is the $q \to qg$ <u>splitting function</u>
- Universal object that relates to $q \| g$

$\sim$ probability of emitting collinear radiation.

– Have derived for specific $q\bar{q}g$ case, but can show that this is universal:

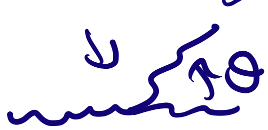
$$\langle |M_{n+g}|^2 \rangle \xrightarrow[g \parallel q]{} \frac{8\pi\alpha_s^2}{s_{qg}} P_{q\to qg} \langle |M_n|^2 \rangle$$

M_n : arbitrary QCD process

* Not quite what we need as also a PS integral to deal with. But it also find this also factorises in $q \parallel g$ limit.

End up with:
$$\sigma_{n+g} \underset{q\parallel g}{=} \sigma_n \int dz \frac{d\theta_{ij}^2}{\theta_{ij}^2} \cdot \frac{\alpha_s}{2\pi} P_{q\to qg}(z)$$

Similar factorisation for other splittings:

$g \parallel g$ 

: $P_{g\to gg}(z)$

$q \parallel g$ 

: $P_{g\to q\bar{q}}(z)$

Only difference is in the splitting functions, P.

- Similar behaviour for when gluon is soft: $\langle |M_{q\bar{q}g}|^2 \rangle \xrightarrow{E_g \to 0} \langle |M_{q\bar{q}}|^2 \rangle \dfrac{2 s_{12}}{s_{13} s_{23}}$

"Eikonal factor"

Summary

QCD has simple & universal factorisation properties in soft & collinear limits.

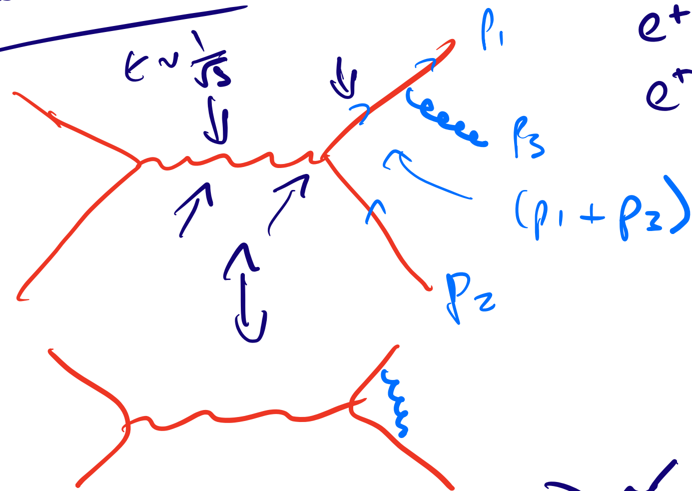
- Required by cancellation of IR divergences.
- Splitting functions ~ prob of emitting collinear radiation. Connect low energy non-pert physics to high energy scattering.

Lecture 5

$e^+e^- \to$ hadrons

$e^+e^- \to q\bar{q}$: LO

$e^+e^- \to q\bar{q}g$: NLO Real



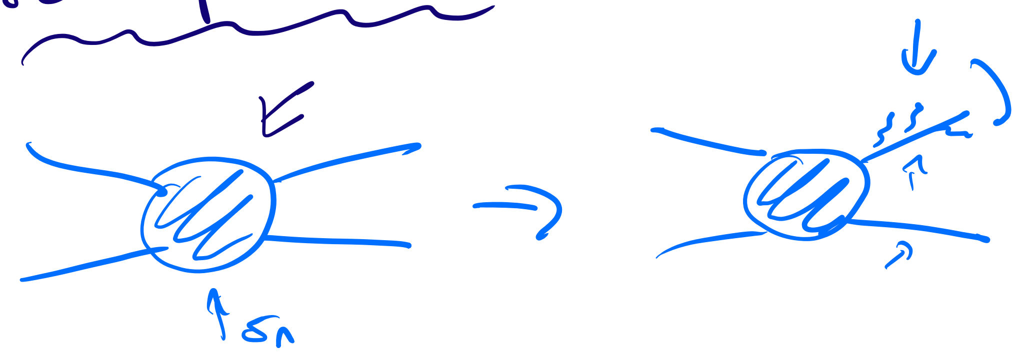
Recall:

$$\sigma_{n+1} = \sigma_n \int dz \, \frac{d\theta_{ij}^2}{\theta_{ij}^2} \frac{\alpha_s}{2\pi} P_{q\to qg}(z)$$

$q \parallel g$

arbitrary

Role in parton showers:

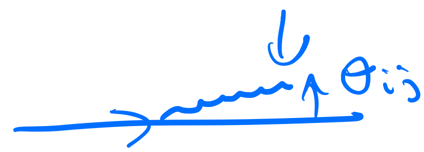


– Showering of outgoing quark line is independent from hard scatter.

