Entanglement from Quantum Matter to Spacetime

Sean Hartnoll (University of Cambridge)

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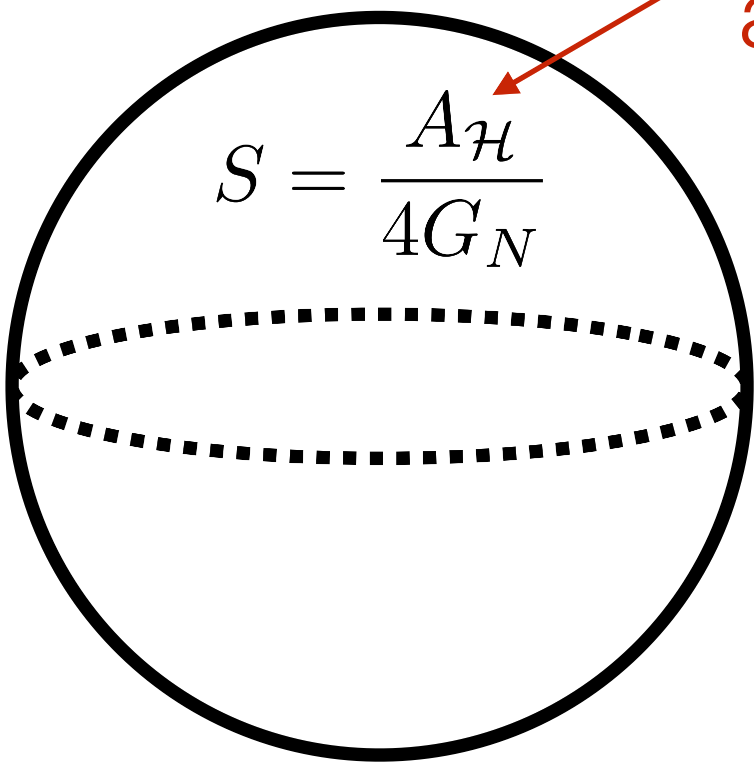
Motivation

Some important themes in 20th century theoretical physics:

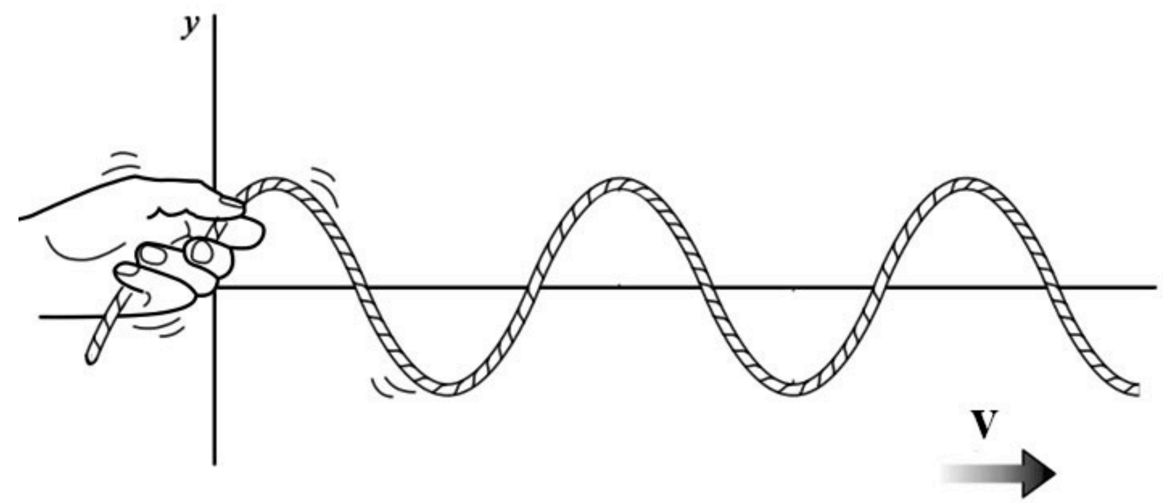
- Entropy counts microscopic degrees of freedom! [Boltzmann]
- Black holes have entropy! [Bekenstein and Hawking]
- In certain cases can be matched microscopically! [Strominger and Vafa: SUSY + CFT]

Motivation

Area of the horizon,
a special sphere



$$S = \frac{A_{\mathcal{H}}}{4G_N}$$



$$S = \log d(P)$$

Number of ways a (highly supersymmetric)
string can carry momentum P

[Hardy-Ramanujan formula for partitions]

Motivation

Quantum entanglement promises to be an organizing principle for 21st century physics.

