Scheme dependence in pQCD at the four loop level

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Overview

We consider perturbative series to the four-loop level in the symmetric momentum subtraction schemes in massless perturbative QCD to investigate scheme dependence as a theory lab.

Renormalization

- Interactions in Quantum Field Theories result in divergent calculations.
- Divergences can be described through regularisation.
- The theory is renormalized by mapping the classical variables onto quantum variables through the inclusion of counter-terms in the Lagrangian.
- There is an ambiguity in the choice of finite counter-terms.
- A scheme provides a set of conditions to ensure the calculation is finite, fixing the counter-terms.

Scheme Difference in Measurables

- Measurable quantities do not depend on the choice of scheme.
- In pQCD measurables are approximated with perturbative series in terms of the running coupling constant for QCD.
- To all orders series will represent measurable. However, due to truncation, perturbative series will have a finite scheme dependence.
- The difference of the values calculated for a measurable ρ calculated in two schemes S_1 and S_2 up to order N is

$$\rho^{S_1}_{(N)}(Q) - \rho^{S_2}_{(N)}(Q) = \mathcal{O}(a_0(Q)^{N+1})$$

where $a_0(Q)$ is the leading order coupling constant.

Minimal Subtraction Schemes

- Schemes differ by the choice of the finite part of the counter-terms and the kinematic configuration in which they are defined.
- The Minimal Subtraction scheme (MS) sets the finite contribution to the counter-terms to zero which is true in all kinematic configurations.
- The modified Minimal Subtraction scheme ($\overline{\text{MS}}$) changes MS by defining counter-terms to remove factor of $4\pi e^{\gamma}$ from calculations.
- These schemes are computationally simple because the finite order contributions from the ϵ expansion in dimensional regularisation do not need to be calculated.
- Why consider other schemes?
  1. No reference to kinematics.
  2. Expensive on the lattice - difficult to match calculations.
  3. Investigate scheme dependence.

Symmetric MOM schemes

Symmetric momentum subtraction schemes are defined by requiring:

1. Gluon two point function has no loop corrections at momenta $p^2 = -M^2$
2. Characteristic vertex function has no finite order corrections in symmetric momentum configuration $p_i^2 = -M^2$

The schemes considered were:

- MOMg using triple gluon vertex
- MOMq using quark-antiquark-gluon vertex
- MOMc using ghost-antighost-gluon vertex

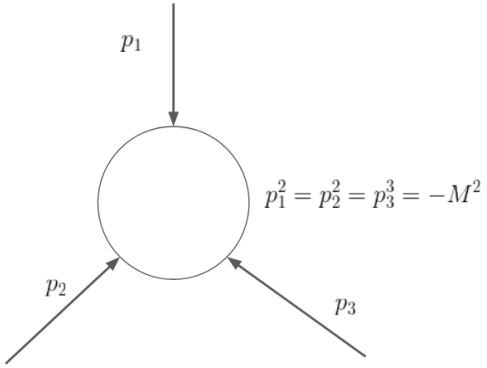


Figure 1: Diagrammatic representation of three point function vertex with momenta labelled.

Changing schemes

Four loop series for measurable ρ in scheme S_1:

$$\begin{aligned}\rho(Q) \approx \rho^{S_1}_{(4)}(Q) = \; & \rho^{S_1}_0 + \rho^{S_1}_1 a_{S_1}(Q) + \rho^{S_1}_2 a^2_{S_1}(Q) + \rho^{S_1}_3 a^3_{S_1}(Q) \\ & + \rho^{S_1}_4 a^4_{S_1}(Q)\end{aligned}$$

Process of changing schemes:

1. Apply coupling constant conversion functions
2. Truncate series
3. Write running in terms of renormalization scale
4. Find common unit with Λ ratio

