HVP lattice calculation for g-2

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(BMWc)

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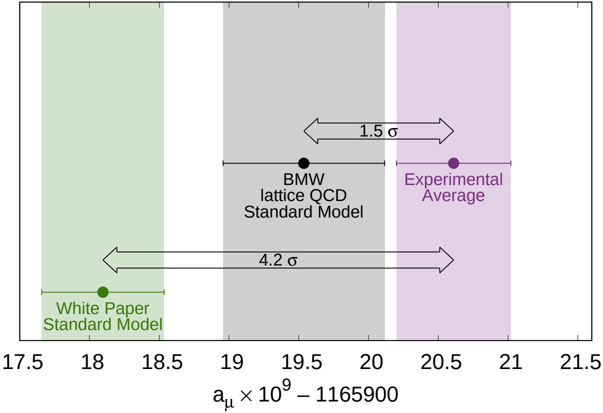
Tensions in $(g-2)_\mu$



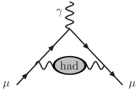
[Muon g-2 Theory Initiative, Phys.Rept. 887 (2020) 1-166]

[Budapest–Marseille–Wuppertal-coll., Nature 593 (2021) 7857]

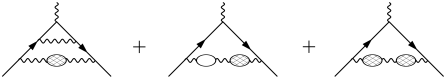
[Muon g-2 coll., Phys. Rev. Lett. 126, 141801 (2021)]

Hadronic contributions

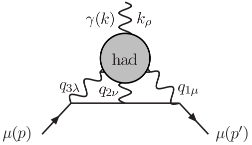
- LO hadron vacuum polarization (LO-HVP, $(\frac{\alpha}{\pi})^2$)



- NLO hadron vacuum polarization (NLO-HVP, $(\frac{\alpha}{\pi})^3$)



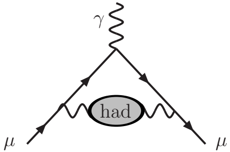
- Hadronic light-by-light (HLbL, $(\frac{\alpha}{\pi})^3$)



- pheno $a_\mu^{\text{HLbL}} = 9.2(1.9)$
  [Colangelo, Hoferichter, Kubis, Stoffer et al '15–'20]
- lattice a_μ^{HLbL}=7.9(3.1)(1.8) or 10.7(1.5)
  [RBC/UKQCD '19 and Mainz '21]

$\mathcal{O}(\alpha^2)$ hadronic contribution: $a_\mu^{\text{LO-HVP}}$

- Hadronic vacuum polarization (HVP) of photon



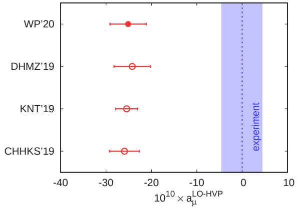
- In QED, $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137} \ll 1 \quad \longrightarrow \quad$ rapidly converging series

$$\left(\frac{g-2}{2}\right) = \left(\frac{\alpha}{\pi}\right) a^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 a^{(2)} + \left(\frac{\alpha}{\pi}\right)^3 a^{(3)} + \ldots$$

- In QCD, at low energies: $\alpha_s = \mathcal{O}(1)$
- Perturbative approach fails $\longrightarrow$ other methods are needed
  - Data driven approach (R-ratio)
  - Lattice QCD

Discrepancy

- $a_\mu^{\text{exp}} - a_\mu^{\text{theory}} = 25.1(6.0)$ around 4.2σ significance



error budget:

$(4.1)_{\text{exp}}(0.1)_{\text{QED}}(0.1)_{\text{weak}}(4.0)_{\text{HVP}}(1.8)_{\text{HLbL}}$

- HUGE: is about 2× electroweak contribution

For new physics:

- FNAL(plan) + same theory errors 6σ
- FNAL(plan) + HLbL 10% + HVP 0.2% 11σ

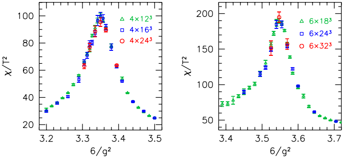
For no new physics:

- 4% larger HVP, $a_\mu^{\text{LO-HVP}} = 720.0(6.8)$
- 360% larger HLbL, $a_\mu^{\text{HLbL}} = 37.9(7.1)$

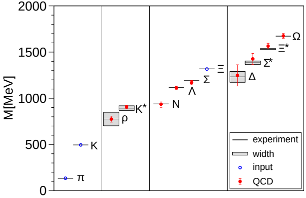
HVP from Lattice QCD

Lattice QCD: examples

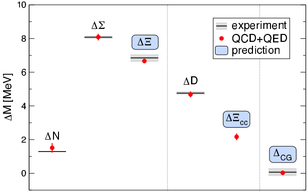
- Wuppertal–Budapest-collaboration, *The order of the quantum chromodynamics transition predicted by the standard model of particle physics*, Nature 443 (2006) 675-678



