Three-loop helicity amplitudes for diphoton production in gluon fusion

arXiv:2111.13595

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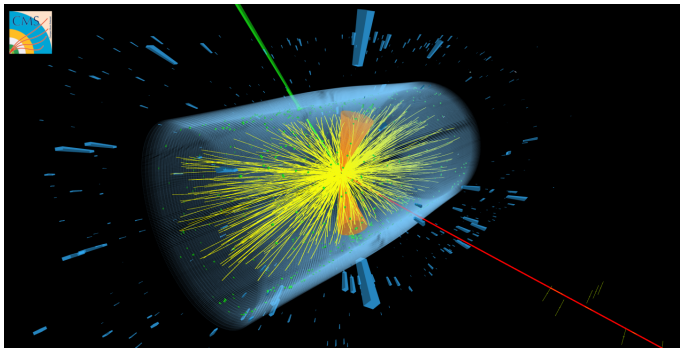
YTF 2021

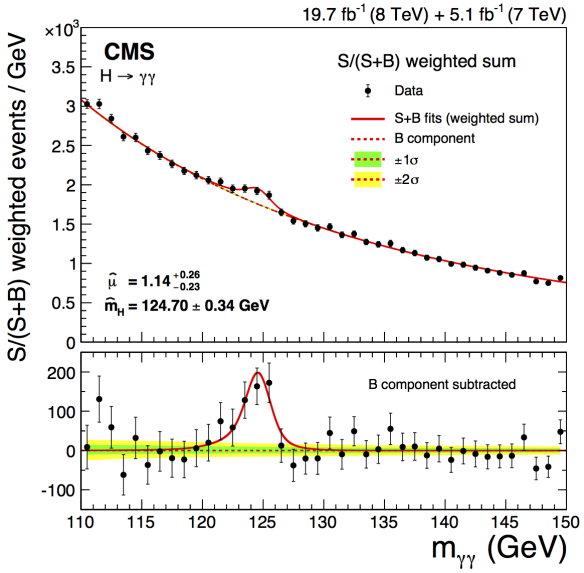
Presentation plan

Motivation

high precision **LHC** measurements

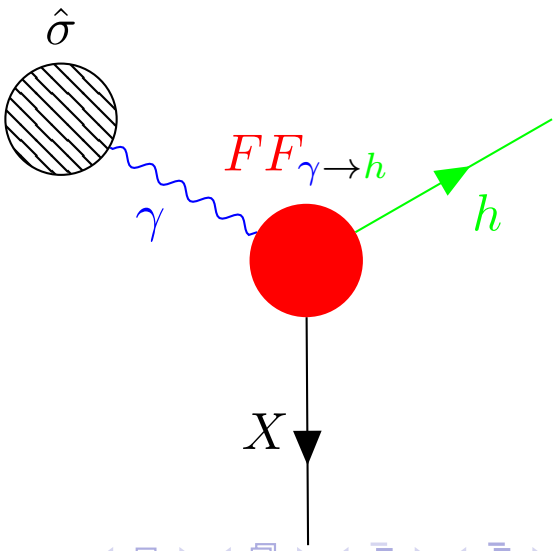


Higgs width



understanding QCD **gauge** structure

photon **fragmentation** functions



Towards 3-loop revolution

$pp \to \gamma\gamma$ milestones

t

- NLO cross section ($q\bar{q}$ channel) [*Binoth, arXiv:9911340*]
- 2L amplitude (gg channel) [*Dixon, arXiv:0109078*]
- NLO cross section (gg channel) [*Dixon, arXiv:0206194*]
- NNLO cross section ($q\bar{q}$ channel) [*Catani, arXiv:1110.2375*]