- How does this work?

Start with outgoing quark:

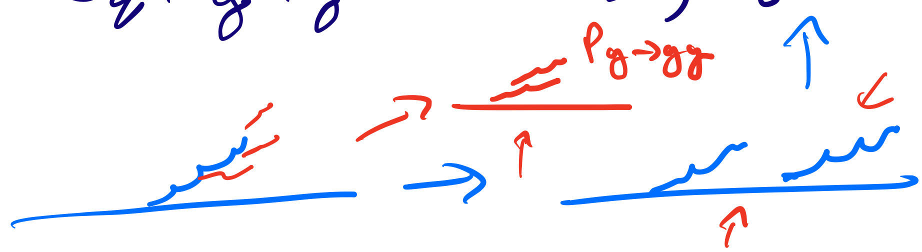
& use $\overset{\sigma_{n+1}}{\sigma_{q+g}} = \overset{\sigma_n}{\sigma_q} \int dz \ldots P_{q\to qg}$



Then iterate: $\sigma_{n+1} = \sigma_{q+g+g}$

$\sigma_n = \sigma_{q+g}$

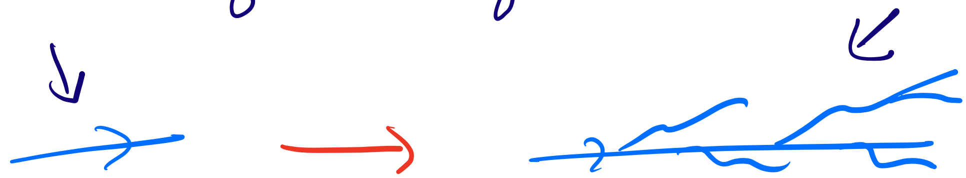
$$\sigma_{q+g+g} = \sigma_{q+g} \int dz \ldots P_{q\to qg}$$



& so on ...

- Parton shower provides a systematic way of doing this iteration

→ collimated 'shower' of QCD radiation (i.e. a jet) + soft emission.



without ever having to calculate the full $\sigma_q + \underbrace{\sigma_{qg} + \sigma_{qgg} + \ldots}$

<u>NB</u>: * Skips details of how this actually works.

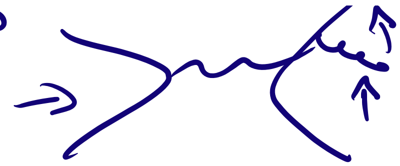
* Shower only gives good approx. to collinear/soft region of emission.

For e.g. $e^+e^- \to q\bar{q} + g$ 3 jet event

need a full NLO calculation

* At NLO and beyond there is an

overlap between σ^{NLO}



& what is generated by parton shower

i.e. double counting.

<u>IR safety</u>: Have discussed IR singularity cancellation in $e^+e^- \to$ hadrons.
When more generally will this occur?
i.e. what questions can we ask of our theory ($=$ pQCD)?

<u>Basic idea</u>: Don't want sensitivity to soft/collinear emission.
$\Rightarrow$ Observable should be invariant under branching:

$$p_i \to p_j + p_k \quad \text{w/} \quad p_j \| p_k$$
$$\text{or} \quad E_{j,k} \to 0$$

Formally:

$$\mathcal{O}(p_1, \dots, p_i, p_j, \dots) \xrightarrow{i \| j} \mathcal{O}(p_1, \dots, (p_i + p_j), \dots)$$

$$\mathcal{O}(p_1, \dots, p_i, p_j \dots) \xrightarrow{E_j \to 0} \mathcal{O}(p_1, \dots, p_i, \dots)$$

$$(p_i + p_j)$$

smoothly.

Have seen how this happens in $e^+e^- \to q\bar{q}(g)$

Problems: to discuss some examples.

* Number of partons?
* Certain jet algorithms?
* ....

