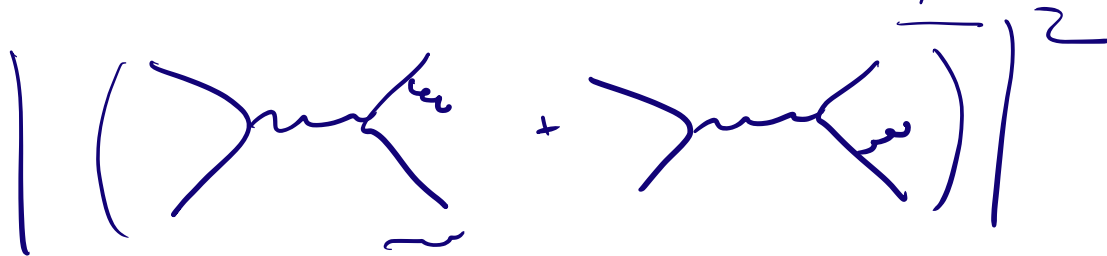


Lecture 3 $e^+e^- \rightarrow g\bar{c}g$ (again)

We have: $\sigma_{NLO}^{real} = \frac{1}{2S_{ab}} \int dPS_3 \langle |M_{g\bar{c}g}^2 \rangle$



$$= L_{\mu\nu} H_{NLO, real}^{\mu\nu}$$

Colour: $\text{loop} = N_c C_F$

Not going to calculate everything:
give some steps.

After some work, we get:

$$\langle |M_{g\bar{c}g}^2 \rangle = 4e^2 f_D^2 g_s^2 N_c C_F \left(\frac{S_{a1}^2 + S_{a2}^2 + S_{b1}^2 + S_{b2}^2}{S_{ab} S_{13} S_{23}} \right)$$

w/ $S_{ij} \equiv (p_i + p_j)^2 = 2(p_i \cdot p_j)$

$e^+(p_a) + e^-(p_b) \rightarrow g(p_1) + \bar{c}(p_2) + g(p_3)$

Often we $S_{ab} \leftrightarrow S$ interchangeably

Next step, cross section:

$$\int dPS_3 = \frac{1}{(2\pi)^5} \int d^4 p_1 d^4 p_2 d^4 p_3$$
$$\delta(p_1^2) \delta(p_2^2) \delta(p_3^2) \delta^{(4)}(p_a + p_b - p_1 - p_2 - p_3)$$

Steps: 1) Eliminate $d^4 p_3$ w/ $\delta^{(4)}(\dots)$

$$2) d^4 p_1 \delta(p_1^2) = \frac{d^3 \vec{p}_1}{2E_1} = \frac{E_1 dE_1}{2} d\cos\theta_1 d\phi_1$$

\mathcal{Q} sin for p_2

3) Define energy fraction, via:

$$E_i \equiv x_i \frac{\sqrt{s}}{2} \quad (E_{a,b} = \frac{\sqrt{s}}{2})$$

$$\Rightarrow dE_i (d\vec{p}_i) = \frac{\sqrt{s}}{2} dx_i \leftarrow$$

4) Eliminate / perform angular integrals.

Finally, result denominator of M^2

$$is \quad \underline{s s_{13} s_{23}}$$

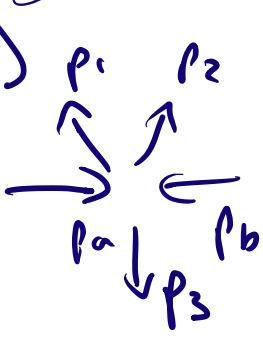
✓

e.g. $S_{13} = 2E_1 E_3 - 2(\vec{p}_1 \cdot \vec{p}_3)$ ✓

But: $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$

$\Rightarrow \vec{p}_2 = -\vec{p}_1 - \vec{p}_3$

$\vec{p}_2^2 = E_2^2 = E_1^2 + E_3^2 + 2(\vec{p}_1 \cdot \vec{p}_3)$



Dust settles: $S_{13} = s(1-x_2)$

↑

So, finally get:

$$\sigma_{\text{e}^+ \text{e}^- \gamma} = \frac{4\pi\alpha^2}{3s} f_2^2 G N_c \frac{\alpha_s}{2\pi}$$

$$- \int_0^1 dx_1 dx_2 \frac{(x_1^2 + x_2^2)}{(1-x_1)(1-x_2)} \leftarrow$$

OK, so perform $\int dx_1 dx_2$ & job done?

Well NO! Observe that this integral

diverges as $x_{1,2} \rightarrow 1$

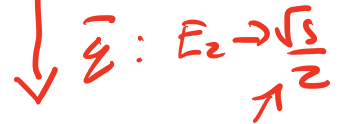
What is going on?

First question: what does $x_{1,2} \rightarrow 1$ mean in terms of our phase space?

• $x_2 \rightarrow 1 \Rightarrow E_2 \rightarrow \frac{\sqrt{5}}{2}$

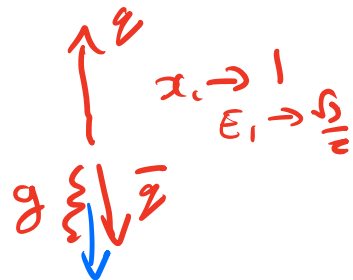


\Rightarrow kinematics '2 body-like'



p_1 & p_3 are going in same direction
i.e. z is collinear to g .

• $x_1 \rightarrow 1 \Rightarrow E_1 \rightarrow \frac{\sqrt{5}}{2}$

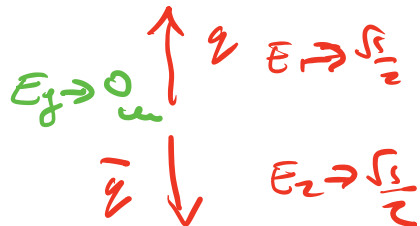


Now \bar{z} & g must be collinear.

• Finally, what if both x_1 & $x_2 \rightarrow 1$

Then \bar{z} & \bar{z} knock-to-knock

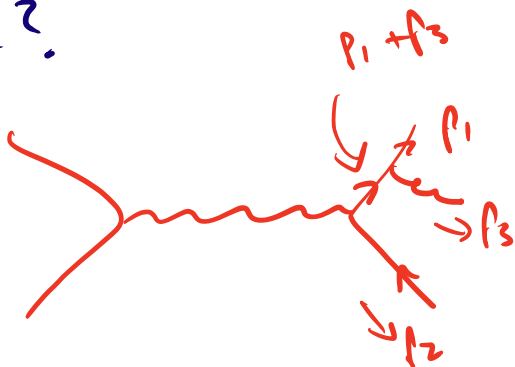
w/ $E = \frac{\sqrt{5}}{2}$



$\& E_g \rightarrow 0$ Syt gluon

In terms of Feynman diagrams, where is this coming from?

• e.g. $x_2 \rightarrow 1$



$$A \sim \frac{1}{(p_1 + p_3)^2}$$

$$\begin{aligned} \text{Now, } (p_1 + p_3)^2 &= 2(p_1 \cdot p_3) \\ &= 2E_1 E_3 - 2\vec{p}_1 \cdot \vec{p}_3 \\ &\rightarrow = 2E_1 E_3 (1 - \cos \Theta_{13}) \end{aligned}$$

So, $p_1 \parallel p_3 \Rightarrow \Theta_{13} \rightarrow 0$ i.e. $\cos \Theta_{13} \rightarrow 1$

$$\& S_{13} \rightarrow 0$$

$\& A \rightarrow \infty$, diverging!

More precisely: Θ_{13} small:

$$\cos \Theta_{13} \approx 1 - \frac{\Theta_{13}^2}{2}$$

$$\& S_{13} \approx E_1 E_3 \Theta_{13}^2$$

$$\& \frac{1}{(p_1 + p_3)^2} \approx \frac{1}{2E_1 E_3 \Theta_{13}^2} \rightarrow \infty$$

α small. for $x_1 \rightarrow 1 \Rightarrow \Theta_{23} \rightarrow 0$

- Also, we can see that $S_{13} \rightarrow 0$

as $E_3 \rightarrow 0$ (soft gluon)

How to interpret this?

