Generalized Symmetries in QFT and String Theory

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Motivation

Symmetries are an essential part of Theoretical/Mathematical Physics.

The last few years have seen an immense broadening of what we call a global symmetry in QFT:

Generalized or higher-form symmetries and higher-group symmetries.

In this talk I will provide some background on these, give motivation in QFT, and discuss their realization in string theory constructions of QFTs.

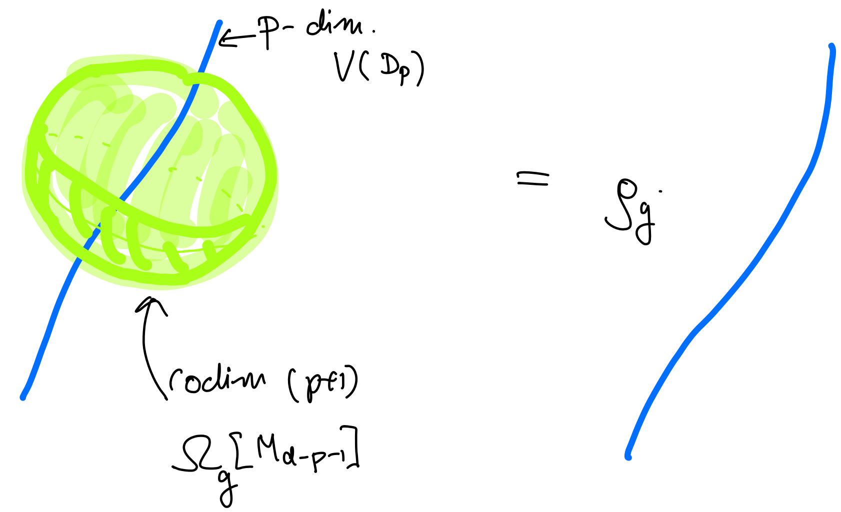
Plan

1. Generalized Symmetries:
   Higher form, higher group, non-invertible.
2. Applications to Confinement 1: Holographic Confinement
3. Applications to Confinement 2: Non-Lagrangrian theories
   $\Rightarrow \mathcal{N} = 1$ deformations of Class S theories with confining vacua

1. Generalized Symmetries

Symmetries from Topological Operators

Consider a global symmetry G under which local operators in a QFT are charged. A way to measure the charge is to surround the local operator by a codim 1 topological operator. The topological nature lets us deform this picture to yield:



A natural generalization is to allow for other topological operators - say of codimension $q+1$: these link with q-dimensional operators: $\Rightarrow$ these form so-called q-form symmetries.

Higher-Form Symmetries

q-form global symmetry Γ, has charged operators of dim q and topological charge operators of co-dimension $q+1$. Charge operators satisfy group law of Γ. [Gaiotto, Kapustin, Seiberg, Willett]

$\underline{q=0}$: local operators charged under a global symmetry are 0-dim, with $d-1$-dim charge-operators that surround the point.

$\underline{q=1}$: charged operators are Wilson or 't Hooft line operators

$\underline{q=2}$: In 6d $(2,0)$ of type $\mathfrak{g}=ADE$: there is a defect group $\text{Ab}(\Gamma_{ADE})$, with charged objects strings.

Main Example: Line Operators and 1-form Symmetry

Pure gauge theory with gauge algebra $\mathfrak{g}$ and simply-connected group G.
The set of line operators are

$$\mathcal{L} = Z_G \oplus Z_G =< W > \oplus < H > \qquad Z_G\text{= center of } G.$$

Not all lines are mutually local (relative theory): Dirac pairing

$$L_\alpha L_\beta = L_\beta L_\alpha e^{2\pi i \langle L_\alpha, L_\beta \rangle} \,.$$

Polarization $\Lambda \subset \mathcal{L}$:
choice of a maximal set of mutually local line operators (absolute theory).

E.g. for $\mathfrak{su}(N)$: $\mathcal{L} = \mathbb{Z}_N \oplus \mathbb{Z}_N$ with

$$\langle W, H \rangle = \frac{1}{N} \,.$$

$\Lambda =< W >$: gauge group $G = SU(N)$
$\Lambda =< H >$: gauge group $G = PSU(N)$

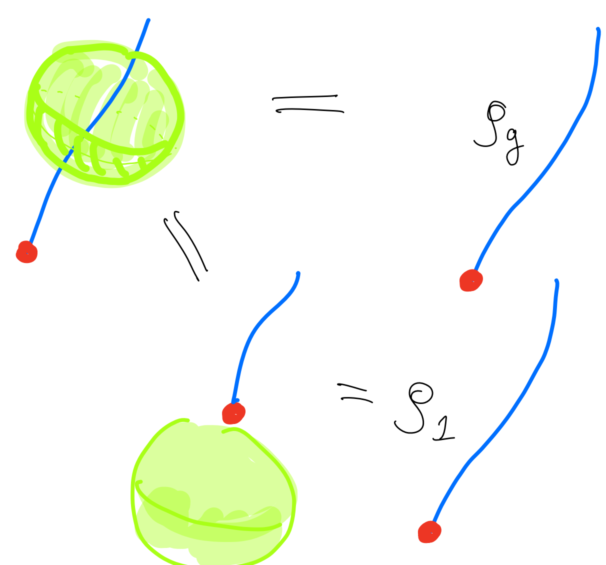
[Gaiotto, Moore, Neitzke][Aharony, Seiberg, Tachikawa]

1-Form Symmetry

$\Lambda \subset \mathcal{L}$ choice of polarization (absolute theory). Then the 1-form symmetry is the Pontryagin dual group to Λ

$$\Gamma^{(1)} = \widehat{\Lambda} = \mathrm{Hom}(\Lambda, U(1)) \,.$$

E.g. $\mathfrak{su}(N)$: $\Lambda = \mathbb{Z}_N$ and $\Gamma^{(1)} = \mathbb{Z}_N$.
Adding matter: line operators can end on local operators and get screened:



Adding Charged Matter

Gauge algebra	Center	Representations	Charges
$\mathfrak{su}(n)$	$\mathbb{Z}_n$	**F**	1 mod n
		Λ^2	2 mod n
		Sym^2	2 mod n
$\mathfrak{so}(4n)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	**V**	(1 mod 2, 1 mod 2)
		S	(1 mod 2, 0 mod 2)
		C	(0 mod 2, 1 mod 2)
$\mathfrak{so}(4n+2)$	$\mathbb{Z}_4$	**V**	2 mod 4
		S	1 mod 4
		C	3 mod 4
$\mathfrak{sp}(n)$	$\mathbb{Z}_2$	**F**	1 mod 2

Background Fields and Gauging

- Backgrounds for $\Gamma^{(p)}$ symmetries are $p+1$-form fields
  $B_{p+1} \in H^{p+1}(X, \Gamma^{(p)})$ for p-form symmetry $\Gamma^{(p)}$.
- Gauging this symmetry requires summing over these backgrounds B.
  E.g. gauge center symmetry of $SU(N)$ results in $PSU(N)$ YM.
- 't Hooft anomalies for higher-form symmetries are formulated in terms of B_{p+1}.

