# Holomorphic Modular Bootstrap

Based on: arXiv:2108.01060 (JHEP 11(2021)195) and arXiv:2012.14939 (JHEP 04(2021)294)

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YTF '21
Online Presentation

16 December 2021

# CFT

- CFTs: QFTs with conformal symmetry - Poincare + Scaling + Spl. Conformal Transformations

- 2D CFTs: CFTs in (1+1) dimensions. Special: Lie Algebra is infinite dimensional, torus partition function is modular invariant,

$$\boxed{\mathcal{Z}(\gamma\tau, \gamma\bar{\tau}) = \mathcal{Z}(\tau, \bar{\tau}) \quad \forall \gamma \in SL(2,\mathbb{Z})} \qquad \text{...(1)}$$

- Why study 2D CFT?
  - String theory is a 2D CFT!
  - Relevant for studying 2D critical systems at the fixed point of RG flow.
  - 2D Holography: $AdS_3/CFT_2$ - many elegant results like Cardy-Calbrese entanglement entropy, Brown-Henneaux-York stress trensor, etc.
  - Representation Theory, Infinite dimensional Lie Algebras, Vertex operator algebras, theory of modular forms, etc.

# RCFT

- 2D CFT with a finite number of primary fields [1] and with central charge $c$ and conformal dimensions $h_i \in \mathbb{Q} \ \forall \ i$.

- The torus partition function can be written as (by modular invariance),

$$\mathcal{Z}(\tau, \bar{\tau}) = \sum_{i,\, j=0}^{n-1} M_{ij}\, \chi_i(\tau)\, \chi_j(\bar{\tau}) \qquad \text{...(2)}$$

where $\mathbb{Z}_0^+ \ni m_{ij} \in M_{ij}$ and the characters are,

$$\boxed{\chi_0(\tau) = \sum_{m=0}^{\infty} a_m^{(0)}\, q^{\alpha_0+m}, \quad \chi_i(\tau) = D \sum_{m=0}^{\infty} a_m^{(i)}\, q^{\alpha_i+m}} \qquad \text{...(3)}$$

with $q = e^{2\pi i \tau}, \alpha_i = h_i - \dfrac{c}{24}$ $(h_0 = 0)$ and $\chi_0(\tau)$: Identity character.

- RCFT Data: $c$, $h_i$s and OPE coefficients. Unitarity: $c > 0,\ h_i \geq 0\ \forall i$.

- Examples: Free boson/fermion, Critical Ising model, WZW CFTs, etc.

[1] G. Anderson and G. Moore, Rationality in Conformal Field Theory

# Modular Forms

- Consider holomorphic $f : \mathbb{H} \to \mathbb{C}$ such that,

$$\boxed{f(\gamma\tau) = f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau),\ \forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})} \qquad \text{...(4)}$$

and $f(\tau)$ is holomorphic at $i\infty \Rightarrow f(\tau)$: weight $k$ modular form.

- Example: Eisenstein series (modular form of weight $k > 2$),

$$E_k(\tau) = \frac{1}{2} \sum_{\substack{p,q \in \mathbb{Z} \\ gcd(p,q)=1}} \frac{1}{(p\tau+q)^k} = \sum_{m=0}^{\infty} E_{k,m}\, e^{2\pi i \tau m} \qquad \text{...(5)}$$

- $M_k(SL(2,\mathbb{Z}))$: Even dimensional $\mathbb{C}$-vector space of weight $k(>2)$ modular forms. Basis = $\{E_4^{\theta} E_6^{\phi} \mid 4\theta + 6\phi = k\}$.

- Serre-Ramanujan derivative $\mathscr{D} : M_k(SL(2,\mathbb{Z})) \to M_{k+2}(SL(2,\mathbb{Z}))$,

$$\boxed{\mathscr{D} := \partial_\tau - \frac{i\pi k}{6} E_2(\tau)} \qquad \text{...(6)}$$

# Modular Linear Differential Equation (MLDE)

- $\chi_i(\tau)$s are weight $0$ modular functions (meromorphic in $\mathbb{H}$ and at $i\infty$).
- It is known [2] that the $n$ characters of a RCFT are the linearly independent solutions of a single $n^{th}$ order ODE in the moduli space of the torus, a modular linear differential equation (MLDE),

$$\boxed{\mathscr{D}^n\chi_i + \sum_{r=0}^{n-1}\phi_r(\tau)\,\mathscr{D}^r\chi_i = 0} \qquad \left(\text{with } \ \phi_r = (-1)^{n-r}\frac{\mathscr{W}_r}{\mathscr{W}}\right) \qquad \ldots(7)$$