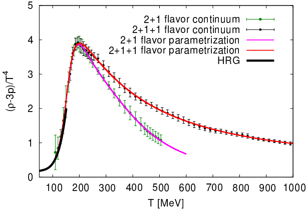
- Budapest–Marseille–Wuppertal-collaboration, *Ab-initio Determination of Light Hadron Masses*, Science 322 (2008) 1224-1227



- Budapest–Marseille–Wuppertal-collaboration, *Ab initio calculation of the neutron-proton mass difference*, Science 347 (2015) 1452-1455

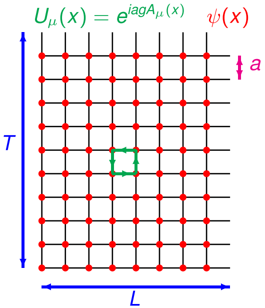


- Wuppertal–Budapest-collaboration, *Lattice QCD for Cosmology*, Nature 539 (2016) 7627, 69-71



Lattice QCD

- Lattice gauge theory: systematically improvable, non-perturbative, 1st principles method
- Discretize space-time with lattice spacing: a



- quarks on sites, gluons on links
- discretize action + operators

$$\int d^4x \longrightarrow a^4 \sum_x$$

$$\partial_\mu \longrightarrow \text{finite differences}$$

- To get physical results, need to perform:
  1. Infinite volume limit $(V \to \infty)$ $\longrightarrow$ numerically or analytically
  2. Continuum limit $(a \to 0)$ $\longrightarrow$ min. 3 different a

Lattice QCD

- Integrate over all classical field configurations

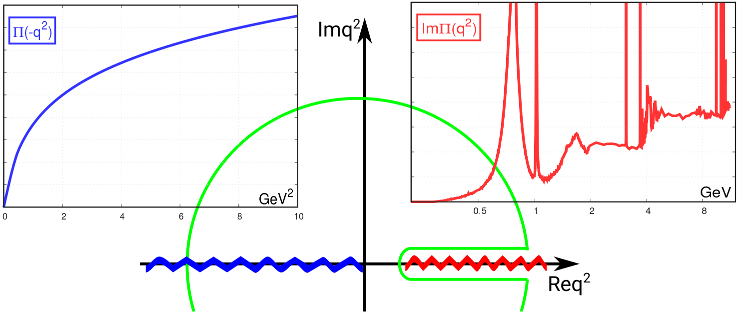
$$\int [\mathrm{d}U]\,[\mathrm{d}\overline{\psi}]\,[\mathrm{d}\psi]\;\mathcal{O}\;e^{-S_{\mathrm{g}}(U)-\overline{\psi}\,M(U)\,\psi}$$

- E.g. $96^3 \times 144$ lattice $\longrightarrow$ $\approx 4 \cdot 10^9$ dimensional integral
- Stochastic integration

- 100000 years for a laptop $\longrightarrow$ 1 year for supercomputer

Hadronic vacuum polarization

- $\Pi_{\mu\nu}(q) = \left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi(q^2)$ analytic + branch-cut



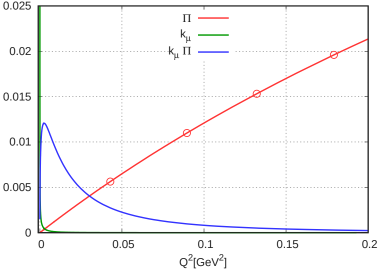
- Minkowski from R-ratio experiments
- Euclidean from lattice QCD or exp. like MUonE
- Minkowski → Euclidean via dispersion relation ($Q^2 = -q^2$)

$$\Pi(Q^2) = \int_{s_{\mathrm{th}}}^{\infty} ds\, \frac{Q^2}{s(s+Q^2)} \frac{1}{\pi} \mathrm{Im}\Pi(s)$$

$a_\mu^{\text{LO-HVP}}$ from lattice QCD

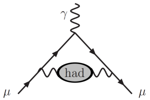
- get Π from Euclidean current-current correlator [Blum '02]

$$\Pi_{\mu\nu} = \int dx\, e^{iQx} \langle J_\mu(x) J_\nu(0) \rangle = \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2 \right) \Pi(Q^2)$$



$$a_\mu^{\text{HVP}} = \frac{\alpha^2}{\pi^2} \int dQ^2\, k_\mu(Q^2)\, \Pi(Q^2)$$