Entanglement is an obstruction to factorizing quantum states. A simple example is: $|\psi\rangle = |\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle$

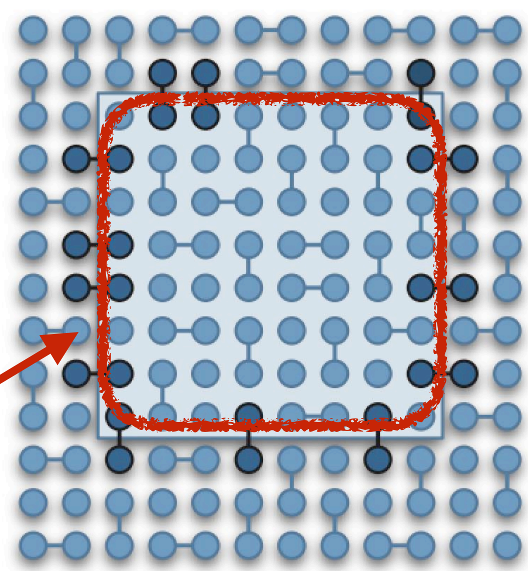
In many-body physics often have a very complicated wavefunction of many spins etc. How to characterize?

The entanglement in the wave function between geometrically separated degrees of freedom has proven to be a very useful piece of information.

Motivation

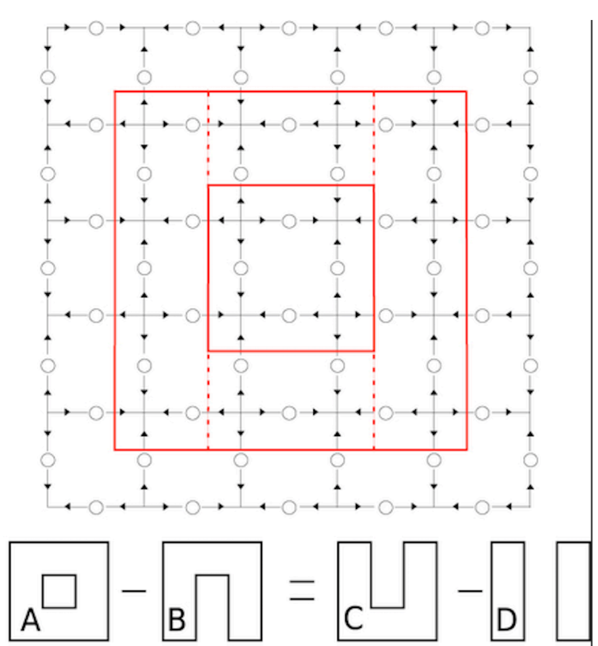
Local interactions ↔ ‘area law’ entanglement [Bombelli et al., Srednicki, …]

$S \sim L$



With local interactions, most entanglement in the wavefunction is short range. Easy to simulate!

Gauge interactions ↔‘topological’ entanglement [Kitaev-Preskill, Levin-Wen, …]



Emergent Gauss law constraints captured by a ‘topological’ term: $S \sim L + \gamma$

Motivation

- In ‘holography’ dynamical spacetime emerges from the quantum state of a non-Abelian gauge theory.
- Gravity is a collective excitation of this state, much as sound waves are collective excitations of solids.
- How to characterize this quantum state?
- Recent-ish development:
  Emergent spacetime requires a lot of entanglement
  [Ryu-Takayangi, Lewkowyck-Maldacena, …]

Motivation

If space is divided by a minimal surface:

$$S_{\mathrm{EE}} = \frac{A}{4G_N} + \text{matter contribution}$$

Major generalization of Bekenstein-Hawking entropy.

What is the microscopic (bulk) origin of this entanglement?
How is spacetime actually made?
<u>Needed: Strominger-Vafa for the 21st century</u>.