Changing schemes

Four loop series for measurable ρ in scheme S_1:

$$\begin{aligned}\rho(Q) \approx\ & \rho_0^{S_1} + \rho_1^{S_1} a_{S_2}(Q) + (\rho_2^{S_1} + c_1^{S_1,S_2}\rho_1^{S1}) a_{S_2}^2(Q) + (\ldots) a_{S_2}^3(Q) \\ & + (\ldots) a_{S_2}^4(Q) + (\ldots) a_{S_2}^5(Q) + +(\ldots) a_{S_2}^6(Q) + \ldots\end{aligned}$$

Process of changing series to scheme S_2:

1. **Apply coupling constant conversion functions**
   $a_{S_1} = a_{S_2} + c_1^{S_1,S_2} a_{S_2}^2 + c_2^{S_1,S_2} a_{S_2}^3 + c_3^{S_1,S_2} a_{S_2}^4$
2. Truncate series
3. Write in terms of renormalization scale
4. Find common unit with Λ ratio

Changing schemes

Four loop series for measurable ρ in scheme S_2:

$$\rho(Q) \approx \rho^{S_2}_{(4)}(Q) = \rho^{S_2}_0 + \rho^{S_2}_1 a_{S_2}(Q) + \rho^{S_2}_2 a^2_{S_2}(Q) + \rho^{S_2}_3 a^3_{S_2}(Q) + \rho^{S_2}_4 a^4_{S_2}(Q)$$

Process of changing series to scheme S_2:

1. Apply coupling constant conversion functions
2. **Truncate series**
   $\rho^{S_2}_n := \mathcal{O}(a^n_{S_2})$ coefficient; $\rho^{S_2}_{n>4} \to 0$
3. Write in terms of renormalization scale
4. Find common unit with Λ ratio

Changing schemes

Four loop series for measurable ρ in schemes S_1 and S_2:

$$\begin{aligned}\rho(Q) &\approx \rho_0^{S_2} + \rho_1^{S_2} a_{S_2}(Q) + \rho_2^{S_2} a_{S_2}^2(Q) + \rho_3^{S_2} a_{S_2}^3(Q) + \rho_4^{S_2} a_{S_2}^4(Q) \\ &\approx \rho_0^{S_1} + \rho_1^{S_1} a_{S_1}(Q) + \rho_2^{S_1} a_{S_1}^2(Q) + \rho_3^{S_1} a_{S_1}^3(Q) + \rho_4^{S_1} a_{S_1}^4(Q)\end{aligned}$$

Process of changing series to scheme S_2:

1. Apply coupling constant conversion functions
2. Truncate series
3. **Write in terms of renormalization scale**

$$\begin{aligned}a^S(Q,\Lambda^S) \approx \frac{1}{\beta_0^S L^s}\Bigg[1 - \frac{\beta_1^S \ln(L^S)}{\beta_0^{S2} L^S} + (\beta_1^{S2}(\ln^2(L^S) - \ln(L^S) - 1) + \beta_0^S \beta_2^S)\frac{1}{\beta_0^{S4} L^{S2}}) \\ + \left(\beta_1^{S3}\left(\ln^3(L^S) - \frac{5}{2}\ln^2(L^S) - 2\ln(L^S) + \frac{1}{2}\right) + 3\beta_0^S\beta_1^S\beta_2^S \ln(L^S) - \frac{1}{2}\beta_0^{S2}\beta_3^S\right)\frac{1}{\beta_0^{S6} L^{S3}}\Bigg]\end{aligned}$$

$$L^S = \ln(Q^2/\Lambda_S^2)$$

4. Find common unit with Λ ratio

Changing schemes

Four loop series for measurable ρ in schemes S_1 and S_2:

$$\begin{aligned}\rho(Q) &\approx \rho_0^{S_2} + \rho_1^{S_2} a_{S_2}(Q) + \rho_2^{S_2} a_{S_2}^2(Q) + \rho_3^{S_2} a_{S_2}^3(Q) + \rho_4^{S_2} a_{S_2}^4(Q) \\ &\approx \rho_0^{S_1} + \rho_1^{S_1} a_{S_1}(Q) + \rho_2^{S_1} a_{S_1}^2(Q) + \rho_3^{S_1} a_{S_1}^3(Q) + \rho_4^{S_1} a_{S_1}^4(Q)\end{aligned}$$