3-loop amplitude milestones

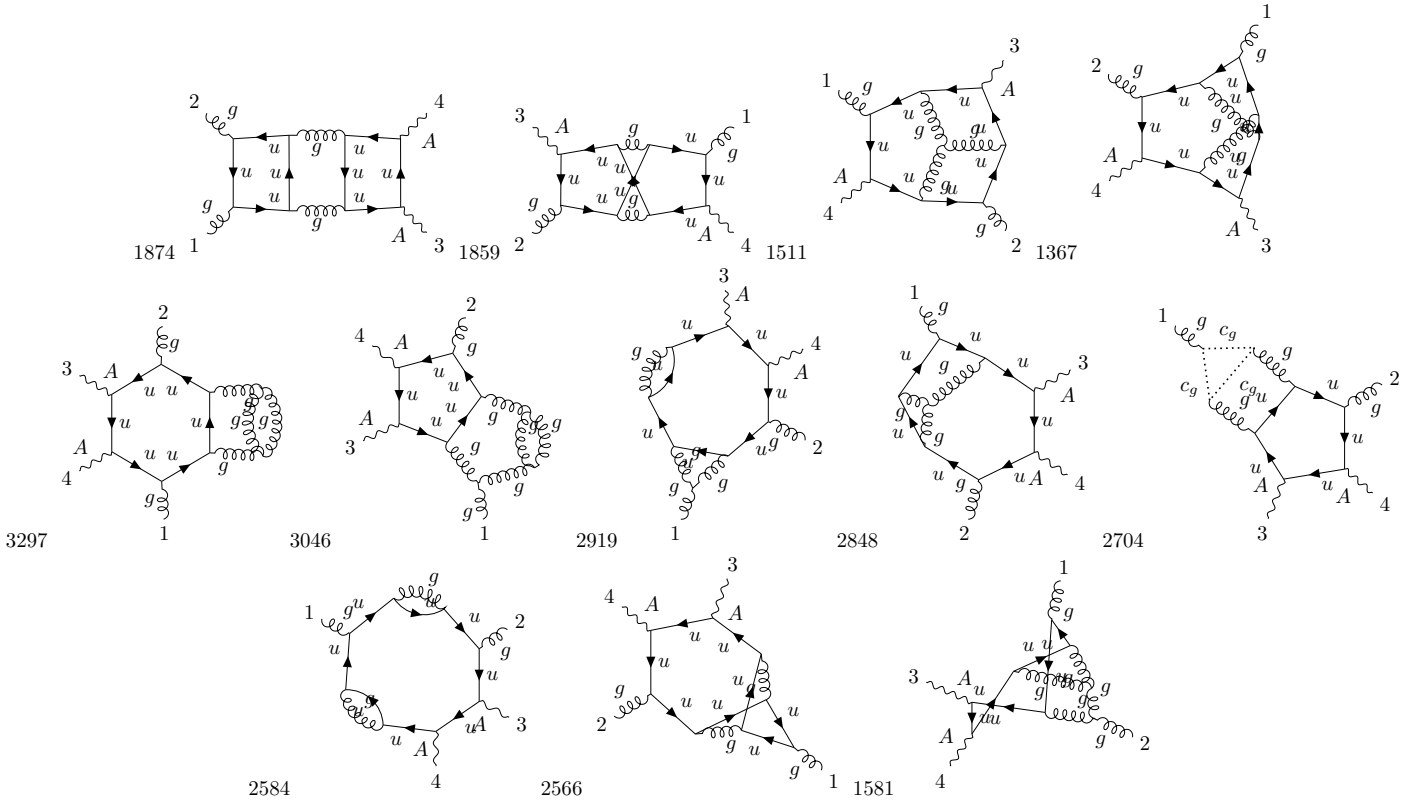
t

- 1→1 QCD [*Tarasov, PRLB 1980*]
- 2→1 QCD [*Vermaseren, arXiv:0508055*]
- 2→2 SYM [*Henn, arXiv:1608.00850*]
- 2→2 QCD [*Tancredi, arXiv:2011.13946*]

we present : **world's** 1st NNLO 3-loop 4-point amplitude
arXiv:2111.13595 with Caola, Manteuffel and Tancredi

The process

$$g(p_1) + g(p_2) \to \gamma(-p_3) + \gamma(-p_4)$$



$\times 275$ pages

Kinematics and color

kinematics

4 on-shell momenta $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$

3 **Mandelstam** variables $s = (p_1 + p_2)^2 > 0\,,\ t = (p_1 + p_3)^2 < 0\,,\ u = (p_2 + p_3)^2 < 0$

with momentum conservation $p_1 + p_2 + p_3 + p_4 = 0 \quad \Rightarrow \quad s + t + u = 0$

1 dimensionless invariant ratio $0 < \quad x = -\frac{t}{s} \quad < 1$

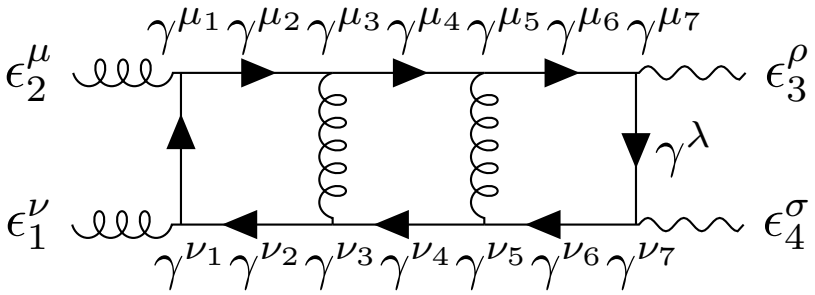
color

2 color indices $\mathcal{A}^{(L)} = \delta^{a_1 a_2} (4\pi\alpha) \left(\frac{\alpha_s}{2\pi}\right)^L A^{(L)}$

2 Casimir invariants $T^a_{ij} T^a_{jk} = C_F \delta_{ik}\,, \qquad f^{acd} f^{bcd} = C_A \delta^{ab}$

3 closed fermion **loop** types $n_f = \sum_f 1, \quad n_f^V = \sum_f Q_f, \quad n_f^{V_2} = \sum_f Q_f^2$

Tensors in d=4−2ϵ dimensions



Σ(diagrams) # Lorentz indices $>$ # all invariant structures

$$\begin{aligned}\# T_i &= 138 \text{ (Lorentz invariant tensors)} - 81 \text{ (by tranversality } \epsilon_i \cdot p_i = 0) \\ &\qquad\qquad\qquad\qquad\qquad\qquad - 47 \text{ (by gauge fixing } \epsilon_i \cdot p_{i+1} = 0) \\ &= 10 \text{ (independent in d dimensions)}\end{aligned}$$