Large N matrices

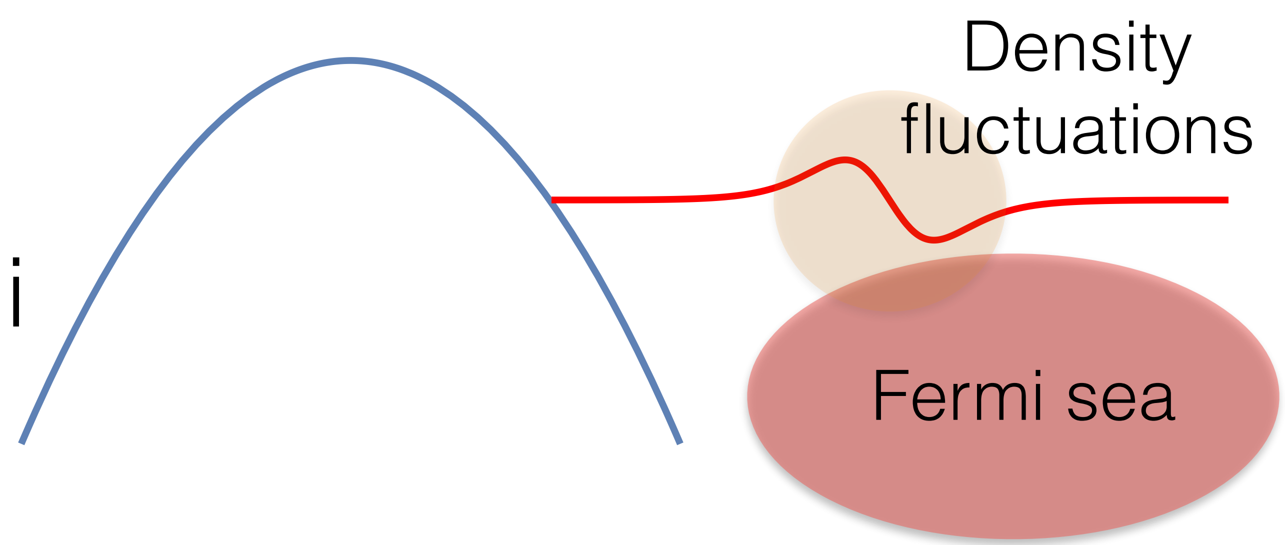
- The simplest version of the non-Abelian gauge theories that appear in holography are large N matrix quantum mechanics.

- Would like to understand the entanglement of matrix wavefunctions. Will look at simple low dimensional models that do not give rise to gravity but do have an emergent space.

- Very simplest example: understood since 90's how a single large N matrix gives rise to an emergent one dimensional space.

Baby model: single matrix

- Singlet sector of $\mathcal{L} \sim \mathrm{tr}\left[\dot{M}^2 - V(M)\right]$ described by eigenvalues {λ} of M.
- Eigenvalues are non-interacting fermions. Fermi sea builds 1d space.

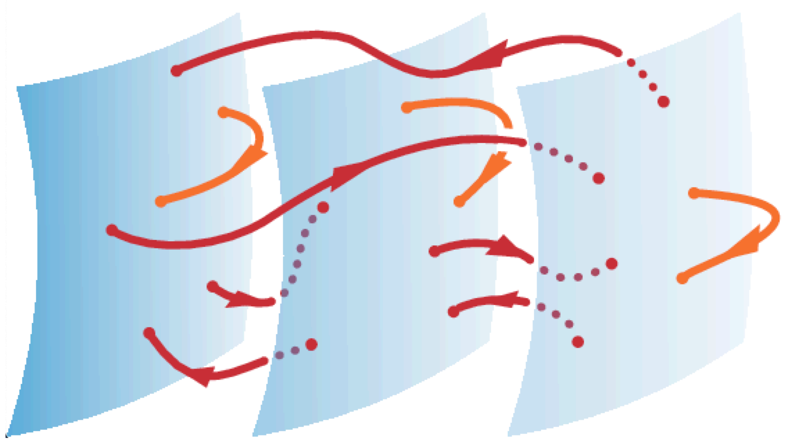


- Entanglement of interval [λ_1,λ_2] using conventional many-body methods. Matches emergent 1+1 ‘tachyon’ field [Das 95, Hartnoll-Mazenc 15]:

$$S_{\mathrm{EE}} = \frac{1}{3}\log\frac{\Delta x}{g}$$

Beyond one matrix

- Eigenvalues are not enough. 'Off-diagonal' modes stretching between coincident branes essential for 'grown up' holography [cf Maldacena].