Process of changing series to scheme S_2:

1. Apply coupling constant conversion functions
2. Truncate series
3. Write in terms of renormalization scale
4. **Find common unit with Λ ratio**

$$a^{S_2}(Q, \Lambda^{S_2}) \approx \frac{1}{\beta_0 \ln\left(\left(\frac{Q}{\Lambda_{S_1}} \frac{\Lambda_{S_1}}{\Lambda_{S_2}}\right)^2\right)}, \quad \frac{\Lambda_{S_1}}{\Lambda_{S_2}} = \exp\left(\frac{c_1^{S_1,S_2}}{2\beta_0}\right)$$

Example: R-Ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = R_0 + R_1 a^{RR}$$

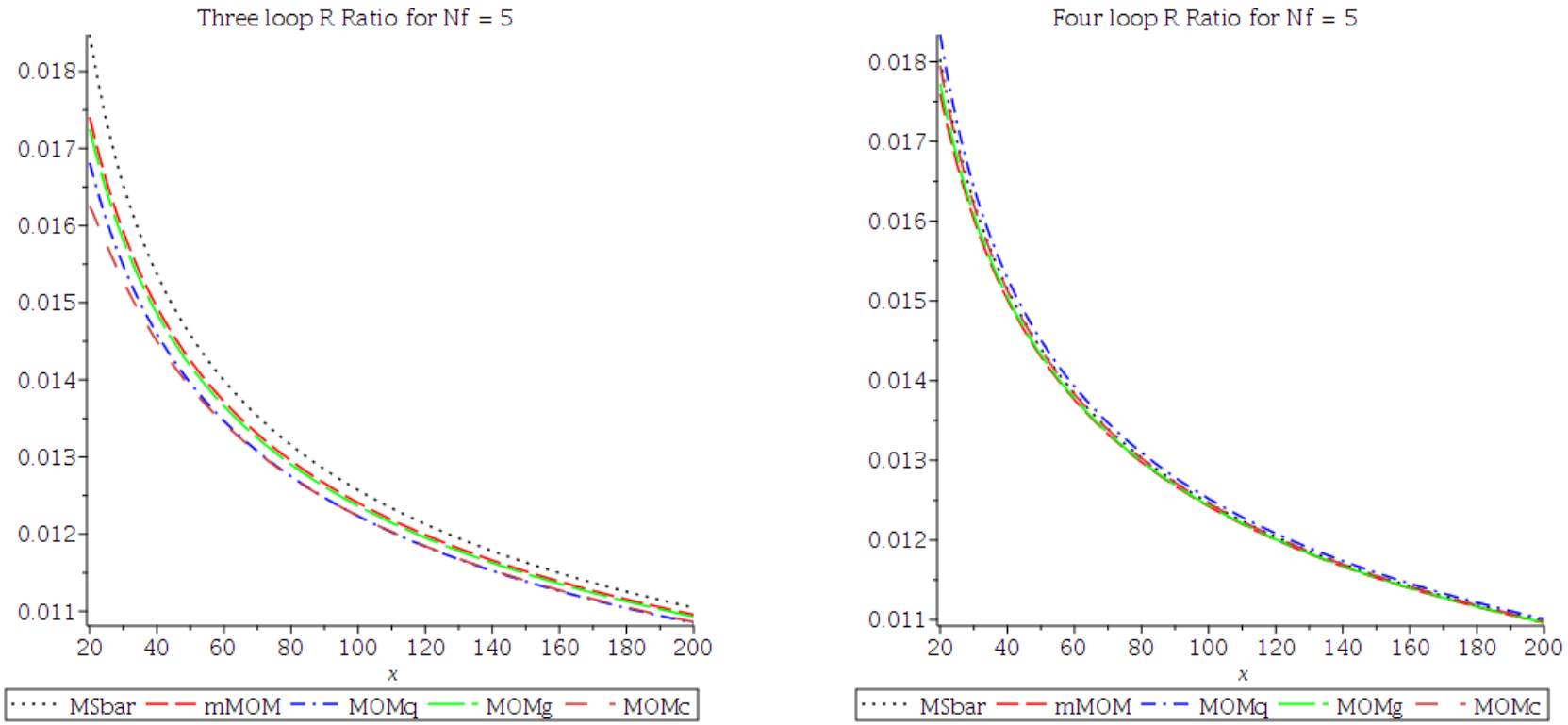
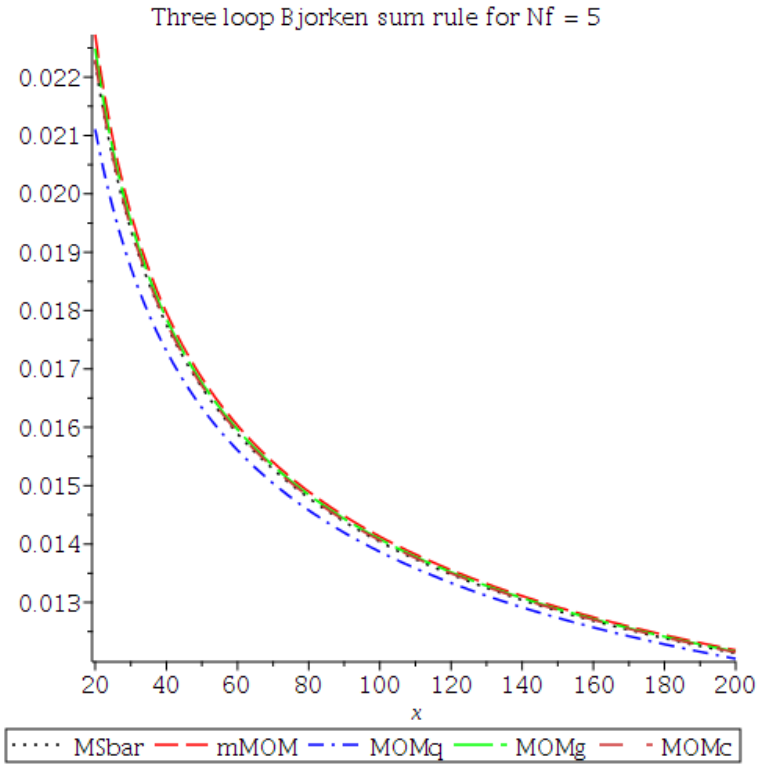