Higher-Group Symmetries

Higher-form symmetries may not always form product groups, but so-called higher groups [Sharpe][Tachikawa][Benini, Cordova, Hsin][Cordova, Dumitrescu, Intriligator]. E.g. 0-form and 1-form symmetries can form a 2-group. This generalizes the concept of an extension group, e.g.

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1$$

2-groups combine e.g. continuous flavor symmetry groups (0-form) and discrete 1-form symmetries.

Mathematically: the background B_2 for the 1-form symmetry is not independent of the 0-form symmetry background.

$$\delta B_2 \neq 0$$

There are numerous examples by now:
6d SCFT (full classification) [Apruzzi, Bhardwaj, Gould, SSN] 5d SCFTs [Apruzzi, Bhardwaj, Oh, SSN], 3d/4d: [Hsin, Lam][Lee, Ohmori, Tachikawa][Apruzzi, Bhardwaj, Gould, SSN], and for class S [Bhardwaj].

Warning:

If you think at this point that these are particularly rare and exotic, we will see the contrary: e.g. that gauge theories with $\mathrm{Spin}(4N+2)$ gauge group and vector matter all have such 2-group symmetries!

Recap: Adding Charged Matter

Gauge algebra	Center	Representations	Charges
$\mathfrak{su}(n)$	$\mathbb{Z}_n$	**F**	1 mod n
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$\mathfrak{so}(4n)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	**V**	(1 mod 2, 1 mod 2)
		S	(1 mod 2, 0 mod 2)
		C	(0 mod 2, 1 mod 2)
$\mathfrak{so}(4n+2)$	$\mathbb{Z}_4$	**V**	2 mod 4
		S	1 mod 4
		C	3 mod 4
$\mathfrak{sp}(n)$	$\mathbb{Z}_2$	**F**	1 mod 2

Spin$(4N+2)$ with Vector Matter

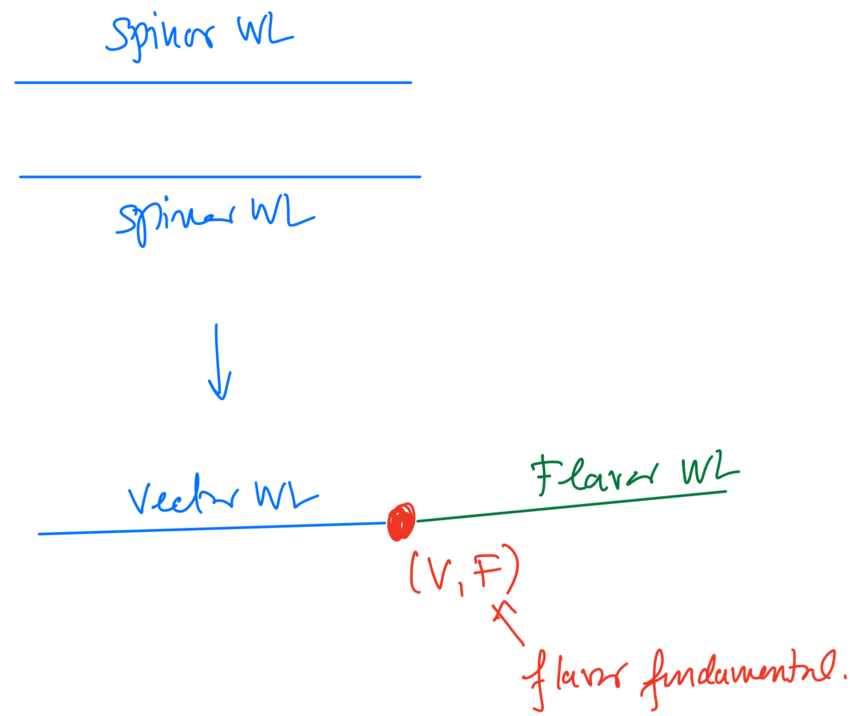
The following works in $d = 3, \cdots, 6$:

[Hsin, Lam][Lee, Ohmori, Tachikawa][Apruzzi, Bhardwaj, Gould, SSN]

Lets focus on 4d YM with Spin$(4N+2)$ gauge group has center $\mathbb{Z}_4^{(1)}$.
Adding $2N_F$vector matter gives $\mathfrak{su}(2N_F)$ flavor symmetry algebra:

Naively:
Wilson line in the vector rep is now screened by the matter: $\Gamma^{(1)} = \mathbb{Z}_2^{(1)}$, generated by the spinor line **S**.

However: The spinor line satisfies $\mathbf{S} \otimes \mathbf{S} = \mathbf{V}$, which gets screened by a particle that is bifundamental $(\mathbf{V}, 2N_F = F)$, and thus charged under the flavor center $\mathbb{Z}_2^F \subset \mathbb{Z}_{2N_F} \subset SU(2N_F)$, i.e. gives rise to a "flavor WL".



$0-$ and $1-$form symmetries are non-trivially interlinked!

The 1-form and 0-form center symmetries fit into the extension sequence:

$$1 \to \mathbb{Z}_2^{(1)} \to \mathbb{Z}_4 \to \mathbb{Z}_2^F \to 1\,.$$

In terms of the background fields for the symmetries:
$\Gamma^{(1)}$-background B_2 and the obstruction $\mathbf{w}_2$ to lifting the flavor symmetry

$$\mathcal{F} = \frac{SU(2N_F)}{\mathbb{Z}_2^F} \;\to\; SU(2N_F)$$

are not independent but form a 2-group

$$\delta B_2 = \text{Bock}(\mathbf{w}_2)\,.$$

where

$$\text{Bock}: \qquad H^2(B\mathcal{F}, \mathbb{Z}_2^F) \;\to\; H^3(B\mathcal{F}, \mathbb{Z}_2^{(1)})\,.$$

This applies to: 6d, 5d, 4d, 3d, with 8 supercharges (or with slightly different global flavor group to 4 supercharges in 4d). In 6d SCFTs: full classification [Apruzzi, Bhardwaj, Gould, SSN].

Non-Invertible Symmetries

Extending the concept of symmetries we can also relax the requirement for the symmetry generators ("topological operators") to form a group.

Well-known in 2d: Verlinde line operators in 2d rational CFTs obey a fusion algebra

$$L_i \times L_j = \sum_k \mathcal{N}_{ij}^k L_k$$

In higher-dimensions this is much less explored. But again examples of so-called "non-invertible" symmetries have recently been constructed in very simple setups. E.g. 4d gauge theory with disconnected gauge group $O(2) = U(1) \ltimes \mathbb{Z}_2$ [Heidenreich et al], and using mixed anomalies and duality walls in [Khaidi, Ohmori, Zheng][Choi, Cordova, Hsin, Lam, Shao].