Basic idea:

$e^+e^- \rightarrow \text{hadrons} \Leftrightarrow e^+e^- \rightarrow q\bar{q} \text{ (} \bar{q}q \text{)}$

Timescale for $\gamma^* \rightarrow q\bar{q} \sim \frac{1}{\sqrt{s}}$

($\Delta E \Delta t \sim \hbar$)

While timescale for hadronisation ($q\bar{q} \rightarrow \text{hadrons}$)

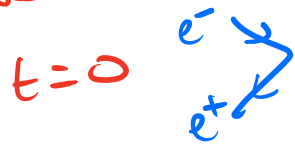
$t_{\text{had}} \sim \frac{1}{\Lambda_{\text{QCD}}}$

Then provided $\sqrt{s} \gg \Lambda_{\text{QCD}}$

Then $t_{\gamma^* \rightarrow q\bar{q}} \ll t_{\text{had}}$

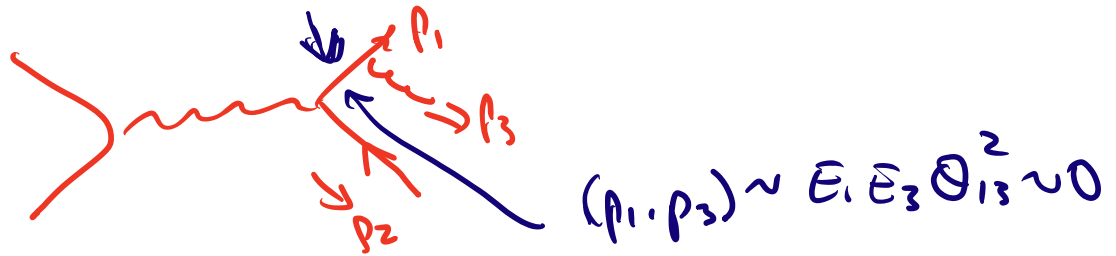
\Rightarrow basis of calculation is that hadronization happens much later than $\gamma^* \rightarrow e^+e^-$ & therefore we can ignore it in total cross section.

Schematically:



\rightarrow long distance physics of hadronization should be independent of / factorized from our $e^+e^- \rightarrow e^+e^-$ ($e^+e^-g \dots$) calculations.

* What about our $\Theta_{13} \rightarrow 0$ limit?



\Rightarrow in $\Theta_{13} \rightarrow 0$ limit our quark propagator is going on-shell, w/ $p_{prop}^2 \sim 0$.

- What does $p^2 = 0$ mean?

On-shell particle \rightarrow externally propagating state.

- So, as $\Theta_{13} \rightarrow 0$, propagator starts to go on-shell & quark is propagating over a long-distance / timescale.

\Rightarrow Timescale for gluon emission

$$t_g \sim \frac{1}{\sqrt{E_1 E_3} \Theta_{13}} \quad \& \text{ this can be } \gg \frac{1}{\Lambda_S}$$

($\sim \frac{1}{1000}$)

\rightarrow Precisely entering the $t \sim \frac{1}{1000}$ regime we ~~are~~ need to be insensitive to.

⇒ Indicates a sensitivity to long distance physics → known as an infrared (IR) divergence

- Now, in reality nothing is ∞ here.
As $t_f \sim \frac{1}{\lambda_{QCD}}$ our pQCD theory cannot be applied.

- Disaster?? In fact, not so.

What have we missed? Virtual corrections

$$M_{q\bar{q}}^V = \text{diagram} \sim \int \frac{d^4l}{(2\pi)^4}$$



The diagram shows a loop with a wavy line and a straight line. The loop momentum is labeled l . The diagram is drawn in red.

Q This also has IR divergences as

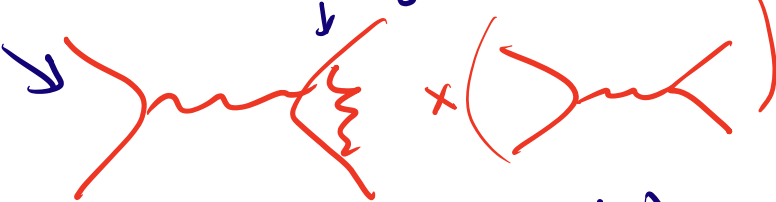

$$\text{Loop } E_l \rightarrow 0 \quad \text{or} \quad p_l \parallel p_1, p_l \parallel p_2$$

Remember our final state is "everything":

$$M_{\text{NLO}} = M^R + M^V$$

\rightarrow  + 

both need to be included:

In particular \rightarrow  \times 

$= \mathcal{O}(\alpha_s)$ (as in real)

Including everything, find that IR divergences in the real emission & the virtual diagrams exactly cancel.

Schematically:

$$\lim_{E_g \rightarrow 0} (R+V) = 0$$

\uparrow \uparrow
 $\nu + \frac{1}{2}$ $\nu - \frac{1}{2}$

$$\lim_{\Theta \rightarrow 0} (R+V) = 0$$

\Rightarrow Total cross section at NLO (NNLO...)

is not sensitive to long distance physics

& our factorized approach is maintained.

Not a coincidence, but a general result:

Kinoshita - Lee - Nauenberg (KLN)

Theorem (1962/63): all IR divergencies
will cancel for suitably defined observable



* Key in HNC physics: Not enough that $\alpha_s(M_Z)$ is small.

In practice:

* $R+V$ IR finite, but R & V
individually not. Could "in principle"
work w/ $R+V$ at all times, but in
practice need to work w/ them individually.

* To do this, we must regulate our

theory. Just as in renormalization:

$$\begin{aligned} \text{Dim. reg.} \quad \sigma_{2\bar{2}g} &\sim \left(\frac{4}{\epsilon^2} + \frac{3}{\epsilon} + \mathcal{O}(\epsilon) \right) \leftarrow \\ &\rightarrow \\ &\rightarrow \sigma_{2\bar{2}}^v \sim \left(-\frac{4}{\epsilon^2} - \frac{3}{\epsilon} + \mathcal{O}(\epsilon) \right) \end{aligned}$$

Dealing with these IR divergences is one of the key issues in higher order calculations.

- In particular, we want a numerical MC generator to produce a finite cross section.

- This goes under the name of "IR subtraction"

Write:

$$\begin{aligned} \sigma_{\text{NLO}} &= \int d\text{PS}_{2\bar{2}g} \left(\overbrace{\sigma_{2\bar{2}g} - \sigma_{\text{IR}}^{\text{sub}}} \right) \\ &\rightarrow + \int d\text{PS}_{2\bar{2}} \left(\sigma_{2\bar{2}}^v + \sigma_{\text{IR}}^{\text{sub}} \right) \end{aligned}$$

Where $\sigma_{\text{sub}} \sim \sigma_{2\bar{2}g}$ in $E_g \rightarrow 0$ limit
 $\mathcal{O} \rightarrow 0$

- solved / automated at NLO

- At NNLO more complicated, various methods on market.

IR limits & matrix element factorisation

The fact that $\sigma_{\mathcal{E}\bar{\mathcal{E}}g} + \sigma_{\mathcal{E}\bar{\mathcal{E}}}$ give finite results for arbitrary \mathcal{E} & $\bar{\mathcal{E}}$ phase space points, seems a little surprising & hints at universal behaviour!

We have:
$$d\sigma_{\mathcal{E}\bar{\mathcal{E}}g} \xrightarrow[\substack{\mathcal{E} \parallel g \\ \bar{\mathcal{E}} \parallel g \\ E_g \rightarrow 0}]{\downarrow} d\sigma_{\mathcal{E}\bar{\mathcal{E}}} \times \text{Universal}$$

This is indeed the case. let's show this.

$$\text{Result: } \langle |M_{\mathcal{E}\bar{\mathcal{E}}g}|^2 \rangle \propto \frac{S_{a1}^2 + S_{a2}^2 + S_{b1}^2 + S_{b2}^2}{S_{a3} S_{13} S_{23}}$$

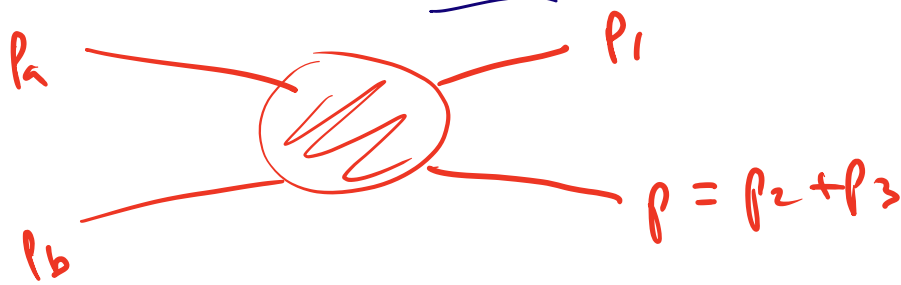
let's investigate how this looks in $\mathcal{E} \parallel \bar{\mathcal{E}}$ ($p_3 \parallel p_2$) limit.

$$p^2 = 0$$

Parametrize that limit: $p_2 = \gamma p$ $p_3 = (1-\gamma)p$

i.e. $p \equiv p_2 + p_3$ (collinear)

We have: $p_a + p_b = p_1 + p_2 + p_3 \equiv p_1 + p$



Consider $(p_a + p_b)^2 = (p_1 + p)^2 = 2(p_1 \cdot p)$
 $\Rightarrow S_{ip} = S_{ab} \quad (\equiv S)$