- The Wronskian $\mathscr{W}$ will be a weight $n(n-1)$ modular function.
- Wronskian index $l$ (from Valence formula),

$$\boxed{\mathbb{Z}_0^+ \ni l\,(\text{order of zeros of } \mathscr{W}) = \frac{n(n-1)}{2} + \frac{nc}{4} - 6\sum_{i=0}^{n-1} h_i} \qquad \ldots(8)$$

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[2] S.D. Mathur, S. Mukhi and A. Sen, On the Classification of Rational Conformal Field Theories}, Phys. Lett. B 213 303 (1988).

# MMS Classification

- A given MLDE is classified by two parameters: $(n, l)$. We can scan different values of $(n, l)$ and seek Frobenius type of character-like solutions of the form (3).
- Integrality constraint: $a_m^{(0)} \in \mathbb{Z}_0^+$ since these Fourier coefficients are state degeneracies at a given level.
- Vacuum constraint: $\chi_0(\tau)$ should have $a_0^{(0)} = 1$ which means that the vacuum state is non-degenerate.
- Stability constraint: $a_m^{(i\neq 0)} \in \mathbb{Q}_0^+$ (as these can be degenerate). Then, one needs to find the LCM of the denominators $(=: D)$ of these rational coefficients which should stabilise for $m \to \infty$. Then, $D\, a_m^{(i\neq 0)} \in \mathbb{Z}_0^+$.
- The remaining solutions will then be called admissible character-like solutions.
- RCFT constraint: One then needs to compute Fusion coefficients for admissible character-like solutions and check if they are positive.
- These will then be the characters of genuine RCFTs.

# (2,0) MLDE: MMS Procedure

- Let us consider the (2,0) MLDE and the corresponding valence formula:

$$\mathcal{D}^2\chi + \mu_{1,0}\, E_4\, \chi = 0, \qquad 1 + \frac{c}{2} - 6h = 0 \qquad \text{...(9)}$$

- Plug (3) and (5) in (9) to get a recursive relation,

$$(m+\alpha)^2 a_m - \frac{1}{6}\sum_{k=0}^{m}(m-k+\alpha)E_{2,k}\, a_{m-k} + \mu_{1,0}\sum_{k=0}^{m} E_{4,k}\, a_{m-k} = 0 \qquad \text{...(10)}$$

- Indicial equation (m=0 in (10)):

$$\alpha^2 - \frac{1}{6}\alpha + \mu_{1,0} = 0 \qquad \text{...(11)}$$

which means $\alpha_1 = \frac{1}{6} - \alpha_0$ and $\mu_{1,0} = \alpha_1\alpha_0 = \alpha_0\left(\frac{1}{6} - \alpha_0\right)$

- Putting $m = 1$ in (10):

$$a_1^{(0)} = \frac{24\alpha_0(60\alpha_0 - 11)}{5 + 12\alpha_0} \qquad \text{...(12)}$$

# (2,0) MLDE: MMS Procedure (contd.)

- (12) implies (with $N := -120\alpha_0$):

$$N^2 + 22N + N a_1^{(0)} - 50\, a_1^{(0)} = 0 \quad \Rightarrow \quad a_1^{(0)} = \frac{N(N+22)}{50-N} \qquad \text{...(13)}$$

- Since $a_1^{(0)} \in \mathbb{Z}_0^+$, we have that $N \in \mathbb{Z}$ (using the Integer Root Theorem) which implies, $5c \in \mathbb{Z}$. Also, as $a_1^{(0)} \geq 0$ we have $c < 10$.
- Now we solve for integral $a_1^{(0)}$ in (13), $\forall\ 1 \leq N < 50$.
- Next we compute $a_m^{(0)}$s by using the recursive relation (10) upto very high orders of $m$, say $m = 2000$ and impose positivity and integrality of the Fourier coefficients $a_m^{(0)}$s.
- Then we check for solutions obeying the stability constraint.
- For $(2,0)$ MLDE, going upto $m = 4$ is sufficient enough to rule out non-character-like solutions.