$$\begin{aligned}T_i = (&p_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3 p_2 \cdot \epsilon_4 p_3 \cdot \epsilon_1\,, \quad \epsilon_3 \cdot \epsilon_4 p_1 \cdot \epsilon_2 p_3 \cdot \epsilon_1 \quad \epsilon_2 \cdot \epsilon_4 p_1 \cdot \epsilon_3 p_3 \cdot \epsilon_1\,, \\ &\epsilon_2 \cdot \epsilon_3 p_2 \cdot \epsilon_4 p_3 \cdot \epsilon_1\,, \quad \epsilon_1 \cdot \epsilon_4 p_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3\,, \quad \epsilon_1 \cdot \epsilon_3 p_1 \cdot \epsilon_2 p_2 \cdot \epsilon_4\,, \\ &\epsilon_1 \cdot \epsilon_2 p_1 \cdot \epsilon_3 p_2 \cdot \epsilon_4\,, \quad \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4, \quad \epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3, \quad \epsilon_1 \cdot \epsilon_3 \epsilon_2 \cdot \epsilon_4)\end{aligned}$$

Tensors in 4 dimensions

recent loop-univeral **claim** in 't Hooft - Veltman scheme [*Tancredi, arXiv:2012.00820*] :
tensors indpt in 4-dim = # indpt helicity states (here $= 2^4/2 = 8$)

$$A = \sum_{i=1}^{10} \mathcal{F}_i T_i = \sum_{i=1}^{8} \overline{\mathcal{F}}_i \overline{T}_i \,, \qquad \overline{T}_i = T_i \,, \quad i = 1, ..., 7 \,, \quad \overline{T}_8 = T_8 + T_9 + T_{10}$$

project out remaining tensors from the **physical** 4-dim subspace

$$\overline{T}_i = T_i - \sum_{j=1}^{8} (\mathcal{P}_j T_i) \overline{T}_j \,, \; i = 9, 10 \,, \qquad \sum_{\text{pol}} \mathcal{P}_i \overline{T}_j = \delta_{ij}$$

resulting tensors live purely in the **unphysical** -2ϵ-dim subspace

$$\overline{T}_9 = T_9 - \frac{1}{3}\left(-\frac{2\overline{T}_1}{su} - \frac{\overline{T}_6}{s} - \frac{\overline{T}_2 + \overline{T}_3 + 2\overline{T}_4 - 2\overline{T}_5 - \overline{T}_6 - \overline{T}_7}{t} + \frac{\overline{T}_3}{u} + \overline{T}_8 \right)$$

$$\overline{T}_{10} = T_{10} - \frac{1}{3}\left(\frac{4\overline{T}_1}{su} + \frac{2\overline{T}_6}{s} - \frac{\overline{T}_2 - \overline{T}_4 - 2\overline{T}_3 + 2\overline{T}_6 + \overline{T}_5 - \overline{T}_7}{t} - \frac{2\overline{T}_3}{u} + \overline{T}_8 \right)$$

and they **vanish** ($\forall\epsilon$) for each fixed helicity configuration

Helicity amplitudes

evaluate tensors at fixed helicity configuration $A_{\vec{\lambda}} = \sum_i \overline{\mathcal{F}}_i \, \overline{T}_{\vec{\lambda}} = \mathcal{S}_{\vec{\lambda}} \, f_{\vec{\lambda}}$

spinor weights

$$\mathcal{S}_{++++} = \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\,, \quad \mathcal{S}_{-+++} = \frac{\langle 12\rangle\langle 14\rangle[24]}{\langle 34\rangle\langle 23\rangle\langle 24\rangle}\,, \quad \mathcal{S}_{+-++} = \frac{\langle 21\rangle\langle 24\rangle[14]}{\langle 34\rangle\langle 13\rangle\langle 14\rangle}\,, \quad \mathcal{S}_{++-+} = \frac{\langle 32\rangle\langle 34\rangle[24]}{\langle 14\rangle\langle 21\rangle\langle 24\rangle}\,,$$

$$\mathcal{S}_{+++-} = \frac{\langle 42\rangle\langle 43\rangle[23]}{\langle 13\rangle\langle 21\rangle\langle 23\rangle}\,, \quad \mathcal{S}_{--++} = \frac{\langle 12\rangle[34]}{[12]\langle 34\rangle}\,, \quad \mathcal{S}_{-+-+} = \frac{\langle 13\rangle[24]}{[13]\langle 24\rangle}\,, \quad \mathcal{S}_{+--+} = \frac{\langle 23\rangle[14]}{[23]\langle 14\rangle}$$

little group **scalars**

$$f_{++++} = \frac{t^2}{4}\left(\frac{2\overline{\mathcal{F}}_6}{u} - \frac{2\overline{\mathcal{F}}_3}{s} - \overline{\mathcal{F}}_1\right) + \overline{\mathcal{F}}_8\left(\frac{s}{u} + \frac{u}{s} + 4\right) + \frac{t}{2}(\overline{\mathcal{F}}_2 - \overline{\mathcal{F}}_4 + \overline{\mathcal{F}}_5 - \overline{\mathcal{F}}_7)\,,$$

$$f_{-+++} = \frac{t^2}{4}\left(\frac{2\overline{\mathcal{F}}_3}{s} + \overline{\mathcal{F}}_1\right) + t\left(\frac{\overline{\mathcal{F}}_8}{s} + \frac{1}{2}(\overline{\mathcal{F}}_4 + \overline{\mathcal{F}}_6 - \overline{\mathcal{F}}_2)\right)\,,$$