- Noted by [Das-Kaushal-Mandal-Trivedi 20] that a class of proto-geometric partitions are obtained by diagonalizing one of the matrices (e.g. X_1).

- Eigenvalues of X_1 dealt with as in the single matrix case. Induces a block decomposition of the remaining matrices.

Plan

- Solvable matrix quantum mechanics with two matrices.

- Compute the entanglement of a geometric partition. Treatment of off-diagonal modes inspired by entanglement in gauge theories. This analogy was emphasized by [Hampapura-Harper-Lawrence 20]

- Obtain emergent 2d ‘area law’ and topological-like subleading correction.

Work with Alex Frenkel: 2111.05967

Quantum Hall Matrix Model

Quantum Hall phases: incompressible droplet supporting emergent Chern-Simons dynamics.

Minimal microscopic realization: discretize the area-preserving diffeos of the droplet into U(N). [Susskind 01]

IR-regulated version by [Polychronakos 01]:

$$H = \mathrm{tr}\left(X^2 + Y^2\right) \qquad [X_{ab}, Y_{cd}] = i\delta_{ad}\delta_{bc}$$

(Gauss law) $\quad -i[X,Y] + \Psi\Psi^\dagger = k$

Quantum Hall Matrix Model

- Ground state [Hellerman-Van Raamsdonk 01]:

$$|\psi\rangle = \left[\epsilon^{a_1\ldots a_N}\Psi^\dagger_{a_1}(\Psi^\dagger Z^\dagger)_{a_2}\cdots(\Psi^\dagger Z^{\dagger\, N-1})_{a_N}\right]^k |0\rangle$$

Here Z = X + iY.

- State simple in terms of variables {x,U,Ψ} where

$$X = UxU^\dagger\,, \qquad \Psi = U\widetilde{\Psi}$$

$$\psi = (\det U)^k \prod_{a<b}(x_a - x_b)^k e^{-\frac{1}{2}\sum_d x_d^2} \prod_c \widetilde{\Psi}_c^k e^{-\frac{1}{2}\sum_d |\widetilde{\Psi}_d|^2}$$

[Karabali-Sakita 01]