Resummation

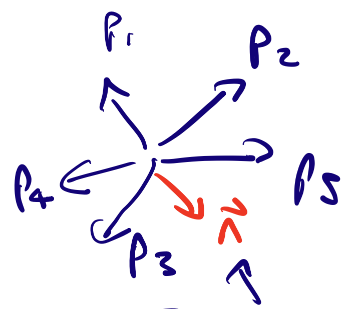
- Stick for e^+e^- for now, but basic concepts carry across to the LHC.

Will consider:

"Event shape" = global measure of event.

Consider "thrust". Classic observable from LEP precision tests.

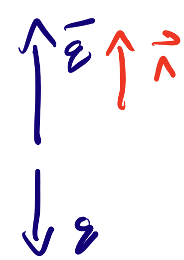
$$T \equiv \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$$



$\vec{n}$: unit vector

$\sum_i$: sum over all partons

e.g. Back-to-back $q\bar{q}$

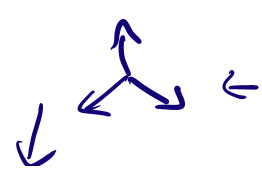


$$\vec{n} \cdot \vec{p}_i = E_i \cos\theta_{in}$$

here: $|\cos\theta_{nq}| = |\cos\theta_{n\bar{q}}| = 1$

$$\Rightarrow \underline{T = 1}$$

spherically symmetric, large multiplicity event: $\underline{T = \frac{1}{2}}$



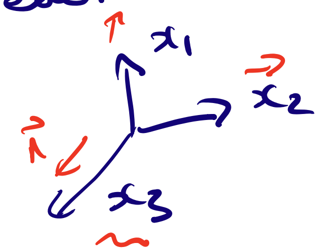
3 particle event: $\frac{2}{3} \le T \le 1$

Start with: $\sigma^{q\bar{q}g} = \sigma_0 C_F \frac{\alpha_s}{2\pi} \int dx_1 dx_2 \frac{(x_1^2 + x_2^2)}{(1-x_1)(1-x_2)}$

$$\left(\sigma^{q\bar{q}} = N_c f_q^2 \frac{4\pi\alpha^2}{s} \right)$$

With a bit of work can express this as

$\frac{d\sigma}{dT}$: for each event



$T = \max(x_1, x_2, x_3)$

i.e. $\frac{d\sigma}{dT} = \sigma^{q\bar{q}g}\, \delta(T - \max(x_1, x_2, x_3))$

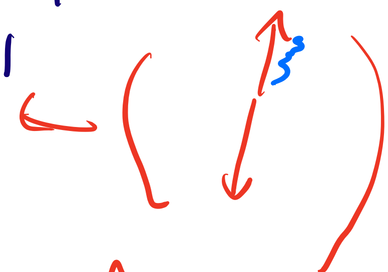
After some work get:

$$\frac{d\sigma}{dT} = \sigma_0 C_F \frac{\alpha_s}{2\pi} \left[\frac{2}{T(1-T)} (3T\underbrace{(T-1)}_{0}+2) \underbrace{\ln\left(\frac{2T-1}{1-T}\right)}_{\ln(1-T)} + \frac{3(3T-2)\overbrace{(T-2)}^{\to -3}}{1-T} \right]$$

Let's consider the $T \to 1$ limit:

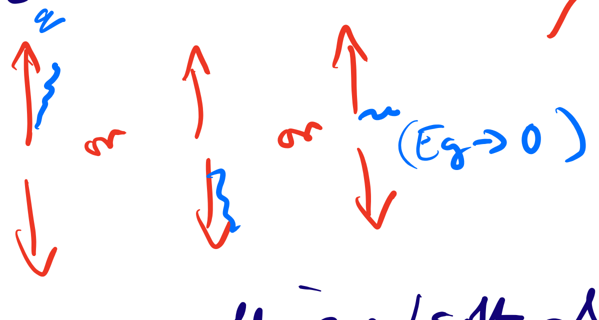
$$\left\{ \frac{1}{\sigma_0}\frac{d\sigma}{dT} \xrightarrow{T\to 1} -C_F\frac{\alpha_s}{2\pi}\left[4\,\frac{\ln(1-T)}{1-T} + \frac{3}{1-T}\right]\right.$$

which diverges as $T\to 1$



Have seen this before:

$T\to 1 \Longrightarrow$

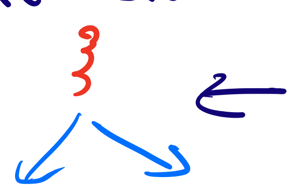


i.e. precisely where collinear / soft gluon emission in $q\bar q+g$ relevant.