$(p_a - p)^2 = (p_1 - p_b)^2 \Rightarrow S_{ap} = S_{b1}$
(& also yet $S_{bp} = S_{a1}$)

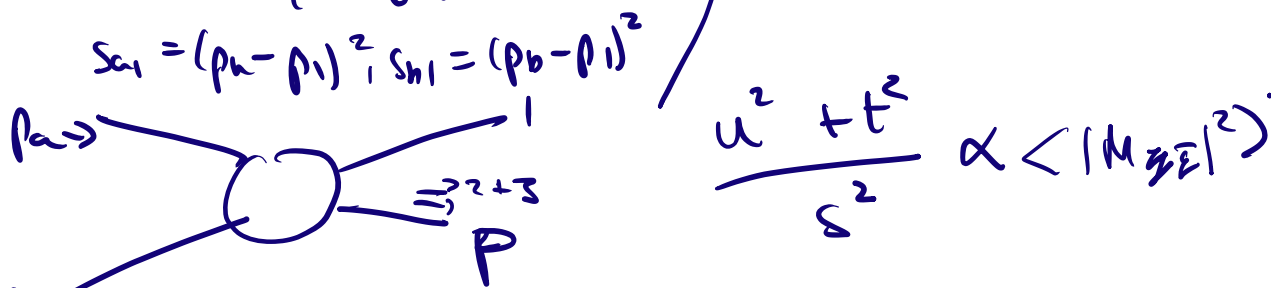
$S_{az} = 2(p_a \cdot p_2) = 2\gamma p_a \cdot p = \gamma S_{ap} = \underline{\gamma S_{b1}}$

& sum $S_{bz} = \gamma S_{a1}$

Finally: $\underline{S_{z3} = (1-\gamma) S_{ip} = (1-\gamma) S}$

$$\Rightarrow \frac{S_{a1}^2 + S_{a2}^2 + S_{b1}^2 + S_{b2}^2}{S_{13} S_{23}} \rightarrow \frac{S_{a1}^2 + \gamma^2 S_{b1}^2 + S_{b1}^2 + \gamma^2 S_{a1}^2}{S (1-\gamma) S_{23}}$$

$$= \frac{(1+\gamma^2)}{(1-\gamma)} \frac{1}{S_{23}} \left(\frac{S_{a1}^2 + S_{b1}^2}{S^2} \right)$$



$$\frac{u^2 + t^2}{s^2} \propto \langle |M_{g \rightarrow gg}|^2 \rangle$$

Including g_s^2 & color:

$$\langle |M_{g \rightarrow gg}|^2 \rangle \xrightarrow{2 \parallel g} 2 g_s^2 C_F \frac{(1+\gamma^2)}{(1-\gamma)} \frac{1}{S_{23}} \langle |M_{g \rightarrow gg}|^2 \rangle$$

We define:

$$P_{g \rightarrow gg}(\gamma) = C_F \frac{(1+\gamma^2)}{(1-\gamma)}$$

- This is the $g \rightarrow gg$ splitting function

- Universal object that relates to $g \parallel g$

\sim probability of emitting collinear radiation.

Only difference is in the splitting functions, P .

- Similar behaviour for $E_g \rightarrow 0$ when gluon is

$$\text{soft: } \langle |M_{g\bar{q}q}|^2 \rangle \rightarrow \langle |M_{g\bar{q}q}|^2 \rangle \frac{2S_{12}}{S_{12}S_{23}}$$

"Eikonal factor"

Summary

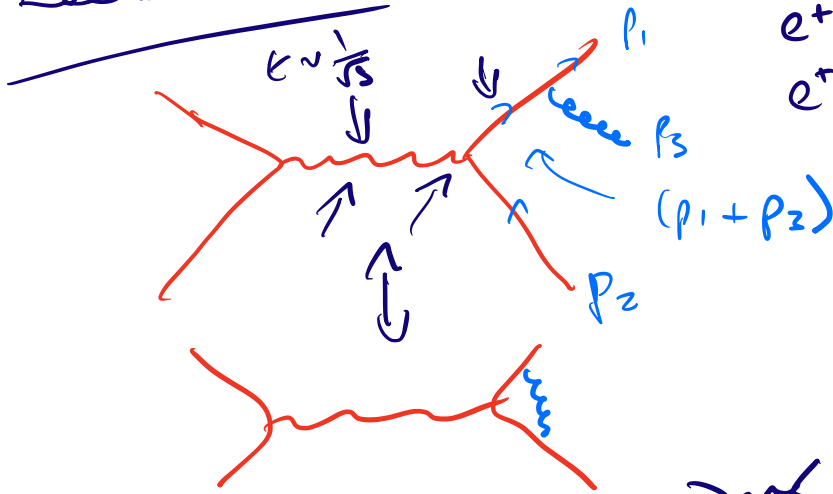
- QCD has simple & universal factorisation properties in soft & collinear limits.
- Required by cancellation of IR divergences.
- Splitting functions \sim probs of emitting collinear radiation. Connect low energy non-part physics to high energy scattering.

Lecture 5

$e^+e^- \rightarrow \text{hadrons}$

$e^+e^- \rightarrow q\bar{q} : \text{LO}$

$e^+e^- \rightarrow q\bar{q}g : \text{NLO Real}$



Recall:

$$\sigma_{n+1} = \sigma_n \int d\Omega \frac{d\theta_{ij}^2}{\theta_{ij}^2} \frac{\alpha_s}{2\pi} P_{q \rightarrow qg}(\beta)$$

$\beta \parallel g$
 arbitrary

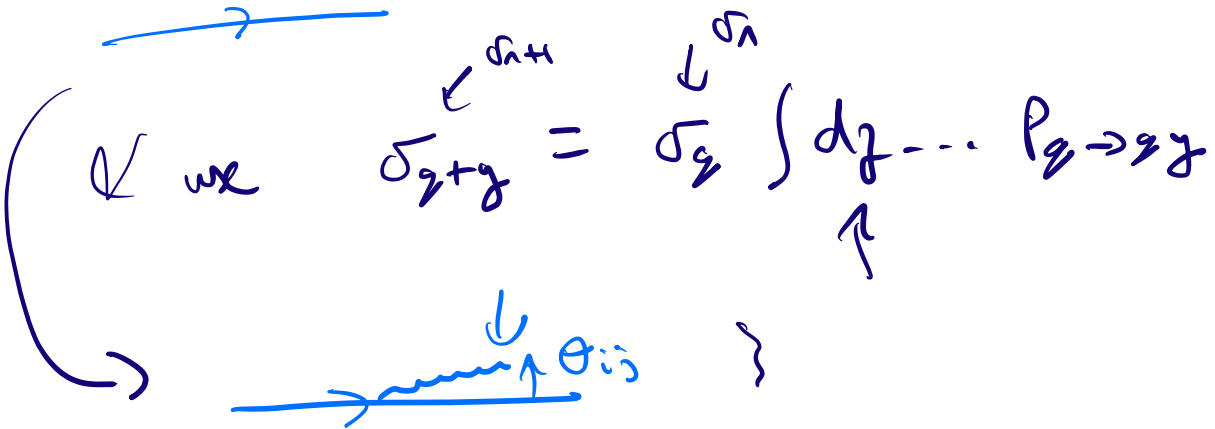
Role in parton showers:



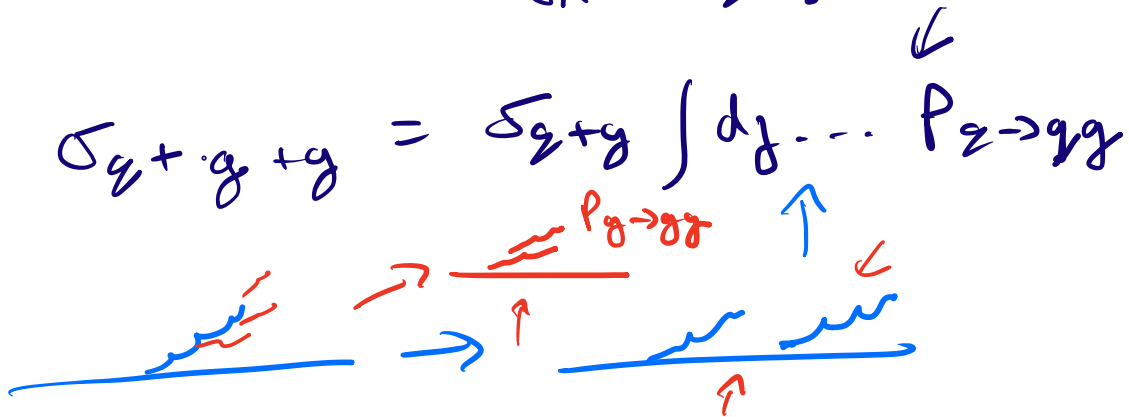
- Showering of outgoing quark line is independent from hard scatter.