$k_\mu(Q^2)$ describes the leptonic part of diagram

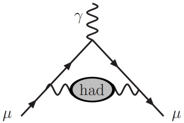


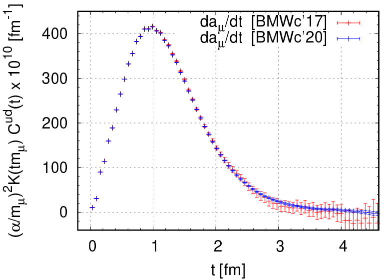
- Q is available at discrete momenta only
- smooth interpolation in Q and prescription for $\Pi(0)$

[Bernecker,Meyer '11], [HPQCD'14], ...

$a_\mu^{\text{LO-HVP}}$ from lattice QCD

- $a_\mu^{\text{LO-HVP}} = \alpha^2 \int_0^\infty dt\, K(t)\, C(t)$





- $C(t)$: Current-current correlator

$$C(t) = \frac{1}{3}\sum_{i=1}^{3}\langle J_i(t) J_i(0)\rangle$$

- $K(t)$ describes the leptonic part of diagram [Bernecker,Meyer '11], [HPQCD'14], ...

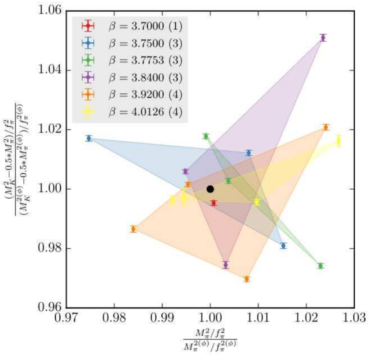
$$K(t) = \int_0^{Q^2_{max}} \frac{dQ^2}{m_\mu^2}\,\omega\!\left(\frac{Q^2}{m_\mu^2}\right)\left[t^2 - \frac{4}{Q^2}\sin^2\!\left(\frac{Qt}{2}\right)\right]$$

$$\omega(r) = [r + 2 - \sqrt{r(r+4)}]^2/\sqrt{r(r+4)}$$

- only integrate up to $Q^2_{max} = 3\,\text{GeV}^2$
- $Q^2 > Q^2_{max}$: perturbation theory

Simulation setup

- 6 lattice spacings: 0.13 fm – 0.064 fm $\longrightarrow$ controlled continuum limit
- Box size: $L \sim 6$ fm
  $L \sim 11$ fm at one lattice spacing $\longrightarrow$ FV effects
  1 fm $= 10^{-15}$ m $\sim$ size of proton
- Quark masses bracketing their physical values



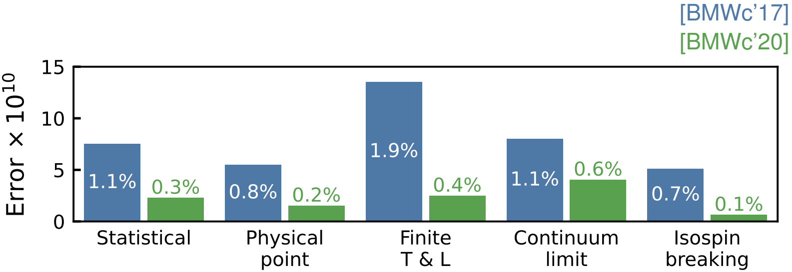
β	a[fm]	$L \times T$	#conf
3.7000	0.1315	48×64	904
3.7500	0.1191	56×96	2072
3.7753	0.1116	56×84	1907
3.8400	0.0952	64×96	3139
3.9200	0.0787	80×128	4296
4.0126	0.0640	96×144	6980

Ensembles for dynamical QED:

β	a[fm]	$L \times T$	#conf
3.7000	0.1315	24×48	716
		48×64	300
3.7753	0.1116	28×56	887
3.8400	0.0952	32×64	4253

Challenges & Improvements

Key improvements



- Incorporated many improvements and recent developments in lattice techniques
- Reduced uncertainty by factor 3.4 compared to [BMWc '17]

Noise reduction – LMA

- noise/signal in $C(t) = \langle J(t)J(0)\rangle$ grows for large distances
- Treat lowest eigenmodes of Dirac operator exactly (LMA) $\longrightarrow$ λ_i, v_i

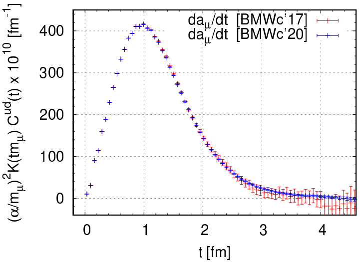
[Neff *et.al.* 2001] [Giusti et.al. 2004] [Li *et.al.* 2010] ...