$$f_{+-++} = -\frac{t^2}{4}\left(\frac{2\overline{\mathcal{F}}_6}{u} - \overline{\mathcal{F}}_1\right) + t\left(\frac{\overline{\mathcal{F}}_8}{u} - \frac{1}{2}(\overline{\mathcal{F}}_2 + \overline{\mathcal{F}}_3 + \overline{\mathcal{F}}_5)\right)\,,$$

$$f_{++-+} = \frac{t^2}{4}\left(\frac{2\overline{\mathcal{F}}_3}{s} + \overline{\mathcal{F}}_1\right) + t\left(\frac{\overline{\mathcal{F}}_8}{s} + \frac{1}{2}(\overline{\mathcal{F}}_6 + \overline{\mathcal{F}}_7 - \overline{\mathcal{F}}_5)\right)\,,$$

$$f_{+++-} = -\frac{t^2}{4}\left(\frac{2\overline{\mathcal{F}}_6}{u} - \overline{\mathcal{F}}_1\right) + t\left(\frac{\overline{\mathcal{F}}_8}{u} + \frac{1}{2}(\overline{\mathcal{F}}_4 + \overline{\mathcal{F}}_7 - \overline{\mathcal{F}}_3)\right)\,,$$

$$f_{--++} = -\frac{t^2}{4}\overline{\mathcal{F}}_1 + \frac{1}{2}t(\overline{\mathcal{F}}_2 + \overline{\mathcal{F}}_3 - \overline{\mathcal{F}}_6 - \overline{\mathcal{F}}_7) + 2\overline{\mathcal{F}}_8\,,$$

$$f_{-+-+} = t^2\left(\frac{\overline{\mathcal{F}}_8}{su} - \frac{\overline{\mathcal{F}}_3}{2s} + \frac{\overline{\mathcal{F}}_6}{2u} - \frac{\overline{\mathcal{F}}_1}{4}\right)\,,$$

$$f_{+--+} = -\frac{t^2}{4}\overline{\mathcal{F}}_1 + \frac{1}{2}t(\overline{\mathcal{F}}_3 - \overline{\mathcal{F}}_4 + \overline{\mathcal{F}}_5 - \overline{\mathcal{F}}_6) + 2\overline{\mathcal{F}}_8$$

IBP reduction

only Lorentz **scalar** structures left

$$\int_d \frac{\mathcal{N}(d;\{p_i\cdot p_j\},\{p_i\cdot k_j\},\{k_i\cdot k_j\})}{D_1^{n_1}\dots D_{10}^{n_{10}}}\,,\quad \int_d = \int\left(\prod_{i=1}^{3} e^{\epsilon\gamma_E}\frac{d^d k_i}{i\pi^{d/2}}\right)$$

$\{p_i\cdot k_j\},\{k_i\cdot k_j\}\to$ additional $D_{11},\dots,D_{15}$ $\quad\Rightarrow\quad$ $\mathcal{I}_{\{n_i\}}=\int_d \frac{1}{D_1^{n_1}\dots D_{15}^{n_{15}}}$

use Integration By Parts (**IBP**) identities [*Chetyrkin, NuclPhysB 1981*]

$$\text{Ward identities}\quad\Rightarrow\quad \int_d \frac{\partial}{\partial k_j^\mu}\frac{v_j^\mu(k,p)}{D_1^{n_1}\dots D_m^{n_m}}=0$$

[*Laporta, arXiv:0102033*] : # new integrals generated < # linear relations between them

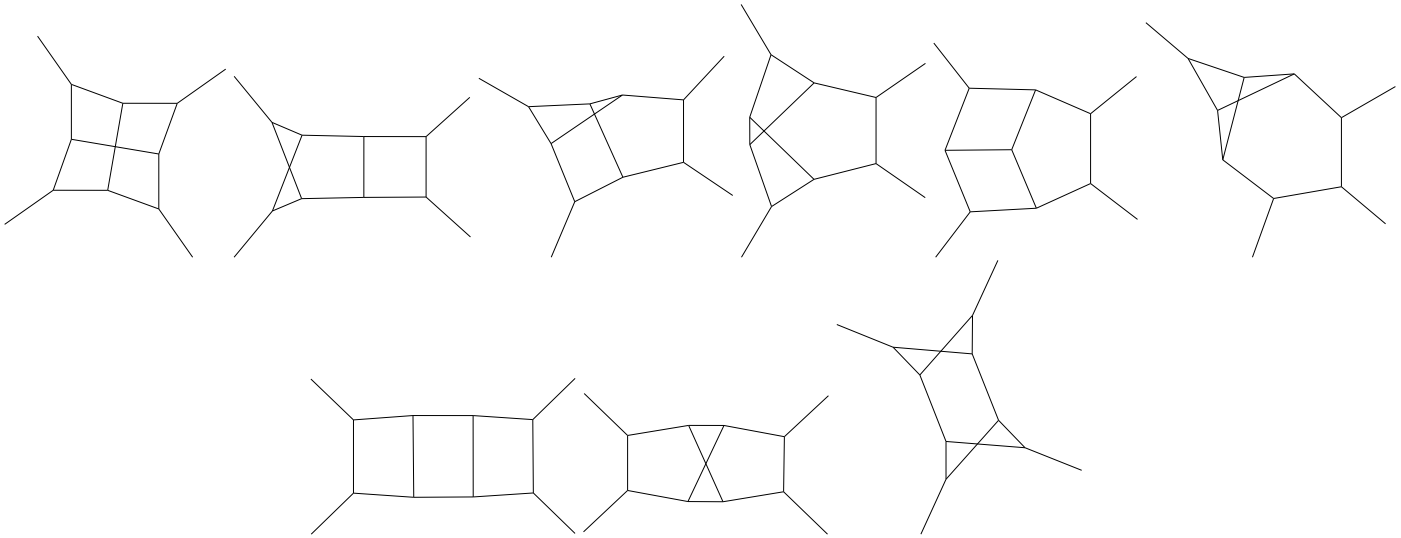
$$\text{loop momentum shift invariance}\quad\Rightarrow\quad \sum_{i\in\text{ext}}\left(p_i^\mu\frac{\partial}{\partial p_i^\nu}-p_i^\nu\frac{\partial}{\partial p_i^\mu}\right)\mathcal{I}_{\{n_j\}}=0$$