- How does this work?

Start with outgoing quark:



Then iterate: $\sigma_{n+1} = \sigma_{q+g+g}$
 $\sigma_n = \sigma_{q+g}$



Keep on ...

- Parton shower provides a systematic way of doing this iteration

→ collimated 'shower' of QCD radiation
(i.e. jet) + soft emission.



without ever having to calculate the

full $\int g_s + n_g + m_g + k \bar{z}$

NB: * Skips details of how this actually works.

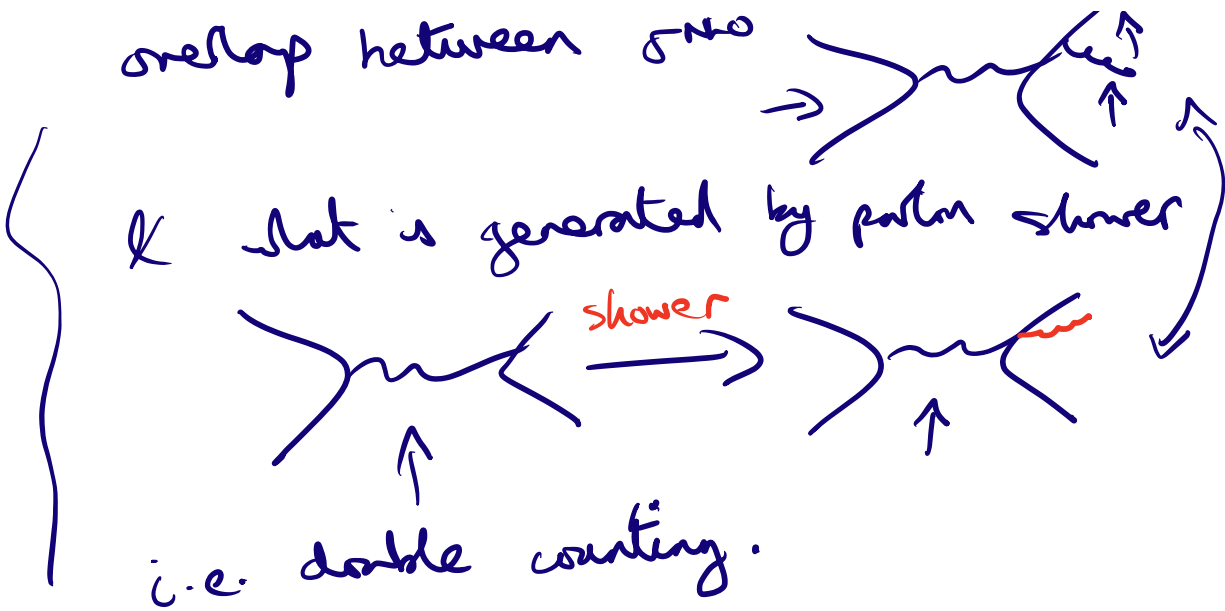
* Shower only gives good approx. to collinear/soft region of emission.

For e.g. $e^+e^- \rightarrow \gamma\bar{\gamma} + g$ 3 jet event



need a full calculation
NLO

* At NLO and beyond there is an



IR safety: Have discussed IR singularity
 cancellation in $e^+e^- \rightarrow$ hadrons.
 When use generally will this occur?
 i.e. what questions can we ask of our
 theory (= pQCD)?

Basic idea: Don't want sensitivity
 to soft/collinear emission.
 \Rightarrow Observable should be invariant
 under branching:

$$p_i \rightarrow p_i + p_R \quad \text{w/ } p_i \parallel p_R$$

$$\text{or } E_{i,R} \rightarrow 0$$

Formally:

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \circlearrowleft & \mathcal{O}(p_1, \dots, p_i, p_j, \dots) & \xrightarrow{\text{coll}} \mathcal{O}(p_1, \dots, (p_i + p_j), \dots) \\ & \downarrow E_j \rightarrow 0 & \\ & \mathcal{O}(p_1, \dots, p_i, p_j, \dots) & \rightarrow \mathcal{O}(p_1, \dots, p_i, \dots) \\ & & \downarrow \\ & & (p_i + p_j) \end{array}$$

smoothly

Have seen how this happens in $e^+e^- \rightarrow \gamma\bar{\gamma}(y)$

Problems: to discuss some examples-

* Number of factors?

* Certain jet algorithms?

* ...

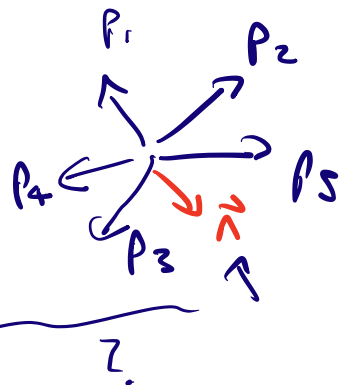
Resummation

- Stick for e^+e^- for now, but basic concepts carry across to the LHC.

Will consider:

"Event shape" = global measure of event.

Consider "thrust". Classic observable from LEP precision tests.

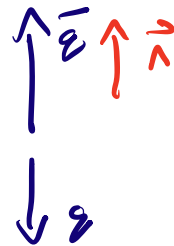
$$T \equiv \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$$


\vec{n} : unit vector

\sum : sum over all particles

e.g. Back-to-back $\vec{z} \bar{\vec{z}}$

$$\vec{n} \cdot \vec{p}_i = E_i \cos \theta_{in}$$

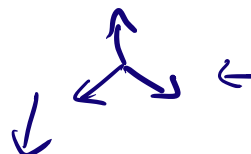


here: $|\cos \theta_{nz}| = |\cos \theta_{n\bar{z}}| = 1$

$$\Rightarrow T = 1$$

* spherically symmetric, large multiplicity event:

$$T = \frac{1}{2}$$



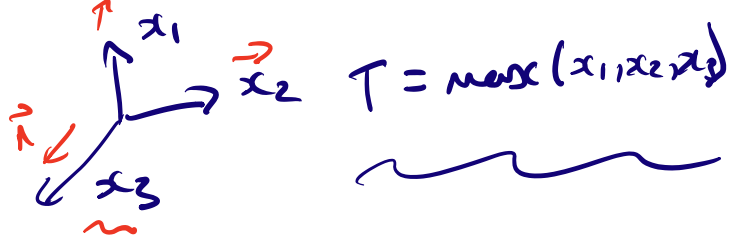
3 particle event: $\frac{2}{3} \leq T \leq 1$

Start with: $\sigma_{2\bar{2}} = \sigma_0 \left(\frac{\alpha_S}{2\pi} \right) \int da_1 da_2 \frac{(x_1^2 + x_2^2)}{(1-x_1)(1-x_2)}$

$(\sigma_{2\bar{2}} = N_c f_{\frac{2}{3}}^2 \frac{4\pi\alpha^2}{s})$

With a bit of work can express this as

$\frac{d\sigma}{dT}$: for each event



i.e. $\frac{d\sigma}{dT} = \sigma_{2\bar{2}} f(T - \max(x_1, x_2, x_3))$

After some work get:

$$\frac{d\sigma}{dT} = \sigma_0 \left(\frac{\alpha_S}{2\pi} \right) \left[\frac{2}{T(1-T)} + \frac{3(3T-2)(T-2)}{(1-T)^3} \right] \left(\frac{3T(T-1)+2}{0} \right) \ln \left(\frac{2T-1}{1-T} \right)$$

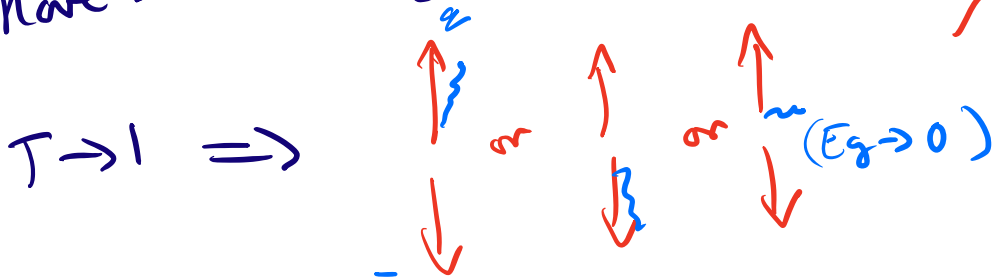
\uparrow
 $\ln(1-T)$

let's consider the $T \rightarrow 1$ limit:

$$\left\{ \frac{1}{\sigma_0} \frac{d\sigma}{dT} \xrightarrow{T \rightarrow 1} -C \frac{\alpha_s}{2\pi} \left[4 \frac{\ln(1-T)}{1-T} + \frac{3}{1-T} \right] \right.$$

which diverges as $T \rightarrow 1$

Have seen this before:

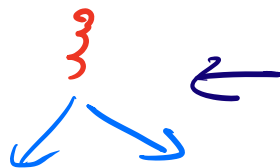


i.e. precisely where collinear / soft gluon emission in $q\bar{q} + g$ relevant.