# (2,0) MLDE: Solutions

| S. No | $N$ | $c$ | $a_1^{(0)}$ | $h$ | RCFT |
|---|---|---|---|---|---|
| 1 | 2 | 2/5 | 1 | 1/5 | $\mathcal{M}(5,2)$ |
| 2 | 5 | 1 | 3 | 1/4 | $(\hat{A}_1)_1$ |
| 3 | 10 | 2 | 8 | 1/3 | $(\hat{A}_2)_1$ |
| 4 | 14 | 14/5 | 14 | 2/5 | $(\hat{G}_2)_1$ |
| 5 | 20 | 4 | 28 | 1/2 | $(\hat{D}_4)_1$ |
| 6 | 26 | 26/5 | 52 | 3/5 | $(\hat{F}_4)_1$ |
| 7 | 30 | 6 | 78 | 2/3 | $(\hat{E}_6)_1$ |
| 8 | 35 | 7 | 133 | 3/4 | $(\hat{E}_7)_1$ |
| 9 | 38 | 38/5 | 190 | 4/5 | $E_{7\frac{1}{2}}$ |
| 10 | 40 | 8 | 248 | 5/6 | $(\hat{E}_8)_1$ |

- The $9^{th}$ entry in the table fails to satisfy fusion rules and hence is not a RCFT (but is of considerable interest to mathematicians).
- The $10^{th}$ entry is actually a $1$-character RCFT which appears as a solution to $(2,0)$ MLDE.

# (3,0) MLDE

- Next we move onto solving the (3,0) MLDE,

$$\mathscr{D}^3\chi_i + \mu_{1,0}\,E_4\,\mathscr{D}\chi_i + \mu_{0,1}\,E_6\,\chi_i = 0 \qquad \text{...(14)}$$

- The corresponding indicial $(m = 0)$ equation for (14) is,

$$\alpha^3 - \frac{\alpha^2}{2} + \left(\frac{\mu_{1,0}}{144} + \frac{1}{18}\right)\alpha + \frac{\mu_{0,1}}{1728} = 0 \qquad \text{...(15)}$$

- The $(m = 2)$ equation is (with $N := -1680\alpha_0$),

$$\boxed{\begin{aligned}&N^4 + ((a_1^{(0)})^2 + 93\,a_1^{(0)} - 2\,a_2^{(0)} + 955)N^3\\ &+(-2380\,(a_1^{(0)})^2 + 28770\,a_1^{(0)} + 7700\,a_2^{(0)} + 167160)N^2\\ &+(1372000\,(a_1^{(0)})^2 + 10760400\,a_1^{(0)} - 7330400\,a_2^{(0)} - 9800\,a_1^{(0)}a_2^{(0)})N\\ &+13720000\,a_1^{(0)}a_2^{(0)} - 329280000\,(a_1^{(0)})^2 = 0\end{aligned}} \qquad \text{...(16)}$$

- Also, $(m = 1)$ equation $\Rightarrow \mu_{1,0} = f(N, a_1^{(0)})$ and $\mu_{0,1} = g(N, a_1^{(0)})$.

# (3,0) MLDE contd.

- Next substituting the expressions of $\mu_{1,0}$ and $\mu_{0,1}$ in (15) results in,

$$\begin{aligned}&(1680\alpha + N)[(8749440N - 19756800a_1^{(0)})\alpha^2\\&+(-5208N^2 + 11760\, a_1^{(0)}N - 4374720N + 9878400\, a_1^{(0)})\alpha\\&+N^3 + 14a_1^{(0)}N^2 + 924N^2 - 29400a_1^{(0)}N + 141120N + 9878400a_1^{(0)}] = 0\end{aligned} \quad \text{...(17)}$$

- The roots of the quadratic equation determine $\alpha_1$ and $\alpha_2$ in terms of $N$ and $a_1^{(0)}$.
- For rational roots: discriminant - square of a rational number but since the discriminant above $\in \mathbb{Z}$, it needs to be a perfect square (say $k^2$) to be able to result in rational roots.

$$\boxed{\begin{aligned}&3457440000\,(a_1^{(0)})^2 - 1657376000\, a_1^{(0)}N - 8232000\,(a_1^{(0)})^2N\\&+55899200\,N^2 + 3528000\, a_1^{(0)}N^2 + 4900\,(a_1^{(0)})^2N^2\\&+52080\,N^3 - 2100\, a_1^{(0)}N^3 - 31\,N^4 = k^2\end{aligned}} \quad \text{...(18)}$$