- Decompose propagator

$$M^{-1} = M_e^{-1} + M_r^{-1}$$

$$M_e^{-1} = \sum_i \frac{1}{\lambda_i} v_i v_i^\dagger \quad \longleftarrow \text{exactly}$$

$$M_r^{-1} = M^{-1}\left(1 - \sum_i v_i v_i^\dagger\right) \quad \longleftarrow \text{stochastically}$$

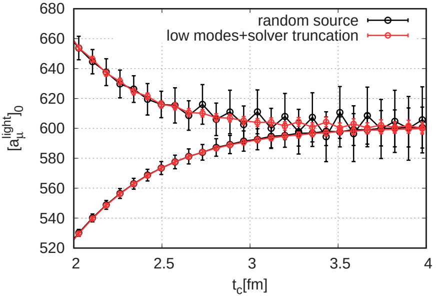


- $L = 6$ fm ≈ 1000 eigenvectors up to $\approx m_s/2$
  $L = 11$ fm ≈ 6000 eigenvectors
- Truncated Solver method (AMA) [Bali *et.al.* 2010] [Blum *et.al.* 2013]
  - Stop after 400 iterations
  - Compute correction to high precision on every 32^{nd} vector

Noise reduction – bounds

- Replace $C(t)$ by upper/lower bounds above t_c [Lehner 2016] [Borsanyi *et.al.* 2017]

$$0 \le C(t) \le C(t_c)e^{-E_{2\pi}(t-t_c)}$$



⟶ factor 5 gain in precision

⟶ bounding t_c: 3 fm → 4 fm

⟶ few permil accuracy on each ensemble

Scale determination

Lattice spacing a enters into a_μ determination:

- physical values of m_μ, m_π, m_K

$\longrightarrow$ $\Delta_{\text{scale}} a_\mu \sim 1.8 \cdot \Delta(\text{scale})$ [Della Morte *et.al.* '17]

1. For final results: M_{Ω^-} scale setting $\longrightarrow$ $a = (aM_{\Omega^-})^{\text{lat}} / M_{\Omega^-}^{\text{exp}}$
   - Experimentally well known: 1672.45(29) MeV [PDG 2018]
   - Moderate m_q dependence
   - Can be precisely determined on the lattice

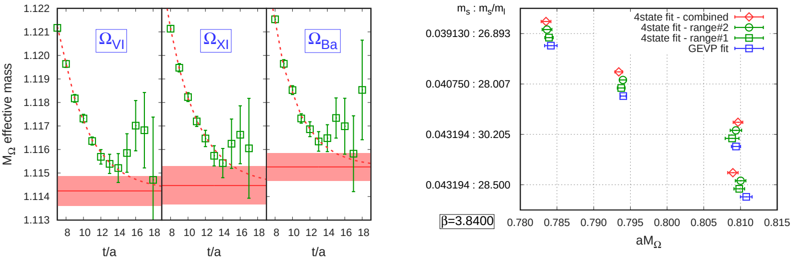
2. For separation of isospin breaking effects: w_0 scale setting
   No experimental value [Lüscher 2010] [BMWc 2012]

   $\longrightarrow$ Determine value of w_0 from $M_\Omega \cdot w_0$

$$w_0 = 0.17236(29)(63)[70]\ \text{fm}$$

M_Ω determination

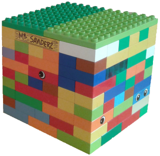
- Staggered baryon operators [Golterman & Smit 1985] [Bailey 2007]
- 2 fit ranges with 4-state fits
- mass extraction using GEVP [Aubin & Orginos 2011] [DeTar & Lee 2015]
- include all $\mathcal{O}(e^2)$ QED effects
- $\approx 0.1\%$ precision on each ensemble



Finite-size effects: lattice

- Typical lattice runs use $L \lesssim 6\,\mathrm{fm}$, earlier model estimates gave $O(2)\%$ FV effect [Aubin *et.al.* '16]
- FV correction in two steps

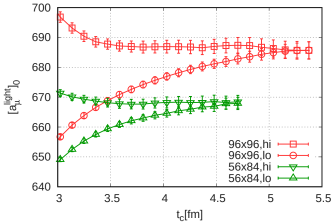
$$a_\mu(\infty) - a_\mu(\mathrm{ref}) = [a_\mu(\mathrm{big}) - a_\mu(\mathrm{ref})]_{\mathrm{4HEX}} + [a_\mu(\infty) - a_\mu(\mathrm{big})]_{\mathrm{XPT}}$$