- Now, as before if we are only interested in $e^+e^- \rightarrow$ hadrons, include virtual & everything OK.

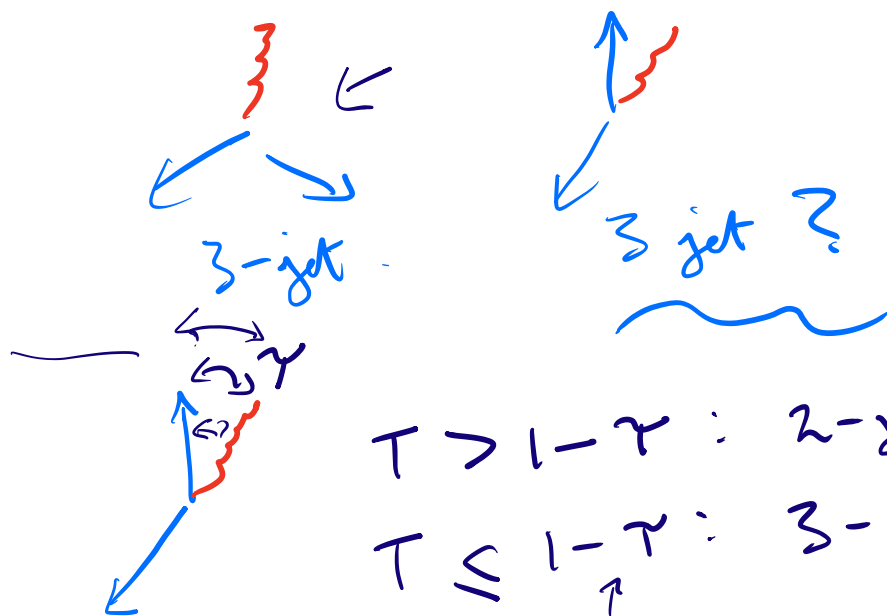
- But what if we want to ask other questions of our theory?

e.g. if a gluon is emitted with large enough energy & not collinear, have "3-jet" event:



Natural observable: fraction of events that
 are 2 or 3 jet-like, R_2 & R_3 .
 w/ $R_2 + R_3 = 1$.

- To do this we need to introduce a
 "resolution parameter" in order to define
 what is meant by 3 or 2 jet-like.



$T > 1 - \gamma$: 2-jet

$T \leq 1 - \gamma$: 3-jet

(c.f. jet radius R)

3-jet fraction: $R_3(\gamma) = \int_{2/3}^{1-\gamma} \frac{1}{\sigma_0} \frac{d\sigma}{dT} dT$

Focus on $T \ll 1$ part of σ : $= -\frac{\alpha_s}{2\pi} \int_{2/3}^{1-\gamma} \left(\frac{4 \ln(1-T)}{1-T} + \frac{3}{1-T} \right) dT$

$$= C_F \frac{\alpha_S}{2\pi} [2\ln^2(1-T) + 3\ln(1-T)]^{1-\gamma} \quad \downarrow_{2,3}$$

$$R_3(\gamma) = C_F \frac{\alpha_S}{2\pi} [2\ln^2 \gamma + 3\ln \gamma - 2\ln^2 \frac{1}{3} - 3\ln \frac{1}{2}] \quad \uparrow$$

IF kept expression:

$$R_3(\gamma) = C_F \frac{\alpha_S}{2\pi} [2\ln^2 \gamma + 3\ln \gamma + \text{regular as } \gamma \rightarrow 0]$$

↑ finite

So, what about $\gamma \ll 1$ limit?
Then, the $\ln^2 \gamma$ dominates &

$$R_3(\gamma) \rightarrow C_F \frac{\alpha_S}{2\pi} 2\ln^2 \gamma$$

$$R_2(\gamma) \rightarrow 1 - C_F \frac{\alpha_S}{2\pi} \ln^2 \gamma$$

\Rightarrow For small (but non-zero) γ
our predicted fraction of 3-jet events
is arbitrarily large & $R_2 < 0$.

What is going on?

Solution: we are expanding in α_s
 \Rightarrow can only trust our results in regime where expansion is reliable.

Here, we have: $R_2(\tau) = 1 - \underbrace{\left(C_F \frac{\alpha_s}{2\pi} 2 \ln^2 \tau \right)}_{\mathcal{O}(\alpha_s^2)} + \underbrace{\dots}_{\text{NLO}}$

So the NLO term for $\tau \rightarrow 0$
 we have $\alpha_s \ln^2 \tau \gg 1$ X

i.e. pQCD is breaking down.

- But all is not lost! At higher orders we will find:

$$R_2(\tau) \stackrel{\tau \rightarrow 0}{=} 1 - C_F \frac{\alpha_s}{2\pi} 2 \ln^2 \tau + \frac{1}{2} \left(C_F \frac{\alpha_s}{2\pi} 2 \ln^2 \tau \right)^2 + \dots$$

i.e. pQCD is breaking down, but the above is a Taylor expansion

$$\left(e^{-x} = 1 - x + \frac{x^2}{2} - \dots \right)$$

⇒ know this $\sim \ln^2 \tau$ dominant (as $\tau \rightarrow 0$)

contribution to all orders in α_s :

$$\Rightarrow R_2(\tau) = \exp\left[-C_F \frac{\alpha_s}{2\pi} 2\ln^2 \tau\right]$$

Order-by-order this will jump around all over the place, but the 'exact' resummed result is OK:

$$R_2(\tau) \rightarrow 0 \quad \text{as } \tau \rightarrow 0$$

Interpretation: Recall NLO term:

$$R_2(\tau) = 1 - C_F \frac{\alpha_s}{2\pi} 2\ln^2 \tau$$

⇒ can interpret $C_F \frac{\alpha_s}{2\pi} 2\ln^2 \tau$ as 'naive'

probability of radiation, leading to 3-jet event.

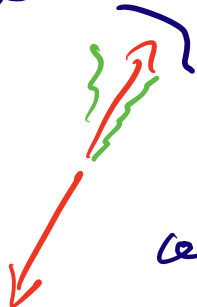
$$\Rightarrow \Rightarrow \exp\left[-C_F \frac{\alpha_s}{2\pi} 2\ln^2 \tau\right] = \exp\left[-P_{\text{emission}}^{\text{naive}}\right]$$

Jargon: "Sudakov Form Factor"

This is a Poissonian probability of no emission.

$$\left(P_{\text{emission}}^{\wedge}(\lambda) = \frac{\lambda^n e^{-\lambda}}{n!} \Rightarrow P(0) = e^{-\lambda} \right)$$

Sr: Probability of no gluon emission in $\alpha \rightarrow 0$ limit is 0. There is always extra radiation, if we look hard enough.

 * Essence of resummation: certain enhanced terms in perturbative expansion can be calculated to all orders, stabilises the result.

$Z_{\beta\perp}$ 

* MC generators implicitly perform this resummation.



* Have considered "leading" $\ln^2 \tau$ term.

$$\text{Recall: } R_3(\tau) = C \frac{\alpha_s}{2\pi} \left[\underbrace{2 \ln^2 \tau}_{\uparrow} + \underbrace{3 \ln \tau}_{\uparrow} \right]$$

$3 \ln \tau$: single log - can / should also resum.

Jargon: $\alpha_s \ln^2 \tau$: "leading" log(LL)
 $\alpha_s \ln \tau$: "next-to-leading" log(NLL).

Hadronic initial states & PDFs

Move to hadron-hadron (hadron-lepton) collisions.

* Main question: how do we deal w/ strongly interacting particles in

the initial state?

Solution: conceptually similar to before
i.e. a Factorisation of long-distance
physics (= binding of quarks/gluons in
~~the~~ initial state) from high energy
interaction.

Jumping ahead a bit, for ep collisions we
write:

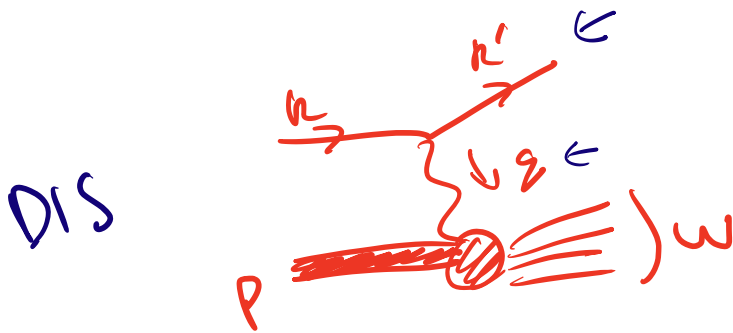
$$\sigma_{ep \rightarrow M} = \sum_i \int dx f_i^p(x, Q^2) \hat{\sigma}_{ei \rightarrow M}(Q^2, x) + \mathcal{O}\left(\frac{1/Q^2}{Q^2}\right)$$

f_i^p : Parton distribution functions (PDFs)

\sim probability of finding a parton w/
momentum fraction x at energy Q^2 .