In 2d these non-invertible symmetries have direct physical implications (2d adjoint QCD: deconfinement of quarks) [Komargodski, Ohmori, Rumpedakis, Seifnashri].

2. Higher-Form Symmetries and Confinement

Application: Confinement

Unbroken 1-form symmetry in a vacuum indicates confinement.

For 4d $\mathcal{N} = 1$ $\mathfrak{su}(N)$ SYM there are N vacua.

For $\mathfrak{su}(2)$, there are 2 vacua, where monopole (m) and dyon (d) condense from $\mathcal{N} = 2$ after breaking to $\mathcal{N} = 1$.

The preserved 1-form symmetry for each polarization is:

Λ	G	$\Gamma_m^{(1)}$	$\Gamma_d^{(1)}$
W	$SU(2)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
H	$SO(3)_+$	$\emptyset$	$\mathbb{Z}_2$
$H + W$	$SO(3)_-$	$\mathbb{Z}_2$	$\emptyset$

Two Extensions/Applications:

1. Holographic Confinement – generalized symmetry perspective

   [Apruzzi, van Beest, Gould, SSN]

2. 4d Non-Lagrangian Theories with confining vacua

   [Bhardwaj, Hubner, SSN]^2

Holographic Confinement and Generalized Symmetries

Holography and Higher-Form Symmetry

Consider $\mathrm{AdS}_5 \times X^5$ with $X_5 = S^5$ (or SE_5) in Type IIB with

$$\int_{X_5} F_5 = N\,, \qquad F_5 = \star F_5\,.$$

For S^5 the dual is 4d $\mathcal{N} = 4$ $\mathfrak{su}(N)$ SYM.

How are different global forms of the gauge group distinguished in the holographic dual?

IIB CS-coupling

$$S_{CS} \supset \frac{1}{4\pi} \int_{\mathrm{AdS}_5 \times S^5} F_5 \wedge B_2 \wedge dC_2\,.$$

[Witten] The 5d effective action governing the fluctuations b_2 and c_2 of the 2-form fields around the background is

$$S_{\text{top}} = \frac{N}{4\pi} \int_{\mathrm{AdS}_5} b_2 \wedge dc_2$$

Eoms:

$$N db_2 = 0 \qquad N dc_2 = 0$$

b_2 and c_2 are flat $\mathbb{Z}_N$ 2-form fields. Construct topological surface operators

$$U_b(M_2) = e^{i \oint_{M_2} b_2}\,, \qquad U_c(M_2) = e^{i \oint_{M_2} c_2}$$

which however are not mutually local

$$U_b(M_2) U_c(M_2') = U_c(M_2') U_b(M_2) e^{\frac{2\pi i}{N} M_2 \cdot M_2'}$$

Line operators are obtained by considering M_2 with $\partial M_2 = \gamma \subset \partial \mathrm{AdS}_5$

$U_b(M_2) =$ F1-strings ending on the boundary, i.e. Wilson lines

$U_c(M_2) =$ D1-strings ending on the boundary, i.e. ’t Hooft lines

B.c. and Global Form of Gauge Group

B.c. on $S_{\text{top}} = \frac{N}{4\pi} \int b_2 \wedge dc_2$:

b_2 Dirichlet (Neumann), c_2 Neumann (Dirichlet)

F1 ending on ∂AdS and U_c are the topological surface operators link with lines, generating the 1-form symmetry of $SU(N)$ (and likewise D1/U_b/$PSU(N)$).

Screening: Bianchi integrated over S^5:

$$\int_{S^5} dF_7 = \int_{S^5} F_5 \wedge H_3 = N H_3$$

so N F1-strings can end on wrapped D5 "baryon vertex".

$\Rightarrow$ N Wilson lines are screened, i.e. $SU(N)$ and $\Gamma^{(1)} = \mathbb{Z}_N$.

Holographic Confinement and 1-form Symmetry

- D3s at the conifold $C(T^{1,1})$ are dual to IIB on $\text{AdS}_5 \times T^{1,1}$, $\int F_5 = N$.
- $T^{1,1} \sim S^3 \times S^2$: wrap D5-branes on S^2, inducing $\int_{S^3} F_3 = M$
  $\Rightarrow$ breaks conformal invariance

[Klebanov-Strassler] (KS) solution: dual to confining phase of 4d $\mathcal{N} = 1$ SYM:

$$ds^2 = \underbrace{\frac{r^2}{R^2} d\mathbf{x}^2 + \frac{R^2}{r^2} dr^2}_{M_5} + R^2 ds^2_{T^{1,1}} \,.$$

r= radial direction, RG-flow; $R(r) \sim \ln(\frac{r}{r_s})^{1/4}$, $r_s = r_0 e^{-N/gM^2 - 1/4}$.

Fluxes pick up r-dependence:

$$\int_{S^2} B_2 = \mathcal{L}(r) = \frac{3 g_s M}{2\pi} \ln \frac{r}{r_0} \,, \qquad \int_{T^{1,1}} F_5 = \mathcal{K}(r) = N + M\mathcal{L}$$

At $r_k = r_0 e^{k/M}$, effective number of D3-branes $\mathcal{K} = N - kM \in \mathbb{Z}$.

$\Rightarrow$ What is dual field theory?

Duality cascade

Far UV: $k = 0$ and the couplings run as $g_{SU(N+M)} \uparrow$ and $g_{SU(N)} \downarrow$

$$\boxed{SU(N+M) \times SU(N)} \overset{\text{Seiberg dualize}}{\longrightarrow} \boxed{SU(N-M) \times SU(N)}$$
$$\overset{\text{Seiberg dualize}}{\longrightarrow} \boxed{SU(N-M) \times SU(N-2M)} \longrightarrow \cdots .$$

$N = kM$ then end of the cascade is pure $\mathcal{N} = 1$ SYM $\boxed{SU(M)}$.

In the far IR the solution is T^*S^3 and conjectured to be dual to the confining phase of 4d $\mathcal{N} = 1$. From now: $N = kM$.

Expectations:

I. 1-form symmetry is unbroken in the IR

II. chiral symmetry breaking

III. IR should contain a gapped TQFT describing the confining phase

In [Apruzzi, van Beest, Gould, SSN] we identified these using the 5d sugra consistent truncation on $T^{1,1}$ of [Cassani, Faedo] and the IR sugra solution.

I. 1-form symmetry in the KS Solution

Topological couplings in the 5d consistent trunction

$$S_{5\text{d}} \supset \text{gcd}(N,M) \int_{M_5} b_2 \wedge d\mathcal{C}_2$$

for NSNS b_2 and $\mathcal{C}_2$ is from RR 2- and 4-form fluctuations.