$L_{\mathrm{ref}} = 6.272\,\mathrm{fm}$  

$L_{\mathrm{big}} = 10.752\,\mathrm{fm}$

1. $a_\mu(\mathrm{big}) - a_\mu(\mathrm{ref})$
   Choose action with small taste splitting
   - 4 steps of HEX smearing
   - DBW2 gauge action
   - $\beta = 0.73$, $a = 0.112\,\mathrm{fm}$
   - $M_\pi = 104\,\mathrm{MeV}$ and $M_\pi = 121\,\mathrm{MeV}$
   - Interpolate to $M_\pi = 110\,\mathrm{MeV}$

$$\longrightarrow \quad M_{\pi,\mathrm{HMS}}^{-2} \equiv \frac{1}{16}\sum_\alpha M_{\pi,\alpha}^{-2} = M_{\pi^0,\mathrm{phys}}^{-2}$$



$$a_\mu(\mathrm{big}) - a_\mu(\mathrm{ref}) = 18.1(2.0)_{\mathrm{stat}}(1.4)_{\mathrm{cont}}$$

Finite-size effects: non-lattice

Comparison to non-lattice approaches

- NLO and NNLO Chiral perturtabion theory (XPT) [Gasser & Leutwyler 1985] [Bijnens *et.al.* 1999]
- MLLGS-model [Gounaris & Sakurai 1968] [Lellouch & Lüscher 2001] [Meyer 2011] [Francis *et.al.* 2013]
- Hansen–Patella approach [Hansen & Patella 2019,2020]
- Rho-pion-gamma model (RHO) [Sakurai 1960], [Jegerlehner & Szafron 2011] [Chakraborty *et.al.* 2017]

	NLO XPT	NNLO XPT	MLLGS	HP	RHO
$a_\mu(\text{big}) - a_\mu(\text{ref})$	11.6	15.7	17.8	16.7	15.2

$$a_\mu(\text{big}) - a_\mu(\text{ref}) = 18.1(2.0)_{\text{stat}}(1.4)_{\text{cont}}$$

2. $a_\mu(\infty) - a_\mu(\text{big})$
   - NLO XPT: 0.3
   - NNLO XPT: 0.6

$$a_\mu(\infty) - a_\mu(\text{ref}) = 18.7(2.0)_{\text{stat}}(1.4)_{\text{cont}}(0.3)_{\text{big}}(0.6)_{I=0}(0.1)_{\text{qed}}[2.5]$$

QCD+QED

- Reach sub-percent level: include isospin breaking effects for
  $\langle jj \rangle$, masses, scale
- Rewrite dynamical QED as quenched QED expectation values

$$\langle O \rangle_{\text{QCD+unquenched QED}} = \frac{\left\langle \left\langle O(U,A)\,\frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{\text{quenched QED}} \right\rangle_{\text{QCD}}}{\left\langle \left\langle \frac{\det M(U,A)}{\det M(U,0)} \right\rangle_{\text{quenched QED}} \right\rangle_{\text{QCD}}}$$

- Expand to first order in e^2 and $\delta m = m_d - m_u$

e_v: valence charge,
e_s: sea charge

  - $O(U,A) \approx O_0(U) + \frac{\delta m}{m_l} \cdot O'_m(U) + e_v \cdot O'_1(U,A) + e_v^2 \cdot O''_2(U,A)$
  - $\left(\prod_{f=u,d,s,c} \frac{\det M^{(f)}[U,A]}{\det M^{(f)}[U,0]} \right)^{1/4} \approx 1 + e_s \cdot \frac{d_1(U,A)}{d_0(U)} + e_s^2 \cdot \frac{d_2(U,A)}{d_0(U)}$

$O(\delta m)$ sea effect vanishes

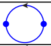
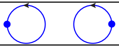
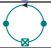
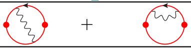
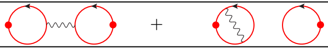
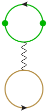
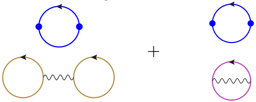
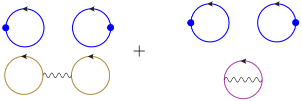
- Strategy:

[De Divitiis *et.al.* 2013] [Eichten *et.al.* 1997]

  - Take isospin symmetric gluon configurations: U
  - Measure $O_0(U)$ and $O'_m(U)$
  - For each gluon field, generate *quenched* photon fields: A
  - Measure $O'_1(U,A)$, $O''_2(U,A)$, $\frac{d_1(U,A)}{d_0(U)}$ and $\frac{d_2(U,A)}{d_0(U)}$