Deep inelastic Scattering

Consider lepton-proton scattering:

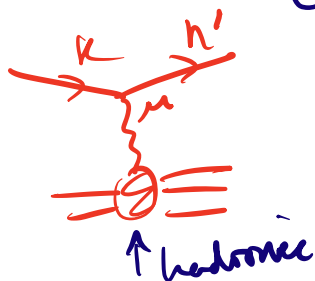


kinematics: $Q^2 = -q^2 > 0, p^2 = m_p^2$
 $W^2 = (q+p)^2 \geq m_p^2$

DIS: $Q^2 \gg m_p^2$ (Deep)
 $W^2 \gg m_p^2$ (inelastic)

Other useful quantities: 'Bjorken' $x \equiv \frac{Q^2}{2(p \cdot q)}$
 Inelasticity $y \equiv \frac{q \cdot k}{k \cdot p}$

What can we say about this process?



$M_{DIS} \propto \bar{u}(k') \gamma^\mu u(k) \cdot J_\mu^{hadronic}$

This defines the unknown object:



$$\Rightarrow \sigma \propto \langle |M|^2 \rangle \propto \text{Tr}(k^\mu \delta^\nu k'^\mu \delta^\nu) \sum_{\text{had.}} (J_\nu^{\text{had}})^2$$

$$\equiv \underline{L}^{\mu\nu} W_{\mu\nu}$$

$L_{\mu\nu}$: leptonic tensor ✓ — ?

$W_{\mu\nu}$: hadronic tensor.

What do we know about $W_{\mu\nu}$?

→ carries Lorentz indices

→ Gauge invariance: $q^\mu W_{\mu\nu}(q) = 0$

Putting this all together, end up with most general form:

$$W_{\mu\nu} = \underline{F_1(x, Q^2)} (g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2})$$

$$+ \frac{2x}{Q^2} \underline{F_2(x, Q^2)} (p_\mu + \frac{q_\mu}{2x})(p_\nu + \frac{q_\nu}{2x})$$

x, Q^2 : only independent Lorentz scalars in the process.

$F_{1,2}$: Structure Functions: encode the internal structure of the proton.

First step:

$$d\sigma_{DIS} \propto L^{\mu\nu} W_{\mu\nu} dPS \leftarrow$$
$$\propto \text{Tr}(k \gamma^\mu k' \gamma^\nu) \left[F_1(x, Q^2) (g_{\mu\nu} + \frac{q_\mu q_\nu}{Q^2}) \right. \\ \left. + \frac{2x}{Q^2} F_2(x, Q^2) (p_\mu + \frac{q_\mu}{2x})(p_\nu + \frac{q_\nu}{2x}) \right]$$

$$\dots \frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{2Q^4} (xy^2 F_1(x, Q^2) + (1-y) F_2(x, Q^2))$$

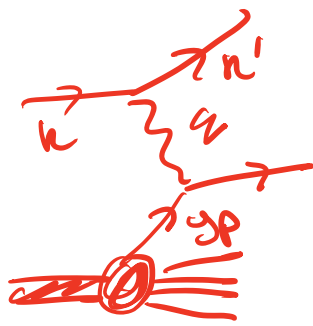
OK, but how do we calculate F_1 & F_2 ?

Parton Model (PM) : Proton consists

of point-like constituents ('partons')

& DIS mediated by scattering of leptons off these.

- Partons carry some momentum fraction q of the proton momentum ($0 < q < 1$)



$$e_q \rightarrow e_q$$

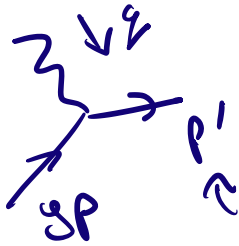
$$\frac{d^2\sigma}{dx dQ^2} = \sum_{q, \bar{q}} \int_0^1 dy f_{21\bar{2}}(y, Q^2)$$

PDF

$$\cdot \frac{d^2 \hat{\sigma}(e_q(y, p))}{dx dQ^2}$$

parton level cross section

One consequence: $x = \frac{Q^2}{2p \cdot q}$



$$y p + q = p'$$

$$(m_p^2, m_q^2 \ll Q^2)$$

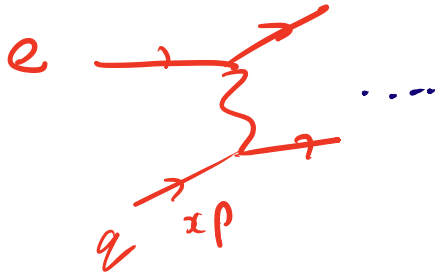
$$c \rightarrow 0$$

$$(y p + q)^2 = p'^2 = 0$$

$$\cancel{y^2 p^2} + q^2 + 2y p \cdot q = 0$$

$$\Rightarrow y = \frac{-q^2}{2p \cdot q} = \frac{Q^2}{2p \cdot q} = x$$

⚡ Calculate lepton-quark cross section



$$\frac{d\hat{\sigma}}{dQ^2} = 2\pi\alpha^2 e_e^2 \frac{(1+(1-y)^2)}{Q^4}$$

Use: $\int_0^1 dx \delta(x-y) = 1$

$$\Rightarrow \frac{d^2\hat{\sigma}}{dx dQ^2} = \frac{2\pi\alpha^2 [1+(1-y)^2] e_e^2 f(x-y)}{Q^4}$$

$$= \frac{4\pi\alpha^2}{Q^4} \left(\frac{y^2}{2} + (1-y) \right) e_e^2 f(x-y)$$

$$\Rightarrow \frac{d^2\sigma(e_p)}{dx dQ^2} = \sum_{\epsilon_1 \bar{\epsilon}_2} \int_0^1 dy f_{\epsilon_1 \bar{\epsilon}_2}(y) \frac{d^2\hat{\sigma}(e_e)}{dx dQ^2}$$

$$= \frac{4\pi\alpha^2}{Q^4} \left[\frac{y^2}{2} + (1-y) \right] \sum_{\epsilon_1 \bar{\epsilon}_2} e_e^2 f_2(x)$$

Recall our general expression:

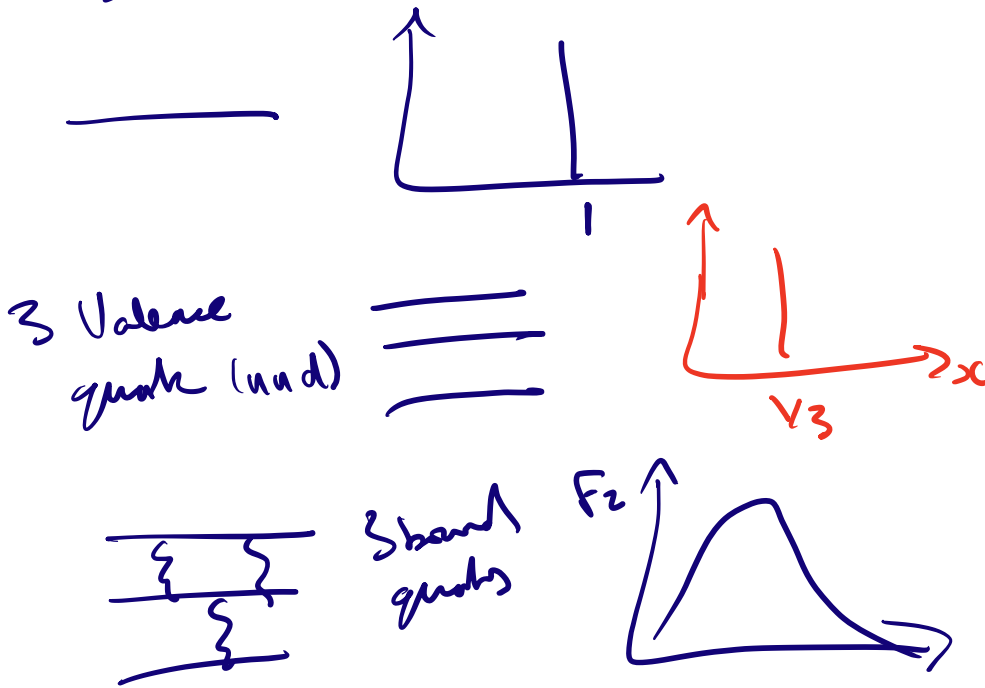
$$\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{xQ^2} \left[x \frac{y^2}{2} F_1(x, Q^2) + (1-y) F_2(x, Q^2) \right]$$

⇒ lead off:

$$\sum_{\alpha} F_{\alpha}(x) = F_2(x) = x \sum_{\alpha} e_{\alpha}^2 f_{\alpha}(x)$$

* Callen-Gross relation

Sketch $F_2(x) : (= e_2^2 \sum f_{\alpha}(x))$



But this is a simplified picture!