- Now, as before if we are only interested in $e^+e^-\to$ hadrons, include virtual & everything OK.

- But what if we want to ask other questions of our theory?

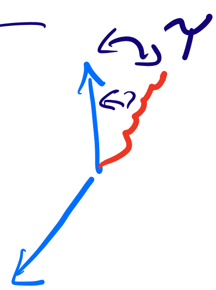
e.g. if a gluon is emitted with large enough energy & not collinear, have "3-jet" event:



Natural observable: fraction of events that are 2 or 3 jet-like, R_2 & R_3.
w/ $R_2 + R_3 = 1$.

- To do this we need to introduce a "resolution parameter" in order to define what is meant by 3 or 2 jet-like.

3-jet.   3 jet?



$T > 1-\tau$: 2-jet

$T \leq 1-\tau$: 3-jet

(c.f. jet radius R)

3-jet fraction: $R_3(\tau) = \int_{2/3}^{1-\tau} \frac{1}{\sigma_0} \frac{d\sigma}{dT} dT$

Focus on $T \sim 1$ part of σ: $= C_F \frac{\alpha_s}{2\pi} \int_{2/3}^{1-\tau} \left(\frac{4\ln(1-T)}{1-T} + \frac{3}{1-T} \right) dT$

$$= C_F \frac{\alpha_s}{2\pi} \left[2\ln^2(1-T) + 3\ln(1-T) \right]_{2/3}^{1-\tau}$$

$$R_3(\tau) = C_F \frac{\alpha_s}{2\pi} \left[2\ln^2\tau + 3\ln\tau - 2\ln^2\tfrac{1}{3} - 3\ln\tfrac{1}{2} \right]$$

If kept expression:

$$R_3(\tau) = C_F \frac{\alpha_s}{2\pi} \left[2\ln^2\tau + 3\ln\tau \right.$$

$$\left. + \text{regular as } \tau \to 0 \right]$$

$\uparrow$ finite

So, what about $\tau \ll 1$ limit?

Then, the $\ln^2\tau$ dominates &

$$R_3(\tau) \to C_F \frac{\alpha_s}{2\pi} 2\ln^2\tau$$

$$R_2(\tau) \to 1 - C_F \frac{\alpha_s}{2\pi} \ln^2\tau$$

$\Rightarrow$ For small (but non-zero) τ our predicted fraction of 3-jet events is arbitrarily large & $R_2 < 0$.

What is going on?

<u>Solution</u>: we are expanding in α_s

$\Rightarrow$ can only trust our results in regime where expansion is reliable.

Here, we have: $R_2(\tau) = \underset{\substack{\uparrow \\ \text{LO } (q\bar{q})}}{1} - \underset{\substack{\uparrow \\ \text{NLO}}}{C_F \frac{\alpha_s}{2\pi} 2\ln^2\tau} + \ldots$

& so the NLO terms for $\tau \to 0$ we have $\alpha_s \ln^2\tau \gg 1$ ✗

i.e. pQCD is breaking down.

- But all is not lost! At higher orders we will find:

$$R_2(\tau) \overset{\tau\to 0}{=} 1 - C_F \frac{\alpha_s}{2\pi} 2\ln^2\tau + \frac{1}{2}\left(C_F \frac{\alpha_s}{2\pi} 2\ln^2\tau\right)^2 + \ldots$$

i.e. pQCD is breaking down, <u>but</u> the above is a Taylor expansion

$$\left(e^{-x} = 1 - x + \frac{x^2}{2} \ldots\right)$$

$\Rightarrow$ Know this $\sim \ln^2\tau$ dominant (as $\tau \to 0$)
contribution to all orders in α_s:

$$R_2(\tau) \underset{\tau\to 0}{=} \exp\left[-C_F \frac{\alpha_s}{2\pi} 2\ln^2\tau\right]$$

Order-by-order this will jump around all over the place, but the 'exact' resummed result is OK:

$$\boxed{R_2(\tau) \to 0 \quad \text{as } \tau \to 0}$$