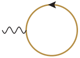
Isospin breaking contributions

$$\langle O\rangle_{\text{QCD+QED}} \approx \langle O\rangle_0 + \frac{\delta m}{m_l}\cdot\langle O\rangle'_m + e_v^2\cdot\langle O\rangle''_{20} + e_v e_s\cdot\langle O\rangle''_{11} + e_s^2\cdot\langle O\rangle''_{02}$$

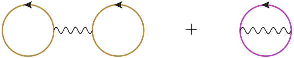
	quark-connected HVP	quark-disconnected HVP
$\langle O\rangle_0 = \langle O_0(U)\rangle_U$		
$\langle O\rangle'_m = \left\langle O'_m(U)\right\rangle_U$		
$\langle O\rangle''_{20} = \left\langle\left\langle O''_2(U,A)\right\rangle_{A,\text{q.}}\right\rangle_U$		
$\langle O\rangle''_{11} = \left\langle\left\langle O'_1(U,A)\cdot\frac{d_1(U,A)}{d_0(U)}\right\rangle_{A,\text{q.}}\right\rangle_U$		
$\langle O\rangle''_{02} = \left\langle\left(O_0(U)-\langle O_0(U)\rangle_U\right)\cdot\left\langle\frac{d_2(U,A)}{d_0(U)}\right\rangle_{A,\text{q.}}\right\rangle_U$		

Isospin breaking calculations

- $\mathcal{O}_1'(U,A)$ and $\mathcal{O}_2''(U,A)$: compute as finite differences
  - Measure $\mathcal{O}(0)$, $\mathcal{O}(\frac{1}{3}e_*)$, $\mathcal{O}(-\frac{1}{3}e_*)$
- $\mathcal{O}_m'(U)$
  - light connected: as derivative
  - disconnected: as finite difference: $\mathcal{O}(m_l, 0)$, $\mathcal{O}(0.9\, m_l, 0)$
- $\frac{d_1(U,A)}{d_0(U)}$ =

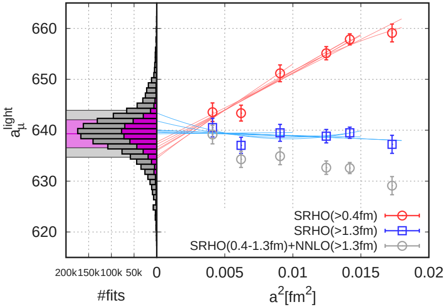


  - 1 photon field on each gluon configuration: same as in $\mathcal{O}_1'(U,A)$
  - Exact trace on low-lying eigenspace
  - 12000 random sources
  - Reduce UV noise by exact rewriting using HPE
- $\frac{d_2(U,A)}{d_0(U)}$ =

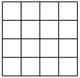


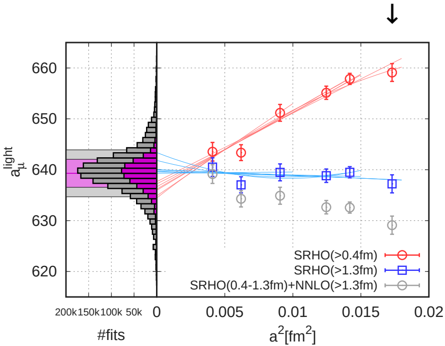
  - 2000 photon fields on each gluon configuration
  - 12 random sources on each photon field
  - Contact term is treated as d_1/d_0