- Next step: consider impact NHO corrections.

In particular, at $\hbar \omega$ f_{112} only depend

2 " " " " " "

on x & at Q^2 : "Broken scaling"
 Works OK, broken by QCD corrections.

Going beyond LO

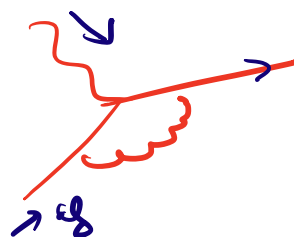
Higher order diagrams:



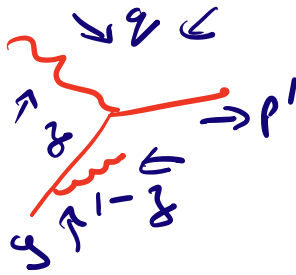
- As in $e^+e^- \rightarrow q\bar{q}\gamma$, IR singularities everywhere!
- As before, much of this cancels between R & V.

However, same w/ diagram C.

We would want cancellation w/ diagram

(B):  $\frac{d\sigma^V}{dx dQ^2} \sim g(x)$

What about C?



We have:

$$(y \delta p + q)^2 = 0$$

$$\Rightarrow y = \frac{x}{\delta} > x$$

$\delta \in 0 < \delta < 1$

$\leftarrow y = x \checkmark$

$$\Rightarrow \frac{d\sigma^R}{dx dQ^2} \sim \int dz g\left(\frac{x}{z}\right)$$

- (B) & (C) are weighted by different PDFs. This means we can never get IR cancellation from this & indeed we don't.

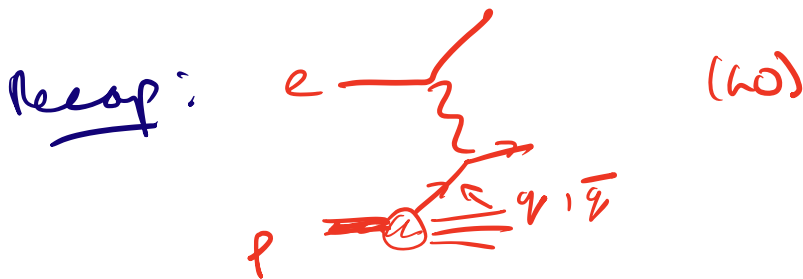
Solution: Refine PDFs such that these collinear in (C) are absorbed into them \Rightarrow assign some amount of long distance collinear emission to PDFs.

Makes sense that we do this:

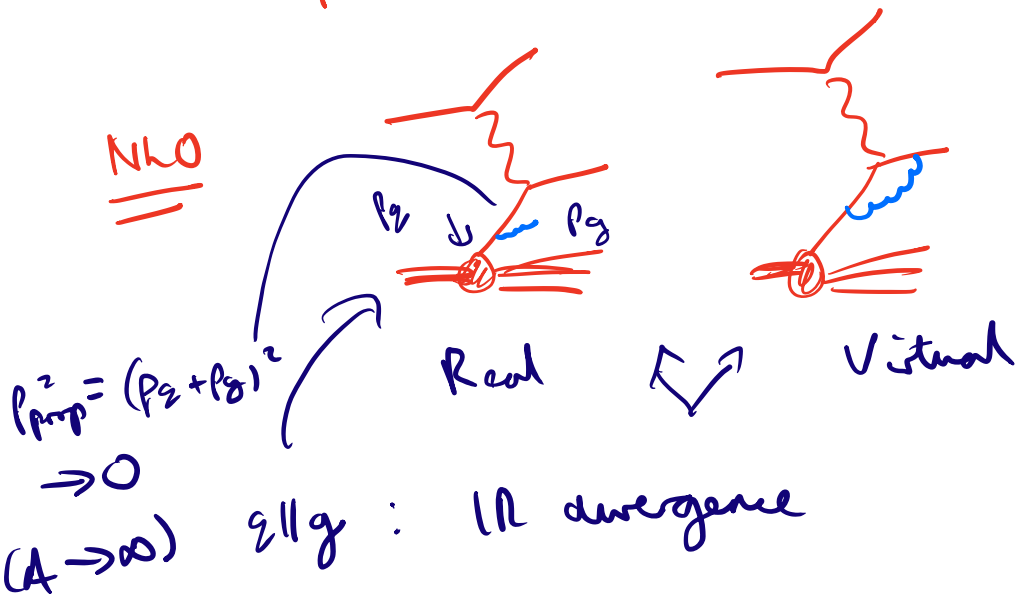


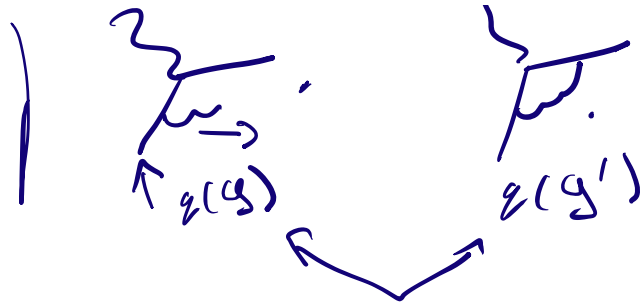
i.e. gluon emission is happening at a time $t \sim \frac{1}{E_g \theta_{gg}} \gg \frac{1}{\sqrt{Q}}$ before the

NLO process \Rightarrow have to assign to PDFs.



NLO





Solution: define PDFs such that these
 linear divergences / effects is absorbed
 into PDFs

Doing this introduces a new "factorisation
 scale", μ_F :

$$f_2(x) \rightarrow f_2(x, \mu_F^2)$$

Why? Give brief picture of \otimes that

Think in terms of transverse momentum, k_\perp
 of gluon wrt quark

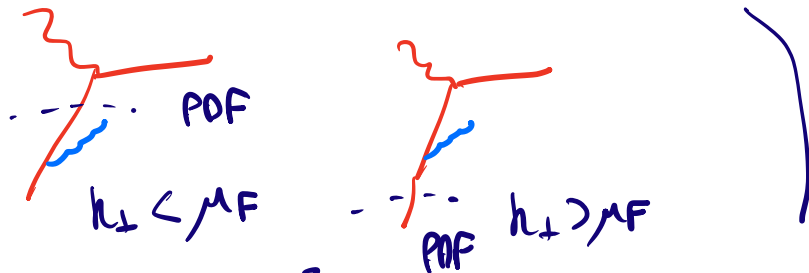
Recall: $d\sigma_{2 \rightarrow 2g} \approx d\sigma_{2 \rightarrow 2} \frac{\alpha_s}{2\pi} \int P_{22}(z) \frac{dk_\perp^2}{k_\perp^2} \frac{dz}{z}$

(collinear: $k_\perp \rightarrow 0$)
 $z \rightarrow zg$

$\int_0^{\infty} \frac{dk_{\perp}^2}{k_{\perp}^2}$: collinear singularity

Essence of PDF factorisation:

- Introduce factorisation scale μ_F , below which contributions to $\int \frac{dk_{\perp}^2}{k_{\perp}^2}$ are assigned to PDFs:

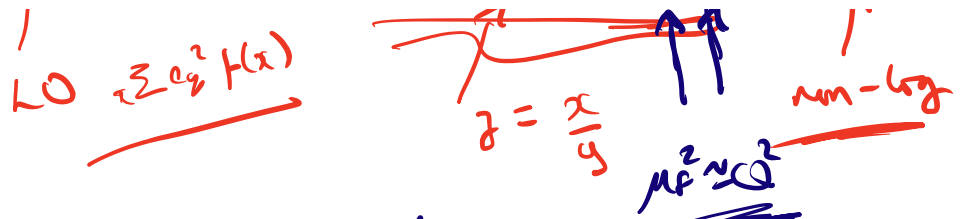


Then $\int_0^{Q^2} \frac{dk_{\perp}^2}{k_{\perp}^2} \rightarrow \int_{\mu_F^2}^{Q^2} \frac{dk_{\perp}^2}{k_{\perp}^2} = \ln\left(\frac{Q^2}{\mu_F^2}\right)$

In terms of our fundamental $F_2 \leftrightarrow f_2$ relation:

$$\rightarrow F_2(x, Q^2) = x \sum e_f^2 \int_x^1 \frac{dy}{y} f_2(y, \mu_F^2)$$

$$\left[\delta\left(1 - \frac{x}{y}\right) + \frac{\alpha_s}{2\pi} P_{22}\left(\frac{x}{y}\right) \ln\left(\frac{Q^2}{\mu_F^2}\right) + R_{22}\left(\frac{x}{y}\right) \right]$$



MF: arbitrary parameter.

$$\frac{\partial f_2}{\partial \ln \mu_F^2} = 0 \text{ to order we calculate in}$$

i.e. prediction should not depend on where
how we assign all-order radiation to the PDFs.