- 1-form symmetries from b.c. $\Gamma^{(1)} = \mathbb{Z}_{\text{gcd}(N,M)}$.
- B.c.: e.g. F1-strings with Dirichlet b.c. correspond to Wilson lines.
- For all slices $r = r_k$ we have

$$\int_{S^3} F_3 = M\,, \qquad \int_{T^{1,1}} F_5 = N - kM$$

  Then integrating the Bianchis gives

$$\int_{T^{1,1}} dF_7 = (N-kM)H_3\,, \qquad \int_{S^3} dF_5 = MH_3$$

  D5s on $T^{1,1}$ and D3s on S^3 screen $\text{gcd}(N,M)$ F1s.

II. Chiral Symmetry Breaking

$U(1)_R$ corresponds to the Reeb vector of the $\mathrm{AdS}_5 \times T^{1,1}$. From the consistent trunction we have a Stückelberg coupling

$$S_{\mathrm{5d}} \supset \frac{1}{2\pi} \int_{M_5} |g_1 + 2MA|^2$$

$g_1 = dc_0$ with shift symmetry $c_0 \to c_0 + 2\pi$ and A is background for $U(1)_R$.

$$A \to A + d\alpha \,, \qquad c_0 \to c_0 - 2M\alpha$$

gauges c_0 away, giving a massive $U(1)$ gauge field, with a remnant chiral symmetry $\mathbb{Z}_{2M}^{(0)}$ left over ($\alpha \in \frac{2\pi}{2M}\mathbb{Z}$).

$\Rightarrow$ Realizes the breaking $U(1)^{(0)} \to \mathbb{Z}_{2M}^{(0)}$

Chiral symmetry breaking in the IR

$$S_{5\mathrm{d}} \supset \frac{1}{2M} \int_{M_5} b_2 \wedge b_2 \wedge A \equiv \omega$$

which is a mixed 't Hooft anomaly $\mathbb{Z}_M^{(1)} - \mathbb{Z}_{2M}^{(0)}$.

If $e^{2\pi i \omega} \neq 1$ then there does not exist a TQFT to saturate the anomaly [Cordova, Ohmori].

Here: $b_2 \wedge b_2 = \mathfrak{P}(b_2) \in 2\mathbb{Z}$ for spin manifolds.

In a confining vacuum: $\mathbb{Z}_M^{(1)}$ must remain unbroken, so in order for the anomaly to be saturated

$$\mathbb{Z}_{2M}^{(0)} \to \mathbb{Z}_2^{(0)}\,, \qquad \oint A \in M\mathbb{Z}.$$

The M vacua are separated by domain walls of wrapped D5s on S^3.

III. IR TQFT

Can we derive the IR theory describing the gapped, confining phase? Remarkably: yes! From the IR supergravity solution

$$S_{5\text{d}} \supset 2\pi \int_{M_5} dc_0 dc_3 + \frac{M}{2} b_2^2\, dc_0$$

Evaluating at the 4d boundary gives $\phi \equiv c_0/M$

$$S_{4\text{d IR TQFT}} = \int M\phi(dc_3 + \frac{M}{2} b_2^2)$$

The M vacua are labeled by $\langle e^{i\phi} \rangle = e^{2\pi i \ell/M}$, $\ell = 0, \cdots, M-1$, and are separated by the domain walls given by $e^{i \oint c_3}$. Agrees with [GKSW].
Supergravity gauge transformations

$$b_2 \to b_2 + d\lambda\,, \qquad c_3 \to c_3 - M b_2 \lambda - \frac{M}{2} \lambda d\lambda$$

have interpretation as 3-group symmetry between 1- and 2-from symmetry backgrounds. Consistent with gauge invariance of $F_4 = dc_3 + \frac{M}{2} b_2^2$ of the S^3-wrapped D5 domain walls.

3. Confinement from 6d

6d – Interlude

Motto: "All good things come from 6d".

- $(1,0)$ 6d SCFTs are believed to be classified by now: F-theory on elliptic CY3.
- 5d SCFTs seem to also come from 6d, upon circle-reduction with benefits
- 4d $\mathcal{N}=1,2$ SCFTs and gauge theories from class S, 3d from class R etc.

Generalized global symmetries also percolate down from 6d: by dimensional reduction of the topological operators.

Alternatively: geometric engineering of 4d, 5d, 6d, theories in IIB/M/F also allows the computation of the generalized symmetries.

Generalized Symmetries of 6d SCFTs

- 6d $(2,0)$ theories of type ADE are relative theories: defect group, which is given by $\mathrm{Ab}(\Gamma_{ADE})$.
  $\Rightarrow$ charged objects are 2d strings ("2-form symmetry")

- 6d $(1,0)$:
  # If the base in F-theory is $\mathbb{C}^2/\Gamma$ with $\Gamma \subset U(2)$ non-trivial, then there is also a defect group [del Zotto, Heckman, Rudelius, Tomasiello].
  # In addition there can be 6d vectors and hypers: gauge groups + matter, which give rise to 1-form symmetries

- These theories can also have 2-group symmetries [Apruzzi, Bhardwaj, Gould, SSN]

1-form symmetry of 4d $\mathcal{N}=2$ Class S

[Tachikawa][Bhardwaj, Hubner, Schafer-Nameki]

6d (2,0) of type $\mathfrak{g}$=ADE is a relative theory, with 2-form symmetry $\mathcal{Z}$.
Compactify on $\mathcal{C}_{g,n}$ to 4d $\mathcal{N}=2$ class S [Gaiotto].

6d surface operators wrapped on $H_1(\mathcal{C}_{g,n},\mathbb{Z})$ give line operators.
Consider $\mathcal{C}_{g,\emptyset}$:

$$\mathcal{L} = H_1(\mathcal{C}_g, \mathcal{Z}) \cong \mathcal{Z}_A \oplus \mathcal{Z}_B$$

from A-/B-cycles. There is a pairing on these line operators

$$a_i \otimes \alpha_i \in H_1(\mathcal{C}_g, \mathbb{Z}) \otimes \mathcal{Z} : \qquad \langle a_1 \otimes \alpha_1, a_2 \otimes \alpha_2 \rangle = (a_1 \cdot a_2)\langle \alpha_1, \alpha_2 \rangle \,.$$

Canonical polarization choices:

$$\mathcal{L} \supset \Lambda = \mathcal{Z}_A \quad \text{or} \quad \mathcal{Z}_B \qquad \Rightarrow \qquad \Gamma^{(1)} = \Lambda \,.$$

Twist lines and punctures [Bhardwaj, Hubner, SSN]; irregular punctures can also carry trapped 1-form symmetry [Bhardwaj, Giacomelli, Hubner, SSN – to appear]

Complementary view: Geometric Engineering

Construct 4d $\mathcal{N} = 2$ theories from IIB on singular, non-compact CY3 X.

Line operators are D3s on non-compact 3-cycles, modulo screening by D3s on compact 3-cycles:

$$\mathcal{L} = \frac{H_3(X, \partial X, \mathbb{Z})}{H_3(X, \mathbb{Z})} \cong \{L \in H_2(\partial X, \mathbb{Z}); L \text{ extends trivially to } X\}$$

Computable for non-Lagrangian theories like Argyres-Douglas.