in-house syzygy & finite fields $\Rightarrow$ **reduce** to Master Integrals (**MIs**)

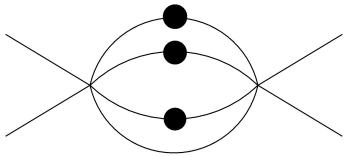
$$f_{\vec{\lambda}}(x)=\sum_{i=1}^{\mathcal{O}(10^6)}\mathcal{C}_{\vec{\lambda},i}(d;x)\,\mathcal{I}_i(x)=\sum_{i=1}^{\mathcal{O}(10^2)}C_{\vec{\lambda},i}(d;x)\,M_i(x)$$

Master topologies

9 distinct top sector topologies



use additional $D_{11}, ..., D_{15}$ & shift loop momenta $\Rightarrow$ only last 3 independent



amazingly, can relate **all** MIs to the 3-loop **sunrise**

Differential equations

differential equation for **canonical** MIs in $\mathrm{d}\log$ form [*Henn, arXiv:1304.1806*]

$$\mathrm{d}\vec{M}(\epsilon; s, t, u) = \epsilon \left[a_s \,\mathrm{d}\log(s) + a_t \,\mathrm{d}\log(t) + a_u \,\mathrm{d}\log(u)\right] \vec{M}(\epsilon; s, t, u)$$

solve with canonical matrices $a_{s_{ij}} \sim \mathcal{O}\left(10^2\right) \times \mathcal{O}\left(10^2\right)$ [*Henn, arXiv:2002.09492*]

$$\frac{d}{dx}\vec{M}(\epsilon; x) = \epsilon \left[\frac{a_t}{x} - \frac{a_u}{1-x}\right] \vec{M}(\epsilon; x) \qquad \Rightarrow \qquad \vec{M}(\epsilon; x) = e^{\epsilon \int dx \left[\frac{a_t}{x} - \frac{a_u}{1-x}\right]} \vec{M}_0(\epsilon)$$

expanding in ϵ leads to Harmonic Polylogarithms (**HPLs**) with letters $\alpha_i \in \{0, 1\}$

$$G(\alpha_n, ..., \alpha_1; x) = \int_0^x \frac{dz}{z - \alpha_n} G(\alpha_{n-1}, ..., \alpha_1; z)\,, \qquad G(\underbrace{0, \ldots, 0}_{n \text{ times}}; x) \equiv \frac{\ln^n x}{n!}$$

relate **all** boundary conditions $M_0(\epsilon)$ to a **single** overall normalization (3L sunrise)

$$\text{require} \quad \lim_{s_{ij} \to 0} \vec{M}(\epsilon; x) \to s^{a_{s_{ij}} \epsilon} \vec{M}_{0, s_{ij}} \quad \text{regular}$$

we've independently **derived** and solved all canonical systems with boundary conditions calculated the on physical Riemann sheet ($s_{ij} \to s_{ij} + i\varepsilon$) (*see results*)

Finite part

coupling **renormalization** $Z[\alpha] = 1 - \frac{\beta_0}{\epsilon}\left(\frac{\alpha_s}{2\pi}\right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right)\left(\frac{\alpha_s}{2\pi}\right)^2 + \mathcal{O}(\alpha_s^3)$

$\overline{MS}$ scheme $S_\epsilon \mu_0^{2\epsilon}\alpha_{s,b} = \mu^{2\epsilon}\alpha_s(\mu) Z[\alpha_s(\mu)]\,, \qquad S_\epsilon = (4\pi)^\epsilon e^{-\gamma_E \epsilon}$

beta function $\beta_0 = \frac{11}{6}C_A - \frac{2}{3}T_F n_f\,, \qquad \beta_1 = \frac{17}{6}C_A^2 - T_F n_f\left(\frac{5}{3}C_A + C_F\right)$

finite part $f_{\vec{\lambda}}^{(3,\mathrm{fin})} = f_{\vec{\lambda}}^{(3)} - \mathcal{I}_2\, f_{\vec{\lambda}}^{(1)} - \mathcal{I}_1\, f_{\vec{\lambda}}^{(2)}$