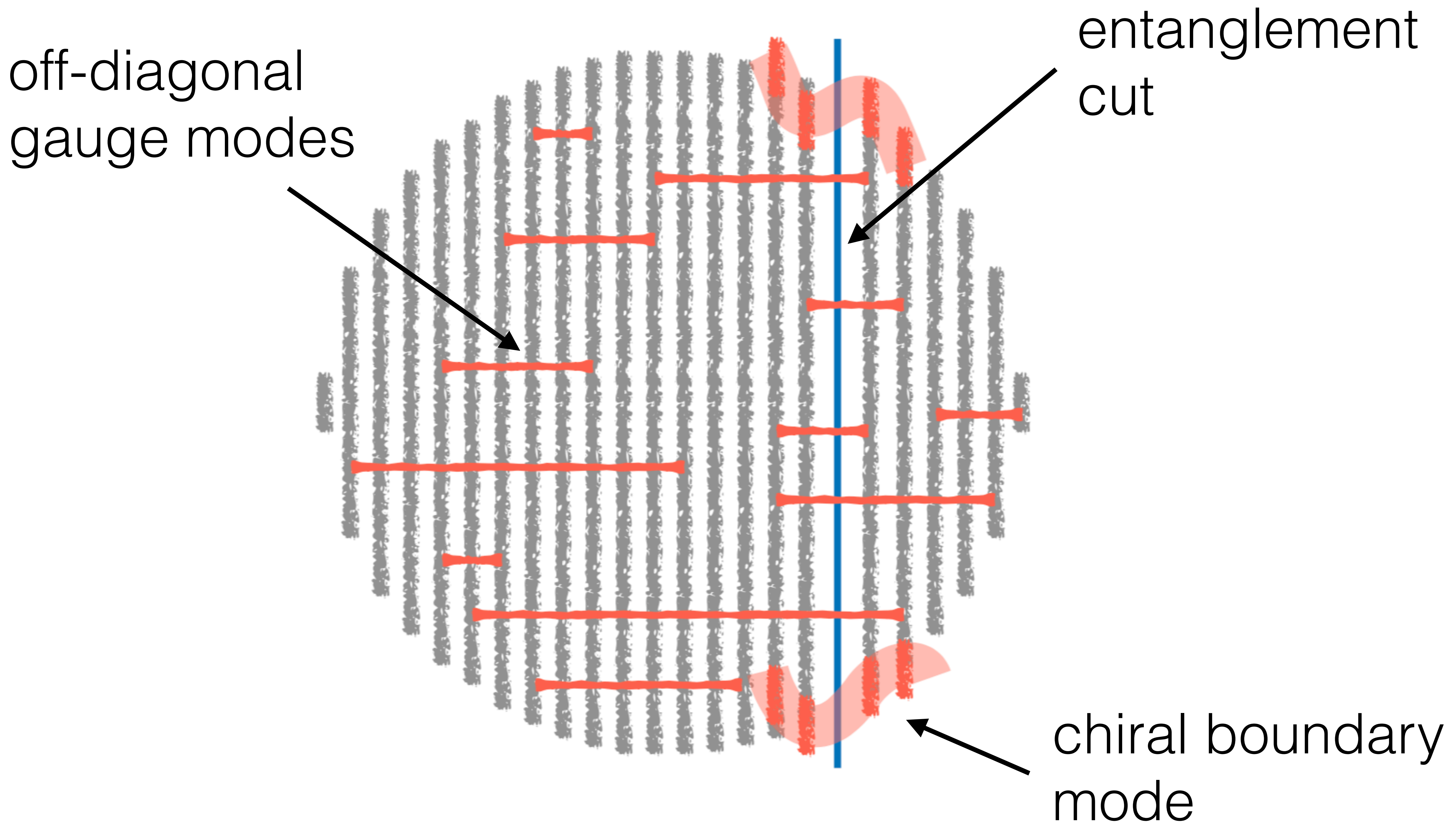
Quantum Hall Matrix Model

- Wavefunction factorizes — allows computation of two contributions to the entanglement due to a vertical (fixed X) partition of the droplet:

  (1) A 'collective field' contribution from fluctuations of the x eigenvalues. Physically: correlations due to chiral boundary mode.

  (2) A 'gauge theoretic' contribution from an associated block partition of the U. Physically: nonlocal correlations due to the Gauss law.

Quantum Hall Matrix Model



Collective field entropy

- Similar to computations of the entanglement in single-matrix models. But we used a new method.

- In terms of the collective field $n(x) \equiv \sum_a \delta(x - x_a)$

  the wavefunction $\psi[n] = e^{S[n]}$

$$S[n] = \frac{k+1}{2} \int dx_1 dx_2 n(x_1) n(x_2) \log |x_1 - x_2| - \frac{1}{2} \int dx n(x) x^2$$

  is strongly peaked on the Wigner semi-circle:

$$n_o(x) = \frac{2N}{\pi R^2} \sqrt{R^2 - x^2}\,, \qquad R^2 = 2N(k+1)$$

Collective field entropy

- Fluctuations about the semi-circle $n(x) = n_o(x) + \delta n(x)$ are described by the Gaussian wavefunction:

$$\psi[\delta n] = e^{\frac{k+1}{2}\int dx_1 dx_2 \delta n(x_1)\delta n(x_2)\log|x_1 - x_2|}$$

- Using steps from [Jackiw-Strominger 81] one can express this wave function in terms of a chiral boson ϕ:

$$\psi[\delta n] = \int \mathcal{D}\phi e^{-\int d\tau d\theta [i\partial_\tau \phi \partial_\theta \phi + (\partial_\theta \phi)^2] - i\int d\theta \phi(\theta)\delta n(\theta)}$$

Collective field entropy

- Can show that the ‘target space’ entanglement of the eigenvalues is equal to the usual entanglement of the chiral boson.