Continuum limit

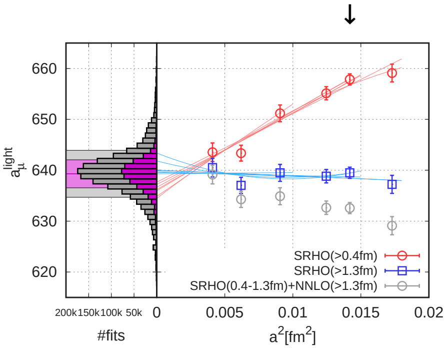


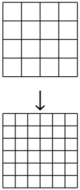
Continuum limit



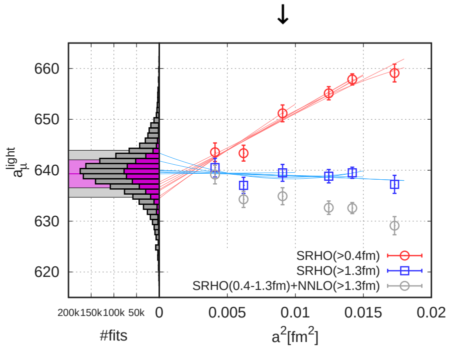


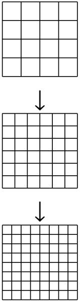
Continuum limit



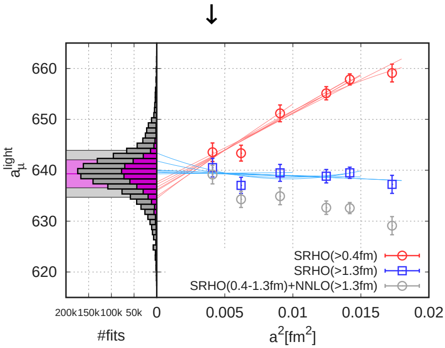


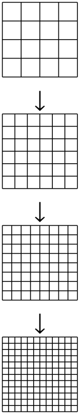
Continuum limit



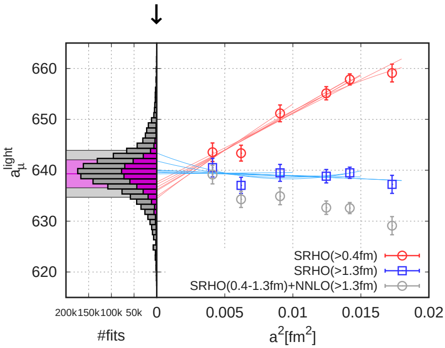


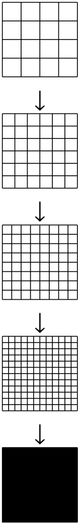
Continuum limit





Continuum limit

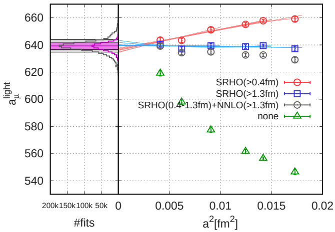


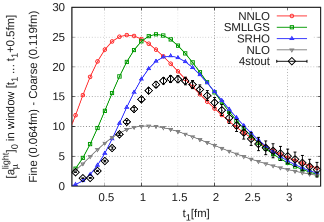


Continuum limit – Taste improvement

Controlled $a \to 0$ extrapolation

- 6 lattice spacings: 0.132 fm $\longrightarrow$ 0.064 fm
- Leading cutoff effects at large t are taste breaking effects $\longrightarrow$ mass effects
- Distortion in spectrum: cured by taste improvement rho-pion-gamma model (SRHO)
  [Sakurai '60][Bijnens *et.al.* '99][Jegerlehner *et.al.* '11][Chakraborty *et.al.* '17]
- Our data confirms: Taste violation according to SRHO describes most of the lattice artefacts in a_μ^{light}
- Central value obtained using SRHO improvement
- At $t > 1.3$ fm add and subtract (NNLO – SRHO)
  [Aubin *et.al.* '20]
- Error corresponding to this variation
  $\longrightarrow$ Add to systematic error in quadrature





Continuum limit – Global fit procedure

- For full result: physical point is set via

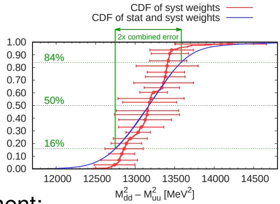
  $M_{\Omega_-}, \quad M^2_{K_\chi} = \frac{1}{2}\left(M^2_{K_0} + M^2_{K_+} - M^2_{\pi_+}\right), \quad \Delta M^2_K, \quad M^2_{\pi_0}$ ⟵ Type-I
- For IB-decomposition: match QCD+QED and $\mathrm{QCD}_{\mathrm{iso}}$ via

  $w_0, \quad M^2_{ss}, \quad \Delta M^2 = M^2_{dd} - M^2_{uu}, \quad M^2_{\pi_\chi} = \frac{1}{2}\left(M^2_{uu} + M^2_{dd}\right)$ ⟵ Type-II
- Expand observable around physical point

$$Y = A + BX_l + CX_s + DX_{\delta m} + Ee_v^2 + Fe_v e_s + Ge_s^2$$

- Combined χ^2 fit for all components
- Several hundreds of thousands of analyses, combined using histogram method