[Heidenreich, Gracia Etxebarria, Regalado][Closset, ± Giacomelli, SSN, Wang][Albertini, del Zotto, Garcia Etxebarria, Hosseini].

Complementary view: Geometric Engineering

A subset of class S theories: $\mathbb{C}^2/\Gamma_{ADE} \to \mathcal{C}_g$.
ALE-fibration is governed by the Higgs field ϕ, which satisfies the Hitchin equations [Gaiotto, Moore, Neitzke]

$$\bar{D}\phi = 0\,, \qquad F_{z\bar{z}} + [\phi, \phi^*] = 0\,.$$

Spectral curve of this Higgs field is the Seiberg-Witten curve $\det(v - \phi) = 0$, which is an N-fold cover of the Gaiotto curve $\mathcal{C}$. Then

$$\mathcal{L} = \mathrm{Ab}(\Gamma_{ADE})^{2g}\,.$$

Generalizable to punctures, regular and irregular, twisted etc.

$\mathcal{N} = 1$ Deformation

4d $\mathcal{N} = 1$ from 6d (2,0) on $\mathcal{C}_{g,n}$ in $V_1 \oplus V_2 \to \mathcal{C}_{g,n}$, with sections (ϕ, ψ).

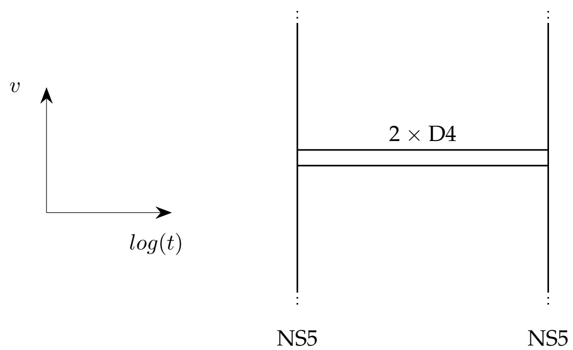
Simple class of configurations: $\phi = (1, 0)$- and $\psi = (0, 0)$-forms on $\mathcal{C}$, with BPS equations [Xie], corresponding to a generalized Hitchin system

$$\begin{aligned} \bar{D}\phi = \bar{D}\psi = 0 \\ [\phi, \psi] = 0 \\ F + [\psi, \psi^*] + [\phi, \phi^*] = 0\,. \end{aligned}$$

(ϕ, ψ) each defines an N-sheeted covers of $\mathcal{C}$.

Strategy: start with $\mathcal{N} = 2$ Higgs field ϕ, and then "rotate" to $\mathcal{N} = 1$ [Barbon][Witten][Hori, Ooguri, Oz][Bonelli, Giacomelli,Maruyoshi,Tanzini] (related [Dijkgraaf, Vafa] curve).

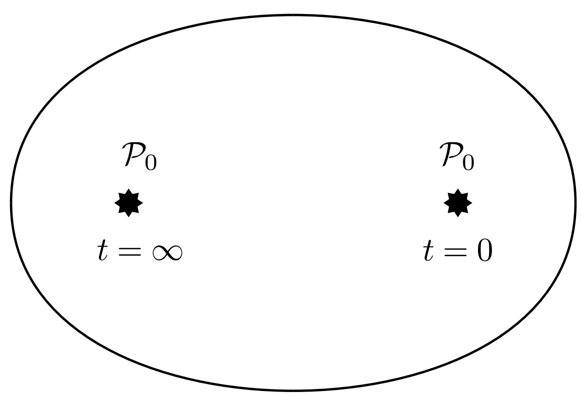
4d $\mathcal{N} = 2$ pure SYM from 6d



Focus at first on $\mathfrak{su}(2)$ for clarity.
The Seiberg-Witten curve is

$$v^2 = \frac{\Lambda^2}{t} + u + \Lambda^2 t\,.$$

t= coordinate on the Gaiotto curve $\mathcal{C} = S^2$.



Class S construction:
$g = 0$, $n = 2$ with two $\mathcal{P}_0$ irregular punctures:

$$\mathrm{Tr}\,\phi^2 \equiv \phi_2 = \frac{v^2}{t^2}dt^2 = \left(\frac{\Lambda^2}{t^3} + \frac{u}{t^2} + \frac{\Lambda^2}{t}\right)dt^2$$

ϕ_2 has poles of order 3 at $t = 0, \infty$.

Rotating to $\mathcal{N} = 1$ SYM

Turn on ψ, where we rotate to $\mathcal{N} = 1$ at $t = \infty$:

$$t \to \infty : \qquad \psi \to \mu \phi_\zeta \,, \qquad \phi_\zeta \frac{dt}{t} = \phi$$
$$t \to 0 : \qquad \psi \to c$$

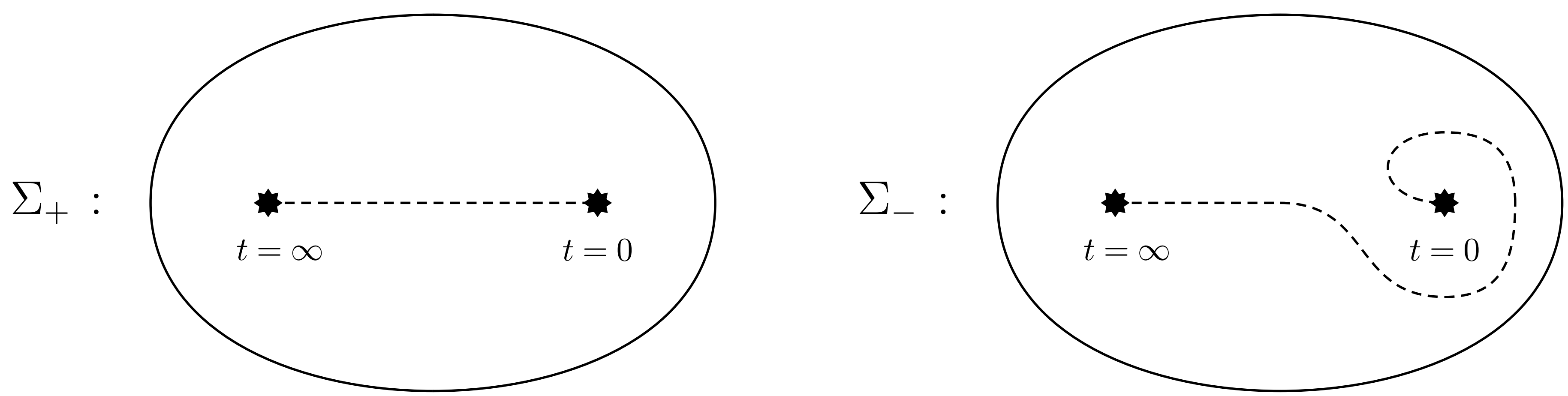
Furthermore for diagonalizable Higgs fields, the BPS equation $[\phi, \psi] = 0$ implies simultaneous diagonalizability. For generic eigenvalue spectrum, this implies that one is a function of the other, and thus the branch-cuts must match.

