

Two-loop five-point amplitudes in massless QCD with finite fields

Ryan Moodie

Turin University

High Precision for Hard Processes

20 Sep 2022



Outline

Two-loop five-point amplitudes in massless QCD with finite fields

Introduction

Processes

Computation

Finite fields

Reconstruction

Performance

Conclusion

Outline

Two-loop five-point amplitudes in massless QCD with finite fields

Introduction

Processes

Computation

Finite fields

Reconstruction

Performance

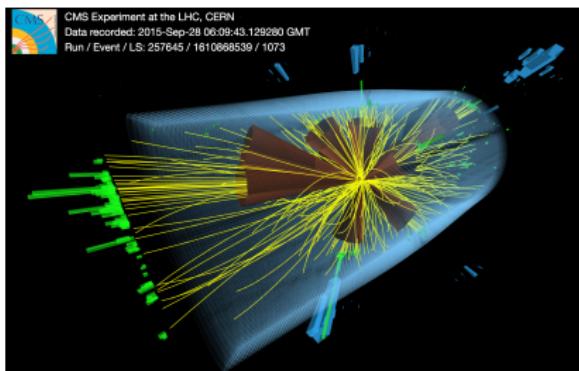
Conclusion

Precision frontier

- Better understand properties of SM
- Indirect probe of new physics by small deviations
- LHC demanding increasing precision
- Theory predictions at 1%
- Requires \geq NNLO QCD

$$\alpha_s \approx 0.1$$

$$\begin{aligned} d\sigma = & d\sigma_{\text{LO}} + \alpha_s d\sigma_{\text{NLO}} \\ & + \alpha_s^2 d\sigma_{\text{NNLO}} + \dots \end{aligned}$$



[McCauley et al. 2015]

Introduction
oo

Processes
●ooo

Computation
ooo

Finite fields
oooo

Reconstruction
ooooo

Performance
oooooooo

Conclusion
oo

Outline

Two-loop five-point amplitudes in massless QCD with finite fields

Introduction

Processes

Computation

Finite fields

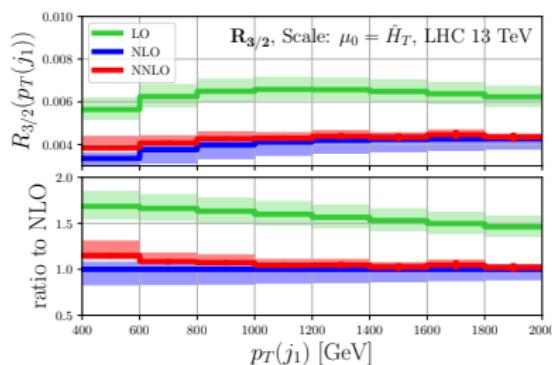
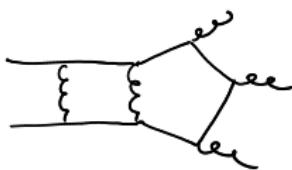
Reconstruction

Performance

Conclusion

Trijet production

- NNLO leading-colour $pp \rightarrow jjj$



[Czakon, Mitov, Poncelet 2021]

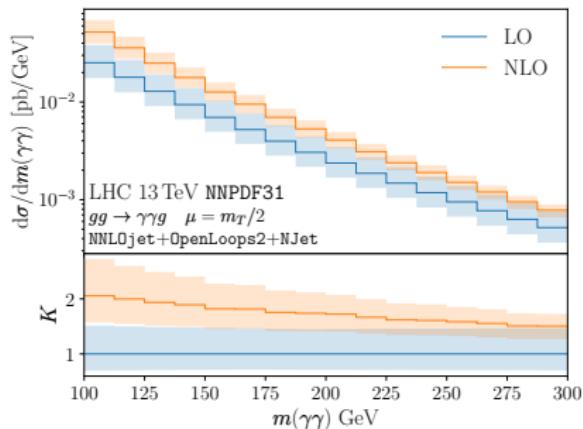
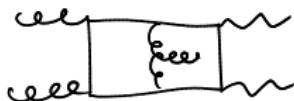
- Probe fundamental parameters of SM
- $R_{3/2} \rightarrow \alpha_s$
- Significant theory uncertainty reduction

[Abreu, Febres Cordero, Ita, Page, Sotnikov 2021]

[Czakon, Mitov, Poncelet 2021]