- This is fixed by conformal invariance.

- Finite N cuts off the mode expansion of the boundary chiral field so that:

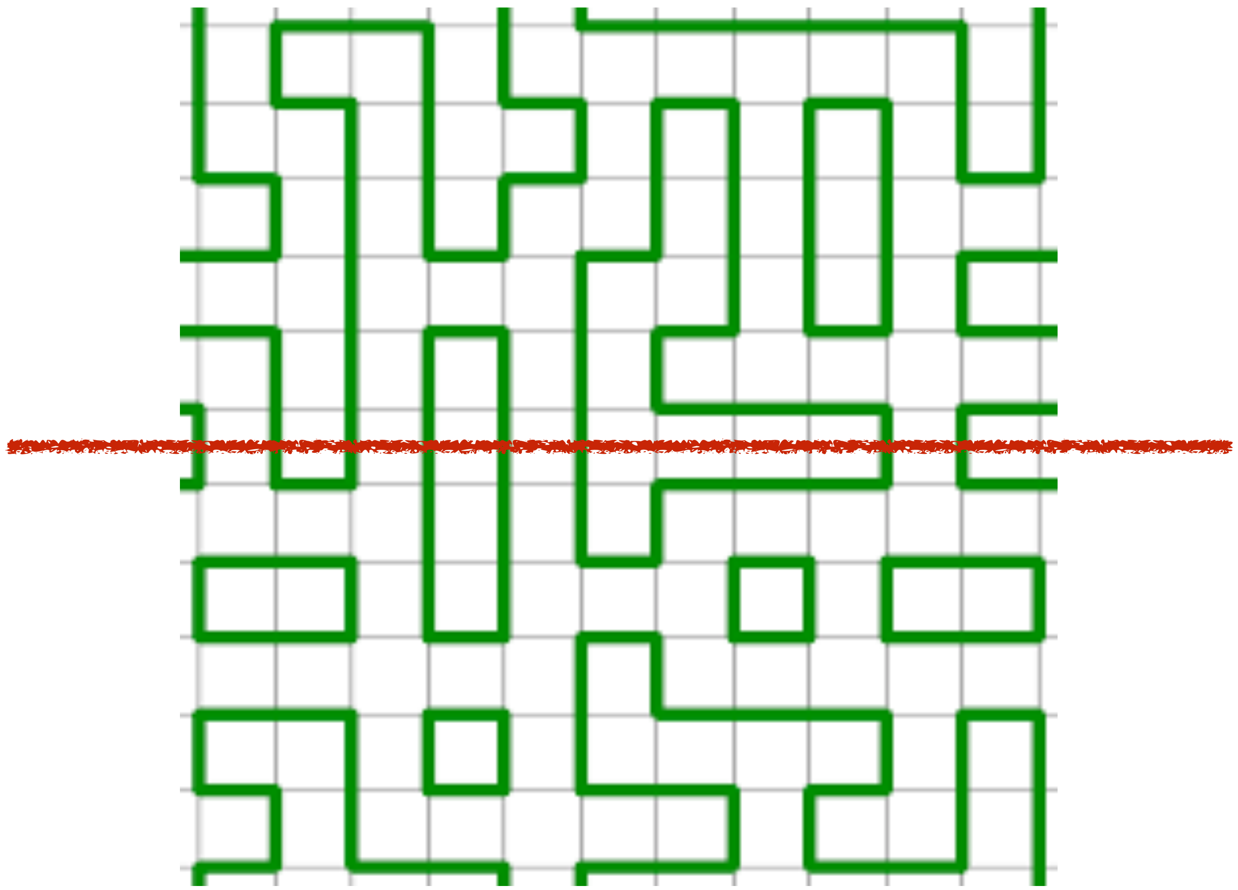
$$s_{\rm bdy} = \frac{1}{6}\log(NL) + \cdots$$

Length of cut with radius normalized to one.