IR operators [*Catani, arXiv:9802439*]

$$\mathcal{I}_1(\epsilon) = -\frac{e^{i\pi\epsilon}e^{\gamma_E\epsilon}}{\Gamma(1-\epsilon)}\left(\frac{C_A}{\epsilon^2} + \frac{\beta_0}{\epsilon}\right)$$

$$\mathcal{I}_2(\epsilon) = -\frac{1}{2}\mathcal{I}_1(\epsilon)\left(\mathcal{I}_1(\epsilon) + \frac{2\beta_0}{\epsilon}\right) + \frac{e^{-\gamma_E\epsilon}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon} + K\right)\mathcal{I}_1(2\epsilon) + 2\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}H_g$$

with

$$K = \left(\frac{67}{18} - \frac{\pi^2}{6}\right)C_A - \frac{10}{9}n_f T_F$$

$$H_g = \frac{1}{2\epsilon}\left[\left(\frac{\zeta(3)}{4} + \frac{5}{24} + \frac{11\pi^2}{288}\right)C_A^2 + T_F n_f\left(\frac{C_F}{2} - \left(\frac{29}{27} + \frac{\pi^2}{72}\right)C_A\right) + \frac{10}{27}T_F^2 n_f^2\right]$$

Complexity comparison

intermediate expressions **complicated**

	1L	2L	3L
Number of diagrams	6	138	3299
Number of inequivalent integral families	1	2	3
Number of integrals before IBPs and symmetries	209	20935	4370070
Number of master integrals	6	39	486
Size of the `Qgraf` result [kB]	4	90	2820
Size of the `Form` result before IBPs and symmetries [kB]	276	54364	19734644
Size of helicity amplitudes written in terms of MIs [kB]	12	562	304409
Size of helicity amplitudes written in terms of HPLs [kB]	136	380	1195