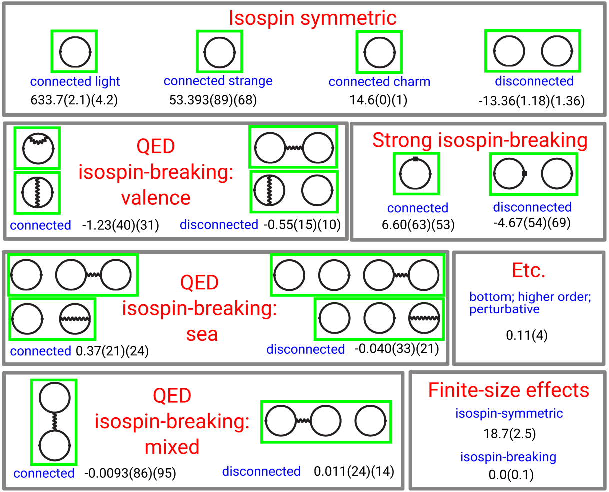
  linear vs. quadratic, a^2 vs $a^2\alpha_s(1/a)^3$ [Husung *et.al* 2020]
  cuts in lattice spacing, hadron mass fit ranges, ...



- Uncertainty arising from choice of taste improvement: Added to systematic error in quadrature

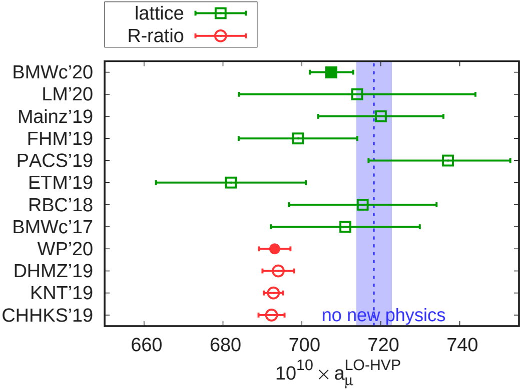
Results

Overview of contributions



$$10^{10}\times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$$

Comparison with other determinations of HVP

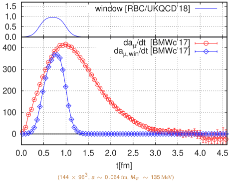


- $a_\mu^{\text{LO-HVP}} = 707.5(2.3)(5.0)[5.5]$ with 0.8% accuracy
- Compatible with other lattice calculations
- First lattice calculation with errors comparable to R-ratio results

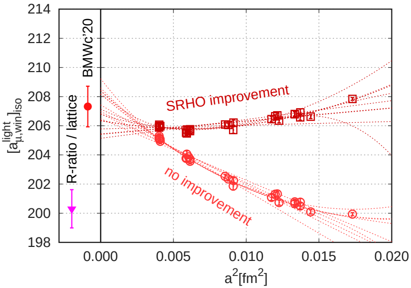
Window observable

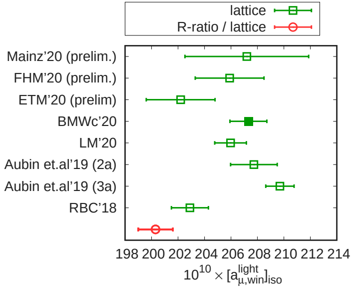
- Restrict correlator to window between $t_1 = 0.4\,\mathrm{fm}$ and $t_2 = 1.0\,\mathrm{fm}$

[RBC/UKQCD'18]



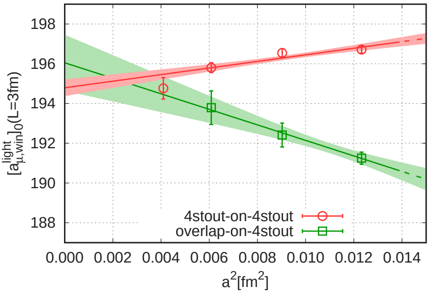
- Less challenging than full a_μ
  - signal/noise
  - finite size effects
  - lattice artefacts (short & long)





Window observable – Overlap crosscheck

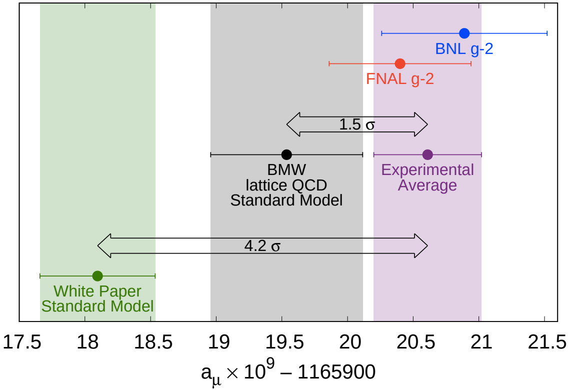
- $L = 3\,\text{fm}$
- Valence: overlap fermions, local current
- Sea: `4stout` staggered



- Continuum limit is consistent with staggered valence

Conclusions & Outlook

Conclusions & Outlook



- Important to have crosschecks from other lattice groups
- Important to understand disagreement with R-ratio, in particular in the window