Figure 2: Three and four loop R-Ratio effective coupling constant with N_f = number of active quarks and x = running momenta in units of $\Lambda^{N_f}_{\overline{\text{MS}}}$.

Example: Bjorken

$$C^{Bjp}(a_s) = \frac{6}{g_A}\Gamma_1^{p-n}(Q^2) = C_0 + C_1 a^C$$



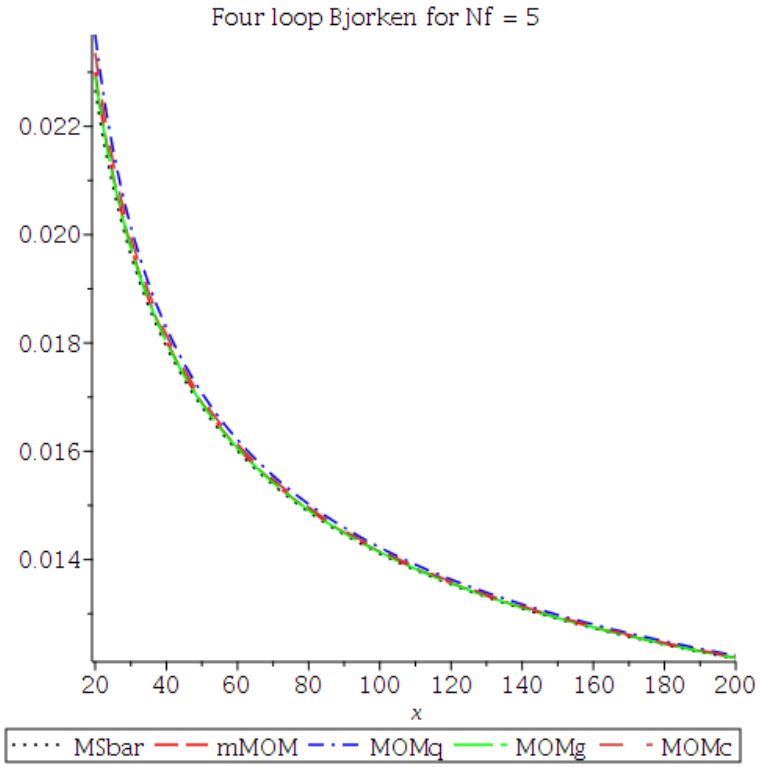


Figure 3: Three and four loop Bjorken sum rule effective coupling constant with N_f = number of active quarks and x = running momenta in units of $\Lambda^{N_f}_{\overline{\text{MS}}}$.

Theory error in pQCD

- Measurable quantities are renormalization group independent, this can be described in the equation

$$\mu \frac{d}{d\mu} \rho(\mu) = 0$$

- A series calculated in pQCD to finite order should have an associated uncertainty.
- Can use expected renormalization group invariance to estimate error.
- Usual measure of error rescales unphysical momenta scale to estimate error

$$\rho \approx \rho_{(N)}^{S_2}(Q)^{+(\rho_{(N)}^{S_2}(Q/2)-\rho_{(N)}^{S_2}(Q))}_{-(\rho_{(N)}^{S_2}(Q)-\rho_{(N)}^{S_2}(2Q))}.$$

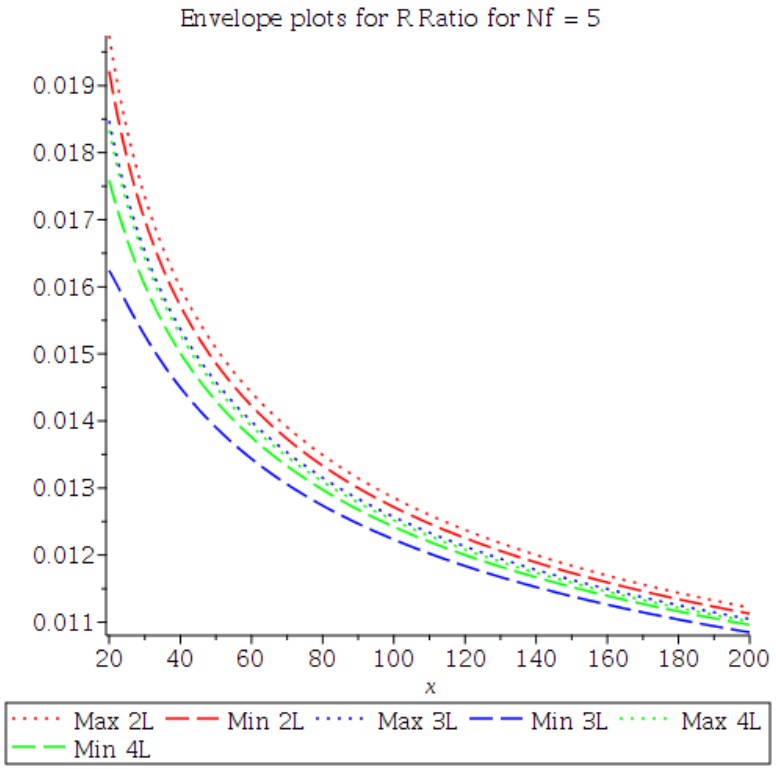
Scheme error

- Can we use scheme difference at finite order to calculate theoretical uncertainty in a series?
- We want to calculate measurable ρ at scale Q

$$\rho(Q) = \rho^{S_1}_{(N)} + \Delta(\rho^{S_1}, N, Q) = \rho^{S_2}_{(N)} + \Delta(\rho^{S_2}, N, Q)$$