→ solving for the curves: v-curve for ϕ and w-curve for ψ.

→ topological matching of branch-cuts

Curves describing the two vacua

The two vacua differ by asymptotics of w, which can be changed by encircling $t = 0$:



Line Operators and 1-form Symmetry of the Vacua

$\mathcal{L}$ and Λ =lines and a polarization of the $\mathcal{N}=2$ class S theory.
Define for each $\mathcal{N}=1$ vacuum with curve Σ_r:

$$\mathcal{I}_r = \{\text{projections of 1-cycles on the } \mathcal{N}=1 \text{ curve } \Sigma_r \text{ onto } \mathcal{C}\} \subset H_1(\mathcal{C}, \widehat{\mathcal{Z}})\,.$$

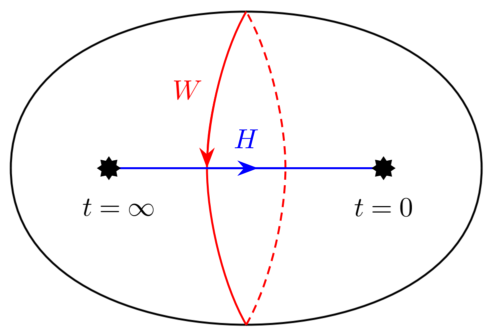
Then lines with perimeter law are $\Lambda_r = \Lambda \cap \mathcal{I}_r$ and the 1-form symmetry $\Gamma_r^{(1)}$ preserved in the vacuum r is

$$\Gamma_r^{(1)} = \left(\frac{\Lambda}{\Lambda \cap \mathcal{I}_r}\right) \subset \Lambda$$

If $\Gamma_r^{(1)} \neq \emptyset$ then the vacuum r is confining.

Why? Confining strings arise from from membranes on relative 1-cycles of the local CY3 and Σ_r[Witten].

Confinement for $\mathfrak{su}(2)$ SYM



$\mathfrak{su}(2)$: Line operators are W and H.
For both vacua $2W = 0$ because of the branch-cut crossing (only going twice around the puncture gives a 1-cycle on the curves). The lines with perimeter law for each vacuum are:

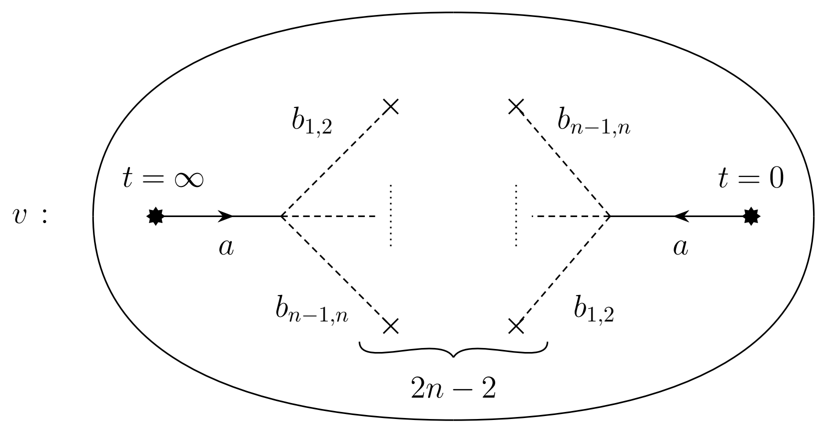
$$\mathcal{I}_+ =< 2W, H > , \qquad \mathcal{I}_- =< 2W, H + W > .$$

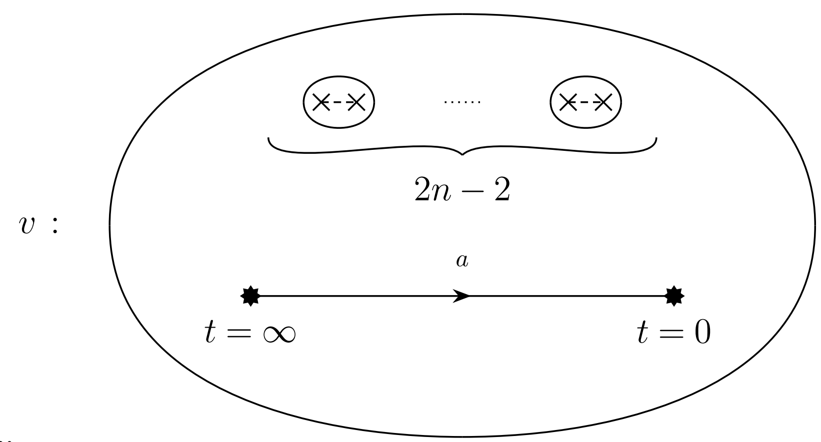
Preserved 1-form symmetry for each polarization:

Λ	G	$\Gamma_+^{(1)}$	$\Gamma_-^{(1)}$
$< W >$	$SU(2)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$< H >$	$SO(3)_+$	$\emptyset$	$\mathbb{Z}_2$
$< H + W >$	$SO(3)_-$	$\mathbb{Z}_2$	$\emptyset$

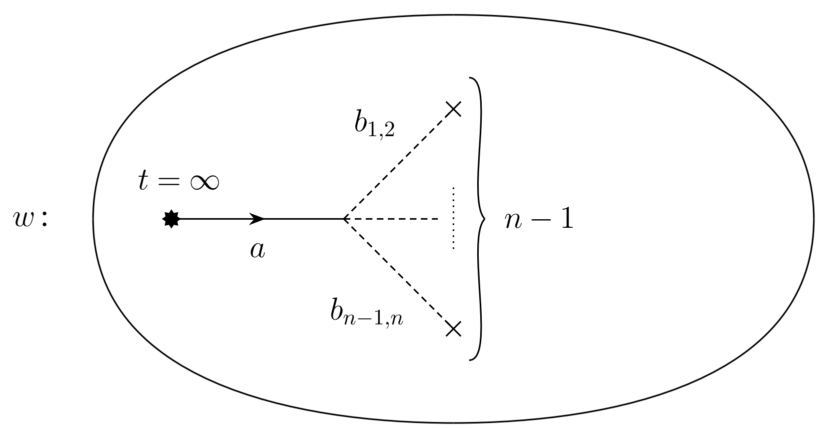
Of course well-known in general [Aharony, Seiberg, Tachikawa], but we derive it purely from the curve.