# (3,0) MLDE: Results

| S. No | $N$ | $a_1^{(0)}$ | $c$ | $h_1$ | $h_2$ |
|---|---|---|---|---|---|
| 0 | 35 | 0 | 1/2 | 1/16 | 1/2 |
| 1 | 840 | 318 | 12 | 1/3 | 5/3 |
| 2 | 1400 | 80 | 20 | 4/3 | 5/3 |
| 3 | 1400 | 728 | 20 | 1/3 | 8/3 |
| 4 | 1400 | 890 | 20 | 2/3 | 7/3 |
| 5 | 1960 | 1948 | 28 | 2/3 | 10/3 |
| 6 | 2520 | 3384 | 36 | 2/3 | 13/3 |

- Now we look for integral solutions for $(a_1^{(0)}, a_2^{(0)}, k)$ by solving (16) and (18) simultaneously with $0 < N \leq 3920$ (that is for $0 < c \leq 56$).
- Next we impose all the relevant constraints. This allowed us to classify all $(3,0)$ RCFTs. We also did a classification of $(3,2)$ RCFTs.
- Finally, we tabulate 6 novel admissible character-like solutions ($0^{th}$ blue row is Critical Ising model)

# Wronskian Indices of some known RCFTs I

- $l\,[\mathcal{M}(p,p')] = 0, \quad l\,[\hat{\mathbf{A}}_1; k] = 0$ and $l\,[\hat{\mathbf{E}}_8; 1] = 2$

- $$l\,[\hat{\mathbf{A}}_2; k] = \begin{cases} \dfrac{k(k^2-4)(3k+4)}{96}, & \text{for even } k \\ \dfrac{(k^2-1)(k+3)(3k-5)}{96} & \text{for odd } k \end{cases} \qquad \ldots(19)$$

- $$l\,[\hat{\mathbf{C}}_2; k] = \frac{(k-1)k(k+1)(k+2)}{8} \qquad \ldots(20)$$

- $$l\,[\hat{\mathbf{G}}_2; k] = \begin{cases} \dfrac{k(k^2-4)(3k+4)}{96}, & \text{for even } k \\ \dfrac{(k^2-1)(k+3)(3k-5)}{96} & \text{for odd } k \end{cases} \qquad \ldots(21)$$

# Wronskian Indices of some known RCFTs II

- We observe a remarkable coincidence between the $(n, l)$ values for the $(\hat{\mathbf{A}}_2)_k$ and $(\hat{\mathbf{G}}_2)_k$ CFTs: for every level, their $(n, l)$ values match. This means they are solutions to the same MLDE (for different sets of parameters in the MLDE). A special case of this is in $(2,0)$ MLDE solutions: both $(\hat{\mathbf{A}}_2)_1$ and $(\hat{\mathbf{G}}_2)_1$ are present.

- We find another remarkable coincidence between the $(n, l)$ values for the $(\hat{\mathbf{B}}_\mathbf{r})_k$ and $(\hat{\mathbf{D}}_\mathbf{r})_k$ CFTs, but for $r \geq 5$. We have checked the matching of the $n$ values $\forall\ r$ analytically and the matching of the $l$ values upto $r = 6$. We conjecture that the $l$ values should match $\forall\ r$ too.

- We also find that every WZW CFT at level one with two or more characters has vanishing Wronskian index.

# Related works

There have been many works in this direction since original MMS (1988). Let me put a few references which are super close to our work:

- Sunil Mukhi and collaborators,
  - arXiv:1510.04478 – 9 novel $(2,2)$ RCFTs
  - arXiv:1810.09472 – Quasi-character approach and 3 novel $(2,4)$ admissible character-like solutions
  - arXiv:2002.01949 – Quasi-character approach to $(3,l>0)$ MLDEs
- Justin Kaidi and collaborators,
  - arXiv:2107.13557 – $PSL(2,\mathbb{Z})$ representation theoretic approach to studying $(n,l)$ MLDEs with $n \leq 5$ and $l \leq 5$. This work came out in parallel to our work and has an overlap with our $(3,0)$ results.
- Jin-Beom Bae and collaborators,
  - arXiv:2010.12392 – MMS approach to Fermionic $(2,0)$ RCFTs
  - arXiv:2108.01647 – Study of Fermionic $(3,0)$ RCFTs. This work too came out in parallel to our work and has an overlap with our $(3,0)$ results.

Thank You for listening..