final expression remarkably **compact**

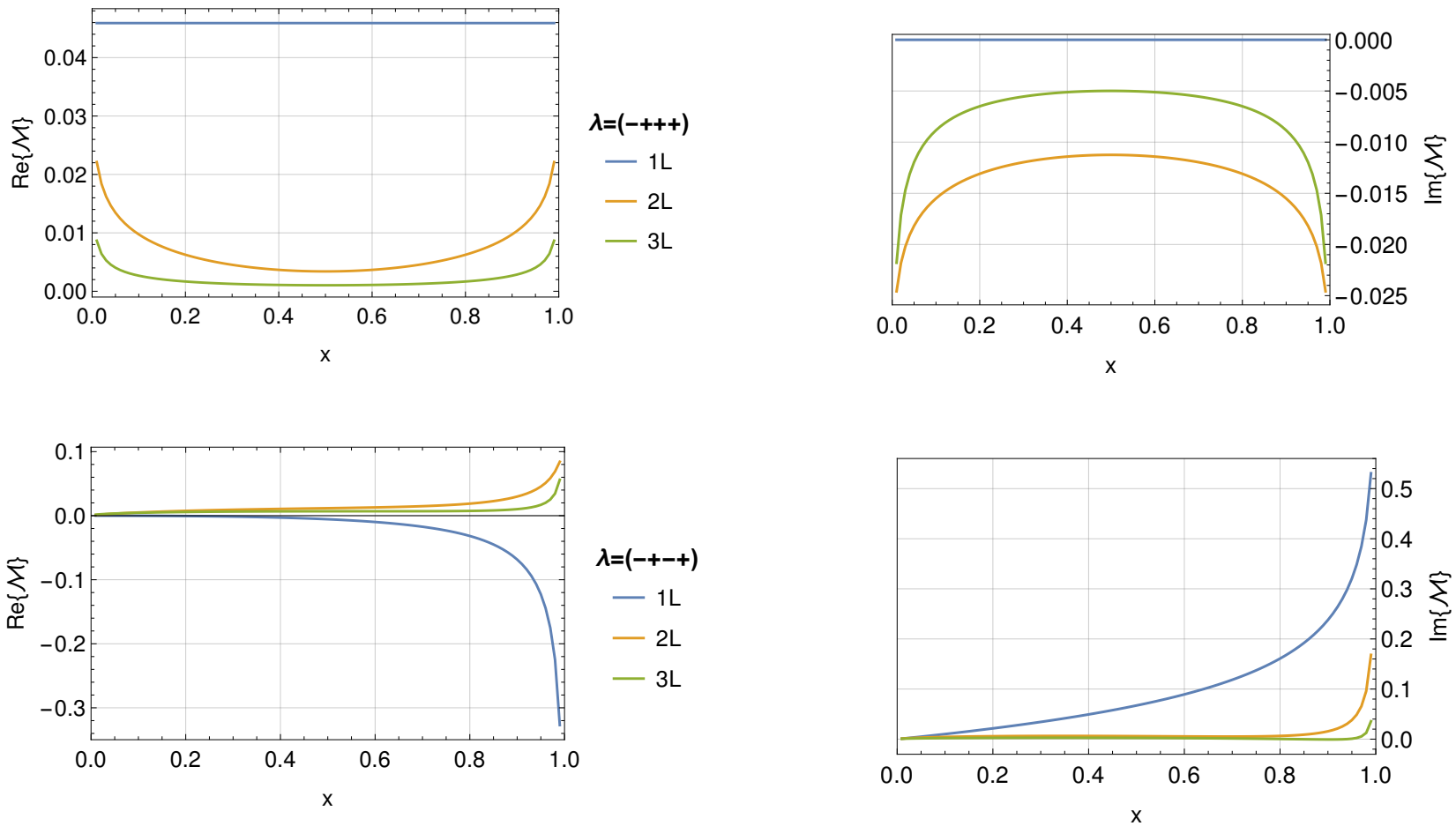
All-plus helicity amplitude

simplest helicity configuration leads to

$$
\begin{aligned}
f^{(3,\mathrm{fin})}_{++++} &= \Delta_1(x)\, n_f^{V_2} C_A^2 + \Delta_2(x)\, n_f^{V_2} C_A C_F + \Delta_3(x)\, n_f n_f^{V_2} C_A + \Delta_4(x)\, (n_f^V)^2 C_A + \Delta_5(x)\, n_f^{V_2} C_F^2 + \Delta_6(x)\, (n_f^V)^2 C_F + \Delta_7(x)\, n_f n_f^{V_2} C_F + \Delta_8(x)\, n_f^2 n_f^{V_2} \\
&\quad + \{(x) \leftrightarrow (1-x)\}\,, \\
\Delta_1(x) &= -\frac{23 L_1(L_1+2i\pi)}{9x^2} + \frac{32 L_1(L_1+2i\pi) - 46(L_1+i\pi)}{9x} - \frac{17}{36}L_0^2 - \frac{19}{36}L_0 L_1 + \frac{1}{9}L_0 - 2i\pi L_0 + \frac{1}{288}\pi^4 \\
&\quad - \frac{373}{72}\zeta_3 - \frac{185}{72}\pi^2 + \frac{4519}{324} + \frac{1}{2}i\pi\zeta_3 + \frac{11}{144}i\pi^3 + \frac{157}{12}i\pi + \frac{43}{9}L_0 x - \frac{7}{9}x^2\left((L_0-L_1)^2+\pi^2\right)\,, \\
\Delta_2(x) &= \frac{8L_1(L_1+2i\pi)}{3x^2} + \frac{16(L_1+i\pi) - 8L_1(L_1+2i\pi)}{3x} - \frac{1}{3}L_0^2 + \frac{5}{6}L_0 L_1 + \frac{17}{3}L_0 + i\pi L_0 - \frac{5}{12}\pi^2 - \frac{199}{6} - 8i\pi - \frac{16}{3}L_0 x + \frac{4}{3}x^2\left((L_0-L_1)^2+\pi^2\right)\,, \\
\Delta_3(x) &= \frac{L_1(L_1+2i\pi)}{18x^2} + \frac{2(L_1+i\pi) - L_1(L_1+2i\pi)}{18x} - \frac{1}{36}L_0^2 + \frac{1}{36}L_0 L_1 - \frac{1}{9}L_0 - \frac{61}{36}\zeta_3 + \frac{475}{432}\pi^2 - \frac{925}{324} - \frac{1}{72}i\pi^3 - \frac{175}{54}i\pi + \frac{2}{9}L_0 x + \frac{1}{36}x^2\left((L_0-L_1)^2+\pi^2\right)\,, \\
\Delta_4(x) &= -\frac{5L_1(L_1+2i\pi)}{4x^2} + \frac{L_1(L_1+2i\pi) - 8(L_1+i\pi)}{2x} + \frac{1}{4}L_0^2 - \frac{1}{4}L_0 L_1 - 2L_0 - 6\zeta_3 + \frac{1}{8}\pi^2 - \frac{1}{2} + 4L_0 x - x^2\left((L_0-L_1)^2+\pi^2\right)\,, \\
\Delta_5(x) &= -\frac{L_1(L_1+2i\pi)}{x^2} + \frac{L_1(L_1+2i\pi) - 2(L_1+i\pi)}{x} - \frac{1}{2}L_0^2 - i\pi L_0 + \frac{39}{4} + i\pi + 2L_0 x - \frac{1}{2}x^2\left((L_0-L_1)^2+\pi^2\right)\,, \\
\Delta_6(x) &= \frac{10L_1(L_1+2i\pi)}{3x^2} + \frac{32(L_1+i\pi) - 4L_1(L_1+2i\pi)}{3x} - \frac{2}{3}L_0^2 + \frac{2}{3}L_0 L_1 + \frac{16}{3}L_0 + 16\zeta_3 - \frac{1}{3}\pi^2 + \frac{4}{3} - \frac{32}{3}L_0 x + \frac{8}{3}x^2\left((L_0-L_1)^2+\pi^2\right)\,, \\
\Delta_7(x) &= \frac{5L_1(L_1+2i\pi)}{3x^2} + \frac{10(L_1+i\pi) - 8L_1(L_1+2i\pi)}{3x} + \frac{2}{3}L_0^2 + \frac{1}{3}L_0 L_1 - \frac{10}{3}L_0 + 2i\pi L_0 + 4\zeta_3 - \frac{\pi^2}{6} + 5 - 3i\pi - \frac{10}{3}L_0 x + \frac{1}{3}x^2\left((L_0-L_1)^2+\pi^2\right)\,, \\
\Delta_8(x) &= -\frac{23}{216}\pi^2 + \frac{5}{27}i\pi\,, \qquad \text{where} \qquad L_0 = \ln(x)\,, \quad L_1 = \ln(1-x)\,.
\end{aligned}
$$

other helicities are at most $\mathcal{O}\left(10^1\right)$ times more complicated (see *ancillary files*)