Matches Chern-Simons result of [Belin-Iqbal-Krutoff]

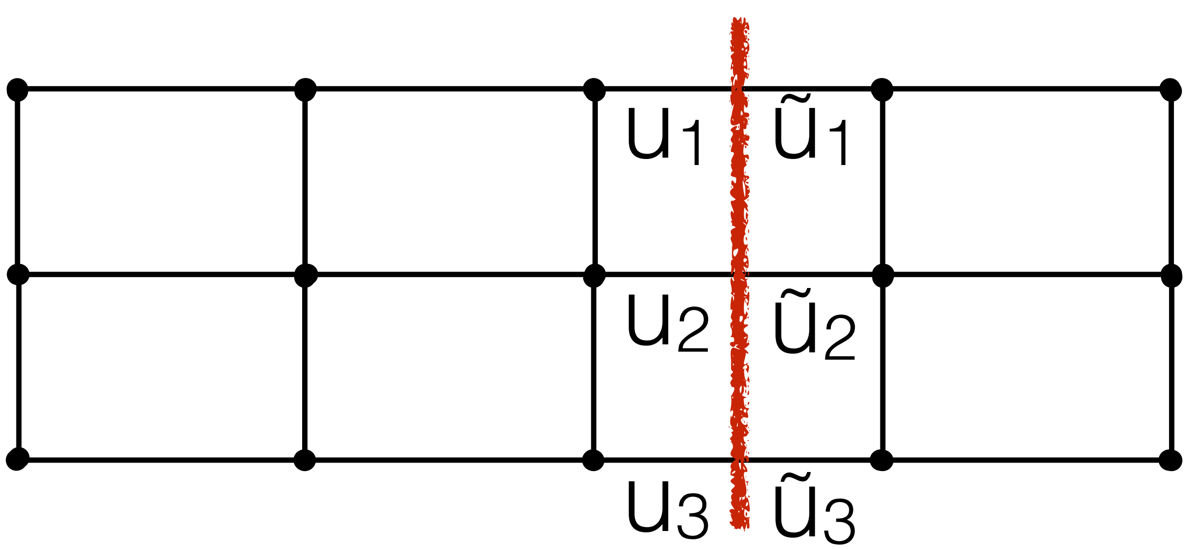
Lattice gauge theory

- The gauge-invariant variables (Wilson loops) of a gauge theory are not local. Defining a geometric partition therefore requires some work.



Lattice gauge theory

- [Donnelly 12]: links crossing the boundary are assigned to both regions. In simple situations the two copies of the holonomy on the link are required by gauge invariance to be maximally entangled.



$$\otimes_i \left(\sum_{u_i} |u_i\rangle |u_i\rangle \right) \quad \Rightarrow \quad \Delta s \sim L \, \log(\dim R)$$

Partitioning the matrices

- In the full theory, the U degrees of freedom are 'pure gauge' and do (almost) nothing.
- We will see however, that when the system is partitioned some of the U's acquire dynamics. We follow the method of [Donnelly-Freidel 16] here.
- For example, suppose in Chern-Simons theory we split the Lagrangian in spatial regions:

$$L = \int_M d^2x \epsilon^{ab} A_a \frac{d}{dt} A_b = \int_{M_1} d^2x \epsilon^{ab} A_a \frac{d}{dt} A_b + \int_{M_2} d^2x \epsilon^{ab} A_a \frac{d}{dt} A_b$$

Partitioning the matrices

- Then if we restrict to a given subregion, pure gauge modes that don't vanish on the common boundary acquire dynamics:

$$\delta A_a = \partial_a \phi \quad \rightarrow \quad \delta L = \int_{\partial M_1} \partial_x \phi \partial_t \phi$$

- A similar things happens in the matrix case. Write the kinetic term in the Lagrangian as a sum (the trace is like the integral previously)

$$L_{\text{kin}} = i\,\mathrm{tr}\left(\Theta Z^\dagger \frac{d}{dt} Z + (1-\Theta) Z^\dagger \frac{d}{dt} Z\right) = L_{L\,\text{kin}} + L_{R\,\text{kin}}$$