- $\Delta(\rho^S, N, Q)$ represents our ignorance in the higher order of the series.
- If the sign of $\Delta(\rho^{S_1}, N, Q)$ and $\Delta(\rho^{S_2}, N, Q)$ are different then $\rho^{S_1}_{(N)}$ and $\rho^{S_2}_{(N)}$ bound the true value $\rho(Q)$.
- Considering more schemes improves probability that true series lies in envelope given by the largest and smallest value of $\rho^{S_i}_{(N)}$

Example



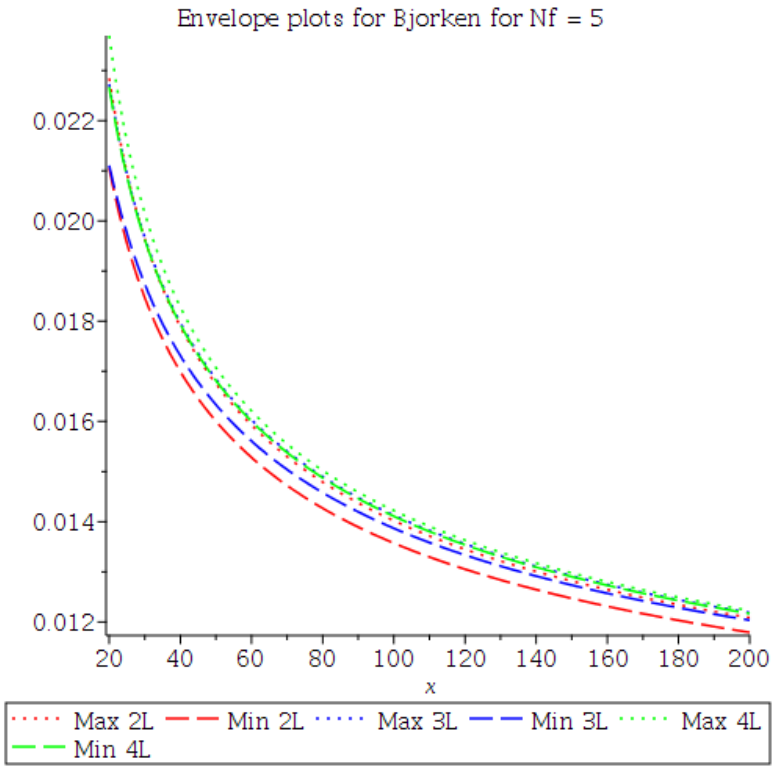


Figure 4: Comparison of envelope graphs constructed by taking the maximum and minimum value of the series in the different schemes (MOMi, mMOM and $\overline{\text{MS}}$) at each scale

Example: R-Ratio

$$a^{RR}|_{N_f=5} = a_{\mathrm{mMOM}} - 2.89086 {a_{\mathrm{mMOM}}}^2 - 299.03413 {a_{\mathrm{mMOM}}}^3 + 407.37433 {a_{\mathrm{mMOM}}}^4 + \mathcal{O}({a_{\mathrm{mMOM}}}^5),$$

$$a^{RR}|_{N_f=5} = a_{\mathrm{MOMc}} - 7.35580 {a_{\mathrm{MOMc}}}^2 - 258.15390 {a_{\mathrm{MOMc}}}^3 + 3883.57250 {a_{\mathrm{MOMc}}}^4 + \mathcal{O}({a_{\mathrm{MOMc}}}^5),$$

$$a^{RR}|_{N_f=5} = a_{\mathrm{MOMg}} - 3.77154 {a_{\mathrm{MOMg}}}^2 - 275.52688 {a_{\mathrm{MOMg}}}^3 + 1671.72909 {a_{\mathrm{MOMg}}}^4 + \mathcal{O}({a_{\mathrm{MOMg}}}^5),$$

$$a^{RR}|_{N_f=5} = a_{\mathrm{MOMq}} - 5.52330 {a_{\mathrm{MOMq}}}^2 - 168.45783 {a_{\mathrm{MOMq}}}^3 + 1337.29074 {a_{\mathrm{MOMq}}}^4 + \mathcal{O}({a_{\mathrm{MOMq}}}^5),$$

$$a^{RR}|_{N_f=5} = a_{\overline{\mathrm{MS}}} + 5.63692 {a_{\overline{\mathrm{MS}}}}^2 - 204.27304 {a_{\overline{\mathrm{MS}}}}^3$$