Single-minus & double-minus amplitudes



note the perturbative convergence

Outlook

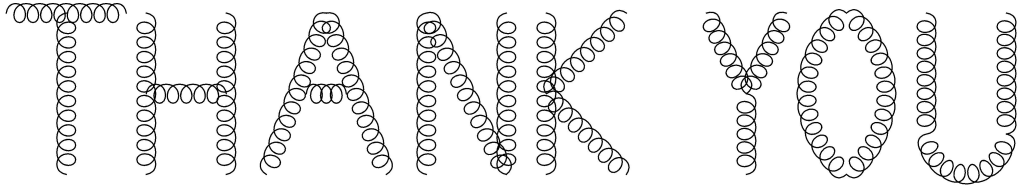
- interfere background with Higgs signal

$$\mathcal{A}_{gg\to H}\ \frac{1}{p^2 - m_H^2 + i m_H \Gamma_H}\ \mathcal{A}_{H\to\gamma\gamma}$$

- differential cross section

$$\sum_{i,j\in flav} \int_0^1 f_i(x_1,\mu_F^2) \otimes f_j(x_2,\mu_F^2) \otimes d\hat{\sigma}(z,\mu_R^2,\mu_F^2)$$

- **hidden** amplitude **structure** awaiting to be unveiled



Appendix : tensor projectors

$$\mathcal{P}_i = \sum_{k=1}^{10} \left(M^{-1}\right)_{ik} \overline{T}_k^\dagger \qquad M_{ij} = \sum_{pol} \overline{T}_i^\dagger \overline{T}_j \qquad \sum_{pol} \epsilon_i^\mu \epsilon_i^{*\nu} = -g^{\mu\nu} + \frac{p_i^\mu q_i^\nu + q_i^\mu p_i^\nu}{p_i \cdot q_i}$$

$$M^{-1} = \begin{pmatrix} \frac{X^{(0)}+dX^{(1)}}{3(d-1)(d-3)t^2} & 0 & 0 \\ 0 & \frac{2}{(d-4)(d-3)} & \frac{1}{(d-4)(d-3)} \\ 0 & \frac{1}{(d-4)(d-3)} & \frac{2}{(d-4)(d-3)} \end{pmatrix},$$

$$X^{(0)} = \begin{pmatrix} -\frac{8}{s^2}+\frac{32}{su}-\frac{8}{u^2} & -\frac{2}{s}-\frac{2}{u} & \frac{4s}{u^2}-\frac{4}{s}-\frac{12}{u} & \frac{2}{s}+\frac{2}{u} & -\frac{2}{s}-\frac{2}{u} & -\frac{4u}{s^2}+\frac{12}{s}+\frac{4}{u} & \frac{2}{s}+\frac{2}{u} & -\frac{2s}{u}-\frac{2u}{s}-7 \\ -\frac{2}{s}-\frac{2}{u} & -2 & \frac{s}{u}+2 & -1 & 1 & -\frac{u}{s}-2 & -1 & -t \\ \frac{4s}{u^2}-\frac{4}{s}-\frac{12}{u} & \frac{s}{u}+2 & -\frac{2s^2}{u^2}+\frac{4s}{u}+4 & -\frac{s}{u}-2 & \frac{s}{u}+2 & -\frac{2s}{u}-\frac{2u}{s}-5 & -\frac{s}{u}-2 & \frac{s^2}{u}+3s+2u \\ \frac{2}{s}+\frac{2}{u} & -1 & -\frac{s}{u}-2 & -2 & -1 & \frac{u}{s}+2 & 1 & t \\ -\frac{2}{s}-\frac{2}{u} & 1 & \frac{s}{u}+2 & -1 & -2 & -\frac{u}{s}-2 & -1 & -t \\ -\frac{4u}{s^2}+\frac{12}{s}+\frac{4}{u} & -\frac{u}{s}-2 & -\frac{2s}{u}-\frac{2u}{s}-5 & \frac{u}{s}+2 & -\frac{u}{s}-2 & -\frac{2u^2}{s^2}+\frac{4u}{s}+4 & \frac{u}{s}+2 & -\frac{u^2}{s}-2s-3u \\ \frac{2}{s}+\frac{2}{u} & -1 & -\frac{s}{u}-2 & 1 & -1 & \frac{u}{s}+2 & -2 & t \\ -\frac{2s}{u}-\frac{2u}{s}-7 & -t & \frac{s^2}{u}+3s+2u & t & -t & -\frac{u^2}{s}-2s-3u & t & t^2 \end{pmatrix},$$

$$X^{(1)} = \begin{pmatrix} \frac{12}{s^2}-\frac{12}{su}+\frac{3(2+d)}{t^2}+\frac{12}{u^2} & \frac{3}{t} & -\frac{6s}{u^2}+\frac{3}{t}+\frac{3}{u} & -\frac{3}{t} & \frac{3}{t} & \frac{6u}{s^2}-\frac{3}{s}-\frac{3}{t} & -\frac{3}{t} & 0 \\ \frac{3}{t} & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{6s}{u^2}+\frac{3}{t}+\frac{3}{u} & 0 & \frac{3s^2}{u^2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{t} & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ \frac{3}{t} & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ \frac{6u}{s^2}-\frac{3}{s}-\frac{3}{t} & 0 & 0 & 0 & 0 & \frac{3u^2}{s^2} & 0 & 0 \\ -\frac{3}{t} & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$