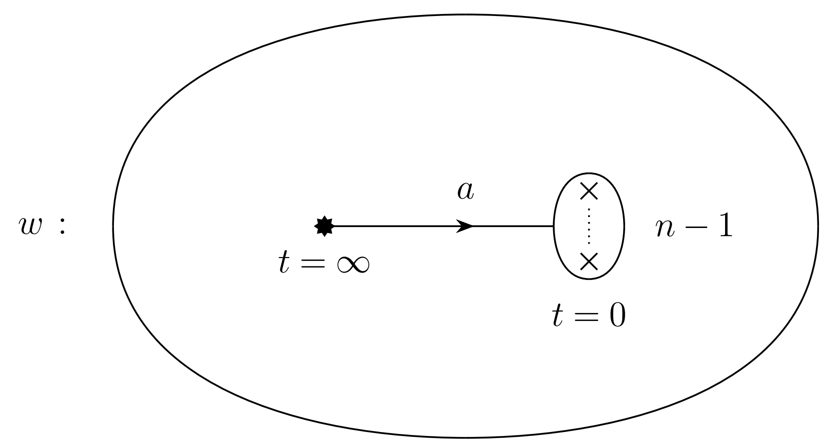
Generalization: Pure $\mathcal{N} = 1$ SYM $\mathfrak{su}(N)$





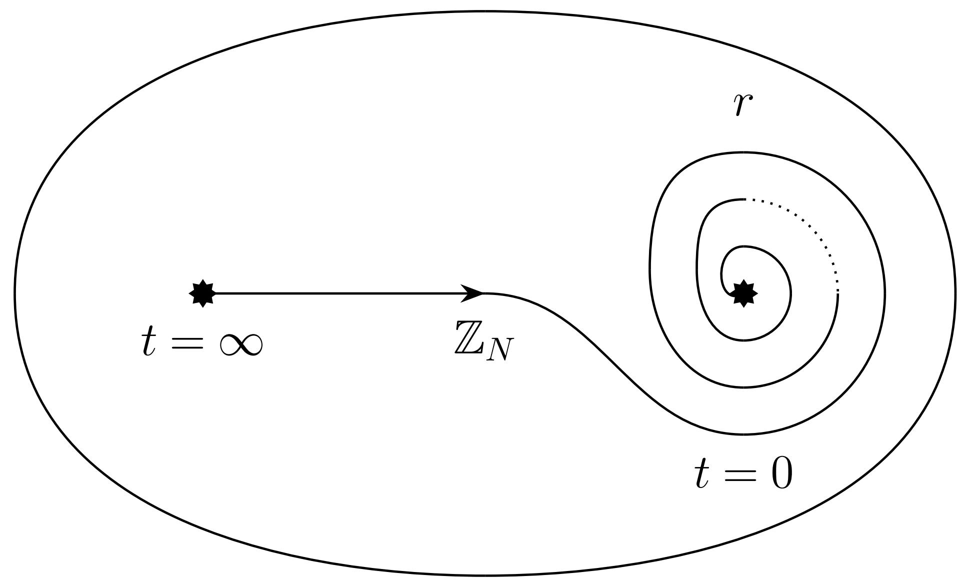
Match Branchcuts:





The N vacua of the $\mathfrak{su}(N)$ SYM theories have curves

Σ_r :



Again choosing polarizations, we can determine the 1-form symmetries in each vacuum for all global forms.

Similarly we can determine the confinement index of $\mathfrak{su}(N)$ SYM with adjoint chirals with superpotential

[Elitzur, Forge,Giveon, Intriligator,Rabinovici][Cachazo,Seiberg,Witten]

$\rightarrow$ see [Bhardwaj, Hubner, SSN]

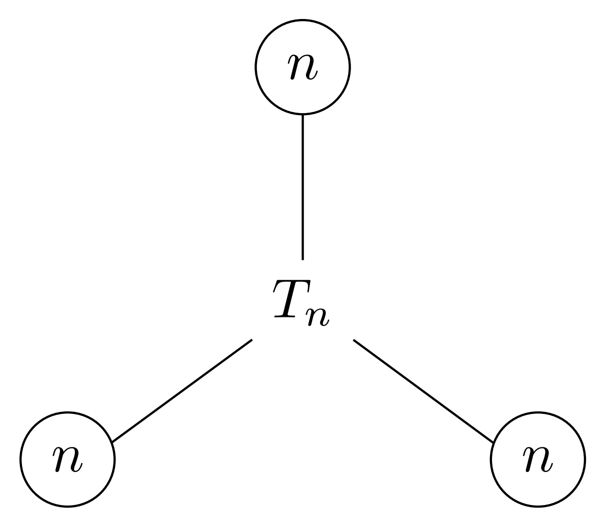
3. Confinement in Non-Lagrangian 4d $\mathcal{N} = 1$ Theories

The main reason for developing this class S based framework was to be able to apply it to theories without UV Lagrangians. In [Bhardwaj, Hubner, SSN] we determined a family of theories with no UV Lagrangian, which have confining vacua:

$$\mathfrak{P}_{n,\alpha} = \text{6d (2,0) } \mathfrak{su}(n) \text{ theory on a sphere with } \alpha \text{ irregular } \mathcal{P}_0 \text{ punctures}$$

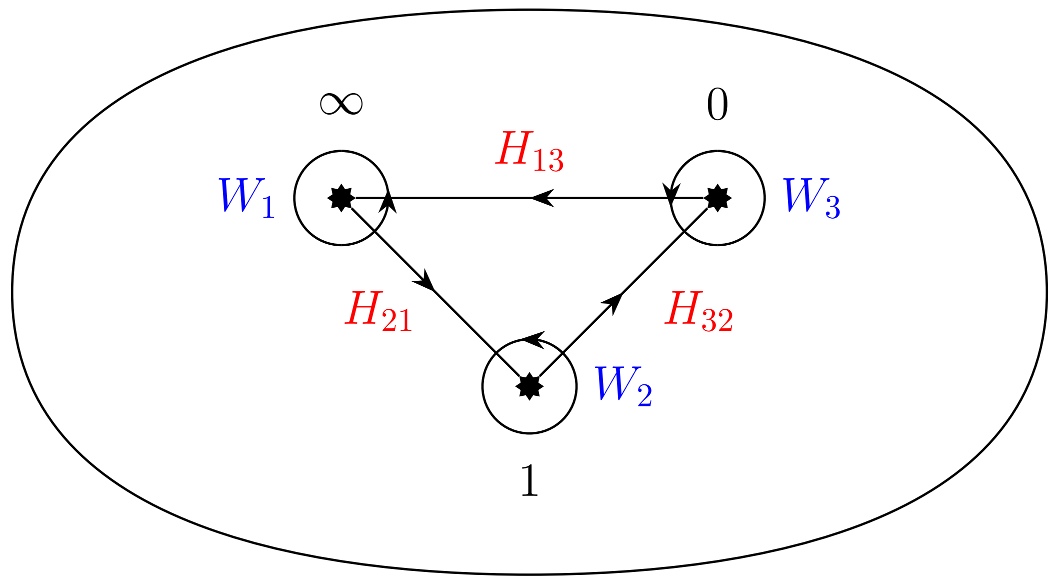
where $\mathcal{P}_0$ has pole of order $1 + 1/n$.
Simplest case $\alpha = 3$: T_n theory with $\mathfrak{su}(n)^3$ flavor gauged:

$\mathfrak{P}_{n,3}$:



Line operators:

v :



with

$$\mathcal{L} = \langle W_i, H_{jk} \rangle / \langle W_1 + W_2 + W_3 = 0;\ H_{21} + H_{32} + H_{13} = 0 \rangle\,.$$

The pairing on these line operators is

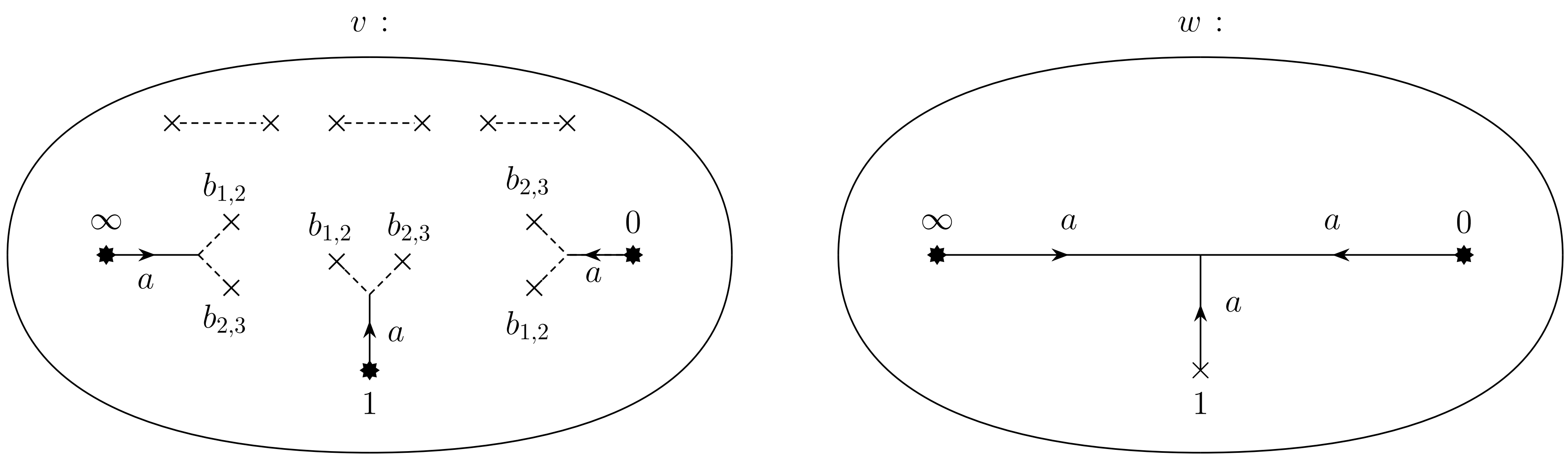
$$\langle W_i, H_{ij} \rangle = 1/n\,, \qquad \langle W_j, H_{ij} \rangle = -1/n$$

Again, many choices of polarization. A simple one is

$$\Lambda = \langle W_i \rangle / \langle W_1 + W_2 + W_3 = 0 \rangle\,.$$

$\mathcal{N}=1$ Curve for $\mathfrak{P}_{3,3}$

The $\mathcal{N}=1$ curve before collision of branch-points is, with punctures rotated at $t=0,\infty$:



The branch-cuts in v-curve can be collided to agree with the cut-structure on the w-curve.
In the electric polarization we find in this vacuum

$$\Gamma_{\Sigma}^{(1)}=\mathbb{Z}_3\times\mathbb{Z}_3\,,$$

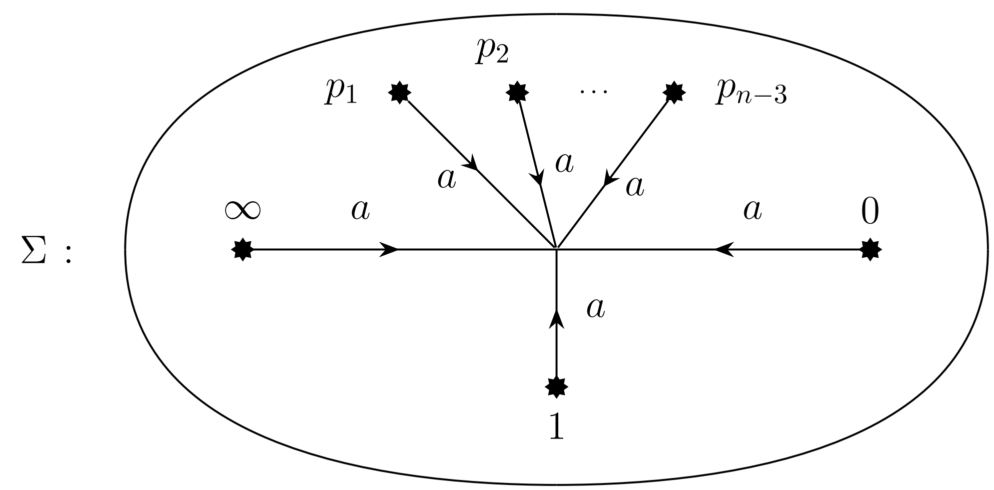
which means this vacuum is confining.

A Family of Confining Theories with no UV Lagrangian

$\mathfrak{P}_{n,n}$ = 6d (2,0) $\mathfrak{su}(n)$ on sphere with n $\mathcal{P}_0$s. Line ops $\mathcal{L} = \mathcal{L}_W \times \mathcal{L}_H$:

$$\mathcal{L}_W = \frac{\langle W_i \rangle}{\langle \sum W_i = 0 \rangle} \simeq \mathbb{Z}_n^{n-1}\,, \qquad \mathcal{L}_H = \frac{\langle H_{i+1,i}, H_{1,n} \rangle}{\langle \sum H_{i+1,i} + H_{1,n} = 0 \rangle} \simeq \mathbb{Z}_n^{n-1}$$

with pairing $\langle W_i, H_{ij} \rangle = \frac{1}{n}$, $\langle W_j, H_{ij} \rangle = -\frac{1}{n}$. Again, on special locus of CB match the branch-cut structure, resulting in the vacuum with curve:



With $\Lambda = \mathcal{L}_W$ polarization this vacuum preserves 1-form symmetry and is confining for n prime:

$$\Gamma_\Sigma^{(1)} = \mathbb{Z}_n^{n-1}\,.$$

Summary

Main takeaway: implementing generalized symmetries in string theory/brane constructions gives insights into strongly-coupled, non-Lagrangian theories.

Omissions today:
lots of progress for generalized symmetries in 3d, 4d, 5d, 6d, using geometric and brane constructions in string theory. This is in particular important for strongly-coupled, like 5d and 6d SCFTs: both symmetries and anomalies can be computed within a string theoretic realization.

Generalized global symmetries – higher-form, higher-group, non-invertible, categorical – are really fundamental, and their implications are only just getting uncovered.