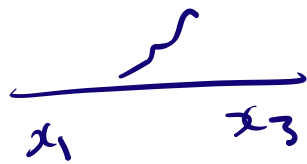
We have:
$$\frac{\partial f_q}{\partial \ln \mu_F^2} = \frac{\alpha_s}{2\pi} \int P_{qq} \left(\frac{x}{y} \right) f_q(y, \mu_F^2)$$

This is the DGLAP evolution eqⁿ.

- Tells us how PDFs change w/ energy
scale due to parton splittings

Encoded in some universal splitting
functions.

DGLAP, general case:



$$q(x_1) \rightarrow q(x_3 < x_1) + g$$

- General trend: as $\mu_F \uparrow$, get more radiation, pushes partons to lower x .

* μ_F^2 , Q^2 ? To predict F_2 measured

at Q^2 , take $\mu_F^2 = Q^2$ (quite a lot of radiation absorbed into PDFs!), we have:

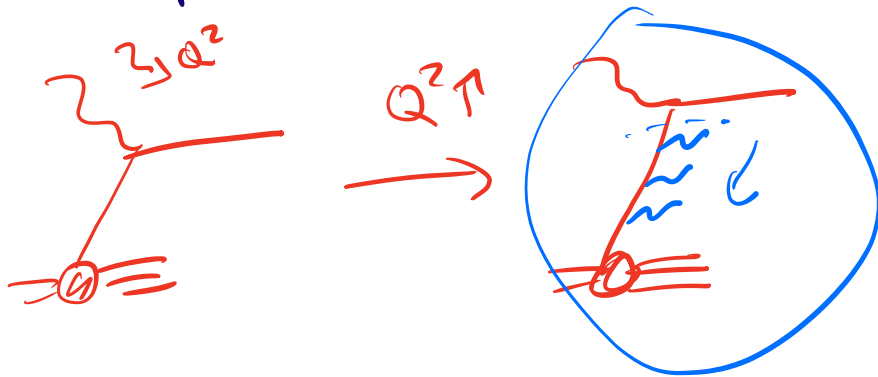
$$F_2(x, Q^2) = x \sum e_f^2 \int_x^1 \frac{dy}{y} f_2(x, Q^2)$$

$\left[\delta(1 - \frac{x}{y}) + R_{22}(\frac{x}{y}) \right]$

$\Rightarrow F_2$ has Q^2 dependence, driven by DGLAP evolution of PDFs

- Note this is a genuine physical effect.
In essence: as Q^2 increases we have

more phase space for extra radiation.

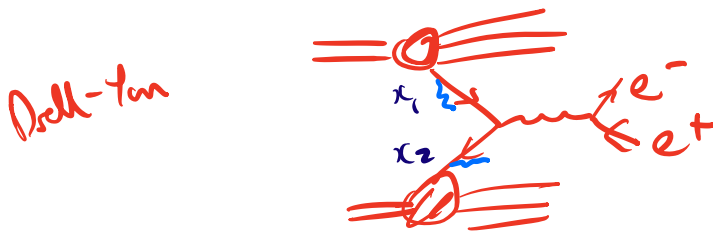


DGLAP is simply a consistent way for us to keep track of that.

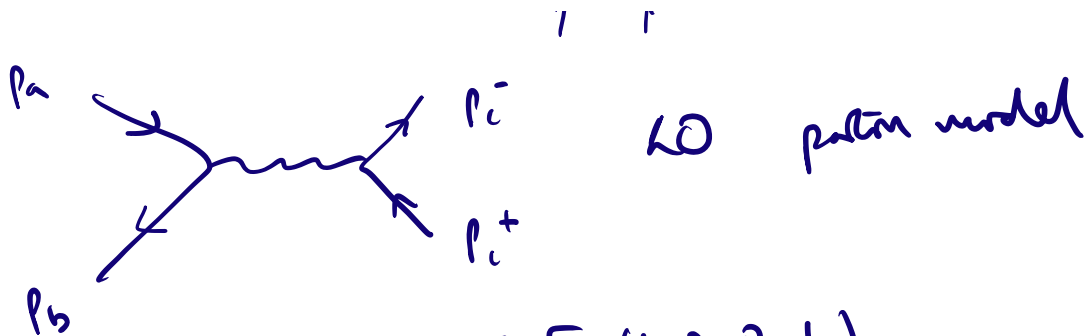
* PDFs themselves are not calculated
 → they are fit.

Hadronic collisions?

Picture much the same:



$$\sigma_{pp} = \sum_{ij} \int dx_1 dx_2 f_i(x_1, \mu_F^2) f_j(x_2, \mu_F^2) \hat{\sigma}_{ij}(x_1, x_2, \alpha_s, Q^2/\mu_F^2)$$



$$p_a = x_1 p_1 = x_1 \frac{\sqrt{s}}{2} (1, 0, 0, 1)$$

$$p_b = x_2 p_2 = x_2 \frac{\sqrt{s}}{2} (1, 0, 0, -1)$$

Find: $x_1 = \frac{M_{\mu}}{\sqrt{s}} e^{Y_{\mu}}$ $x_2 = \frac{M_{\mu}}{\sqrt{s}} e^{-Y_{\mu}}$

w/ $Y_{\mu} = \frac{1}{2} \ln \left(\frac{E_{\mu} + p_{\mu}^z}{E_{\mu} - p_{\mu}^z} \right)$

Beyond LO, mapping not so simple.

↳ Use to constrain PDFs.

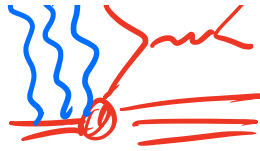
Factorization: The $f_2(x, p^2)$ in e.g. σ^{04}

is same object as σ^{015} .

⇒ Can fit PDFs to one process & predict for another.

↓ showing there don't
~~1.0.0~~ effect unambiguously
 1 PDF

Key issue:



of 1000.

Jet algorithms:

* Experimentally: collimated cluster of hadronic energy. Often know them when you see them.

* But want to be more quantitative

Jet algorithm: Reduces complexity of

final state by combining large number of particles into a jet.



- Allows us to trace back to hard scattering process in a way that (should be) insensitive to soft / collinear radiation.

At LHC: - different ways of doing this.

All rely on a measure of distance between

$$\Delta R_{ij}^2 = \Delta y_{ij}^2 + \Delta \phi_{ij}^2$$

→

distance measure: $d_{ij} = \min(p_{i\perp}^p, p_{j\perp}^p) \frac{\Delta R_{ij}^2}{R^2}$

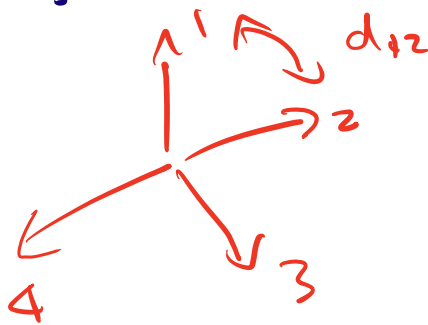
$p=1$: "k_t-algorithm"

$p=0$: "Cambridge - Aachen"

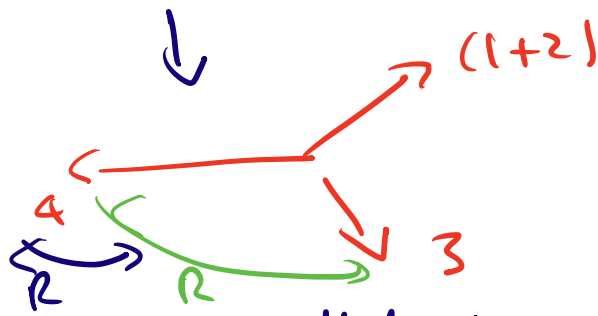
$p=-1$: "anti-k_t"

- All cluster particles according to what is closest in terms of distance measure d_{ij} .

Simple picture:



- If d_{12} is smallest R want jets w/
 $R > \Delta R_{12}$



d_{43} next smallest : combine or not depending on size of R .

EW-physics

Crucial point : SM is a gauge theory
 $\&$ this entails non-trivial relationships
 between unrelated observables:

$M_W, M_Z, G_F \dots$

$$\sin^2 \theta_W = 1 - \frac{M_W^2}{M_Z^2}$$

→ SM over-constrained. Idea behind
 EW precision tests:

- 2 useful examples: - $\sin^2\theta_w$ @ LNC
- M_w