Partitioning the matrices

- Here Θ projects to the space of lowest M eigenvalues of X. One finds:

$$L_{L\,\mathrm{kin}} = y_L \frac{d}{dt} x_L + \frac{i}{2}\,\mathrm{tr}\left(Y_{RL}^\dagger \frac{d}{dt} Y_{RL} \right)$$

$$L_{R\,\mathrm{kin}} = y_R \frac{d}{dt} x_R + \frac{i}{2}\,\mathrm{tr}\left(Y_{LR}^\dagger \frac{d}{dt} Y_{LR} \right)$$

These are the off-diagonal modes that live in both regions

- The Gauss law fixes

$$Y_{LR}^\dagger = Y_{RL} = U_R Y_{RL}^{\mathrm{cl}} U_L^\dagger \qquad\qquad Y_{ab}^{\mathrm{cl}} = y_a \delta_{ab} - i \frac{\widetilde{\Psi}_a^\dagger \widetilde{\Psi}_b}{x_a - x_b}\,.$$

(It is important only to consider $U = U_L U_R$ that respect the block partition)

Partitioning the matrices

To make a long-ish story short (details on arXiv!):

- On each side of the cut U_L and U_R become families of harmonic oscillators, constrained to fixed energy.
- The energy is set by the singular values of Y^{cl}_{RL}.
- Gauge invariance forces these oscillators to be maximally entangled across the cut.
- The oscillators are 'identical'. This is inherited from the fact that U must not include permutations that re-order the eigenvalues.

Gauge theoretic entropy

- The gauge-theoretic entanglement entropy can be computed with Hardy-Ramanujan formula to count the dimension of the entangled oscillator space [same counting problem arises in CS theory, chiral modes on the entanglement cut!]. Find:

$$s = \frac{(Nk)^{1/2}\log(NL)}{\sqrt{6}} L + \cdots$$

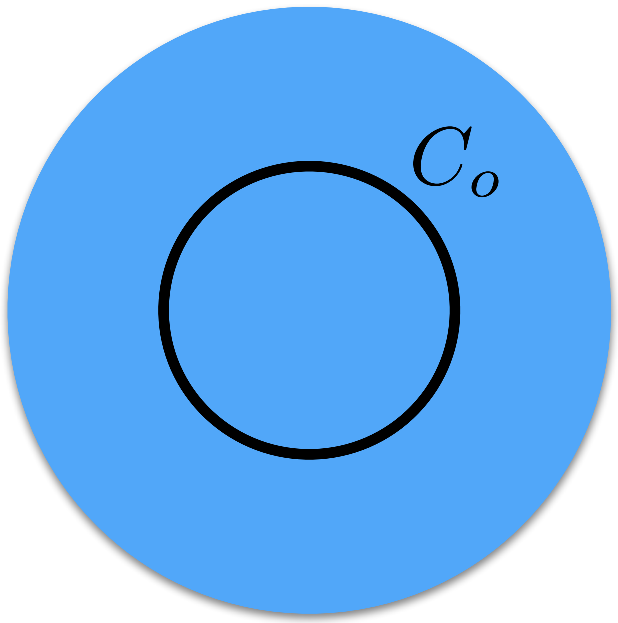
- Area law, regulated by finite N — emergent geometry!

- Logarithmic violation is related to cut intersecting the boundary of the droplet.

Radial partition

- We also looked at a radial partition, using a similar framework. Find:

$$s = \frac{(Nk)^{1/2}}{\sqrt{6}} C_o - 2 \log[(Nk)^{1/2} C_o] + \cdots$$



- No logarithmic violation of area law here, related to cut not intersecting the boundary.

- Trust subleading term in this case. Reminiscent of topological entanglement terms.

Conclusions

- What partition of matrix degrees of freedom captures the partition of an emergent geometry?

- The matrix Hamiltonian does not have spatial locality. But the wavefunction should contain an emergent locality.

- We have defined a partition in a very simple two-matrix model and computed the corresponding entropy (subtleties: role of permutations and ‘partial gauge fixing’).

Conclusions

- We found two contributions that match the expected emergent locality:

  1. a logarithmic entanglement from eigenvalues
     → chiral boundary mode
  2. an area-law gauge-theoretic entanglement
     → bulk Chern-Simons field

- Now have the understanding to move on to a more complicated model with compressible bulk dynamics.