Using experimental data

- Proof of concept: Extract theoretical quantities directly from experimental values for comparison between schemes.
- Suppose we have a value related to the measurable ρ_{Q_E} which has been measured at momentum Q_E, we can solve numerically to find Λ_S.
- We can run this value up to the

$$a_S(M_Z) \approx a^S(M_Z, \Lambda_S) \approx \frac{1}{\beta_0 \ln\left(\left(\frac{M_Z}{\Lambda_S}\right)^2\right)} + \ldots$$

- Convert all such values to a single scheme using the coupling constant conversion functions.

Example: R-Ratio

$$\alpha_{RR} = 0.13697 \pm 0.01225$$
$$\text{at } Q_E = 82.15838 \text{ GeV}$$

Scheme	N	$\alpha_{\overline{\text{MS}}}(M_Z)$	Scheme	N	$\alpha_{\overline{\text{MS}}}(M_Z)$
$\overline{\text{MS}}$	2	$0.12725^{+0.01066}_{-0.01053}$	MOMc	2	$0.12677^{+0.01054}_{-0.01039}$
	3	$0.12982^{+0.01126}_{-0.01124}$		3	$0.13201^{+0.01203}_{-0.01237}$
	4	$0.13056^{+0.01149}_{-0.01154}$		4	$0.13059^{+0.01150}_{-0.01156}$
mMOM	2	$0.12635^{+0.01044}_{-0.01027}$	MOMg	2	$0.12637^{+0.01045}_{-0.01028}$
	3	$0.13085^{+0.01162}_{-0.01174}$		3	$0.13115^{+0.01171}_{-0.01187}$
	4	$0.13081^{+0.01159}_{-0.01169}$		4	$0.13077^{+0.01157}_{-0.01166}$
MOMq	2	$0.12650^{+0.01048}_{-0.01031}$	Average	2	$0.12680 \pm 0.00045^{+0.01066}_{-0.01027}$
	3	$0.13216^{+0.01203}_{-0.01230}$		3	$0.13099 \pm 0.00117^{+0.01203}_{-0.01124}$
	4	$0.13024^{+0.01137}_{-0.01137}$		4	$0.13053 \pm 0.00028^{+0.01159}_{-0.01137}$

[5] R. Marshall. A Determination of the Strong Coupling Constant α_s From e^+e^- Total Cross-section Data. Z. Phys. C, 43:595, 1989.

Thank you for listening

What have we done?

- Investigated scheme dependence in perturbative series up to the four loop level
- Series include: Bjorken sum rule and R-Ratio
- Schemes include:MOMg, MOMq, MOMc, $\overline{\text{MS}}$ and mMOM
- Explore the use of scheme difference as a measure of error
- Applied this idea to calculate fundamental theoretical quantities in different schemes based on experimental data.

References

[1] J. A. Gracey. Two loop QCD vertices at the symmetric point.Physical Review D, 84(8), Oct 2011
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[3] Alexander Bednyakov and Andrey Pikelner. Four-loop QCD MOM beta functions from the three-loop vertices at the symmetric point.Physical Review D, 101(7), Apr 2020.
[4] William Celmaster and Richard J. Gonsalves. Renormalization-prescription dependence ofthe quantum-chromodynamic coupling constant. Phys. Rev. D, 20:1420–1434, Sep 1979
[5] R. Marshall. A Determination of the Strong Coupling Constant α_s From e^+e^- Total Cross-section Data. Z. Phys. C, 43:595, 1989.