Diphoton-plus-jet production via gluon fusion

- NLO full-colour $gg \rightarrow g\gamma\gamma$ (N^3LO $pp \rightarrow g\gamma\gamma$)

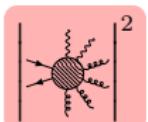
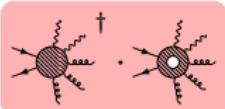
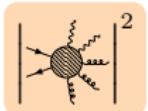
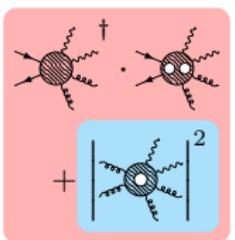
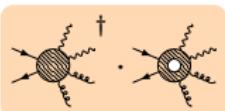
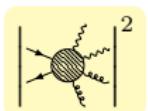
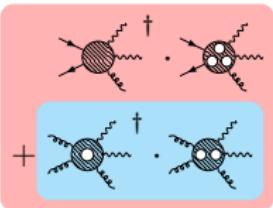
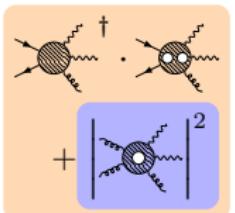
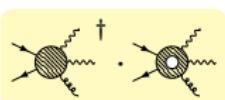
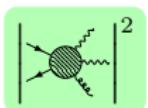


[Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, RM, Peraro, Zoić 2021]

[Badger, Gehrmann, Marcoli, RM 2022]

- Important background for measuring Higgs properties
- Significant NLO corrections, enhances NNLO $pp \rightarrow g\gamma\gamma$

[Chawdhry, Czakon, Mitov, Poncelet 2021]

$pp \rightarrow g\gamma\gamma$ 

LO

NLO

NNLO

N³LO

Loop induced:

LO

NLO

Introduction
oo

Processes
oooo

Computation
●○○

Finite fields
oooo

Reconstruction
ooooo

Performance
oooooooo

Conclusion
oo

Outline

Two-loop five-point amplitudes in massless QCD with finite fields

Introduction

Processes

Computation

Finite fields

Reconstruction

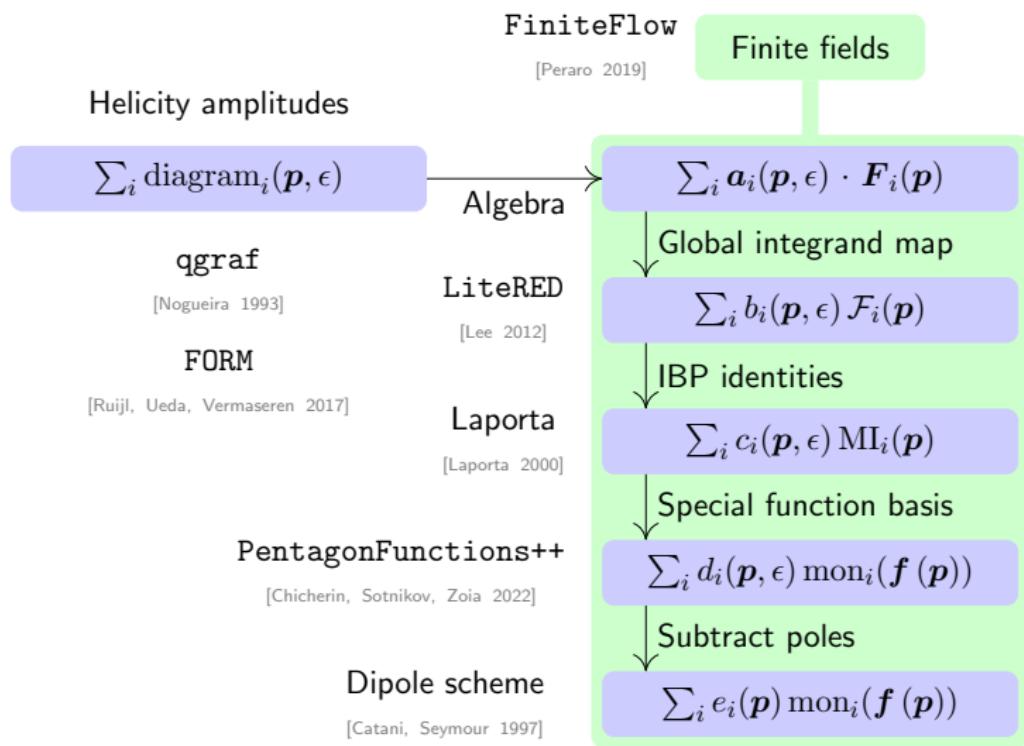
Performance

Conclusion

Five-point two-loop computation

- Major theoretical challenge
 - New methods
- Reconstruct over finite fields
 - [Peraro 2019]
- Pentagon function basis
 - [Gehrmann, Henn, Lo Presti 2018]
 - [Chicherin, Sotnikov 2020]
 - [Chicherin, Sotnikov, Zoia 2022]
- Fast and stable implementations in NJet
 - [Badger, Biedermann, RM, Uwer, Yundin 2021]

Method overview



Introduction
oo

Processes
oooo

Computation
ooo

Finite fields
●ooo

Reconstruction
ooooo

Performance
oooooooo

Conclusion
oo

Outline

Two-loop five-point amplitudes in massless QCD with finite fields

Introduction

Processes

Computation

Finite fields

Reconstruction

Performance

Conclusion

Finite field arithmetic

- Set of $n \in \mathbb{P}^p$ non-negative integers

$$\mathbb{F}_n = \{0, \dots, n - 1\}$$

- Arithmetic operations module n
 - Addition
 - Multiplication
 - Inverses
- Modular multiplicative inverse $x = a^{-1} \pmod{n}$

$$ax = 1 \pmod{n} \quad a \neq 0$$

- One-to-many map $\mathbb{Q} \rightarrow \mathbb{F}_n$

$$\frac{a}{b} \rightarrow a b^{-1} \pmod{n}$$

Numeric representation for computation

`float` Catastrophic cancellation

`int` Overflow

\mathbb{Q} Slow

- \mathbb{F}_n
 - No precision loss (n large)
 - Fast: fixed size integer
 - Recover $\mathbb{F}_n \rightarrow \mathbb{Q}$
 - Chinese Remainder Theorem: $\{\mathbb{F}_n\} \rightarrow \mathbb{F}_{\prod n}$

- Amplitudes

- Large intermediate expressions
 - Bypass with numerical evaluation over \mathbb{F}_n
 - Reconstruct analytic expression

Rational on-shell parametrisation

- Momentum twistor variables

$$\langle ij \rangle, [ij] \rightarrow x_i$$

$$Z = \begin{pmatrix} \lambda_1 & \cdots & \lambda_n \\ \mu_1(\tilde{\lambda}) & \cdots & \mu_n(\tilde{\lambda}) \end{pmatrix}$$

- Rational functions
 - $p_i(x), s_{ij}(x), \text{tr}_5(x), \dots$
- x unconstrained
 - On-shell
 - Momentum conserving

Introduction
oo

Processes
oooo

Computation
ooo

Finite fields
oooo

Reconstruction
●oooo

Performance
oooooooo

Conclusion
oo

Outline

Two-loop five-point amplitudes in massless QCD with finite fields

Introduction

Processes

Computation

Finite fields

Reconstruction

Performance

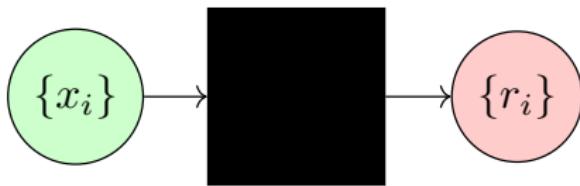
Conclusion

Reconstruction

- Amplitude components

$$F(\mathbf{x}) = \sum_i r_i(\mathbf{x}) \text{mon}_i(f)$$

- Have numerical algorithm for r_i
- **FiniteFlow** [Peraro 2019]



- Reconstruct expression from sufficient evaluations
- Strategies to optimise and compactify

Linear relations in the coefficients

- Linearise the r_i

$$\{r_i\}_{i \in S} \rightarrow \{r_i\}_{i \in T \subseteq S}$$

- Numerically solve:

$$\sum_i a_i r_i(\mathbf{x}) = 0$$

and choose lowest degrees

- Ansätze, eg permutations of 5g r_i for 3g2γ:

$$\sum_i a_i r_i(\mathbf{x}) + \sum_j b_j e_j(\mathbf{x}) = 0$$

Matching factors

- Coefficient ansatz

$$r(\mathbf{x}) = \frac{n(\mathbf{x})}{\prod_k \ell_k^{e_k}(\mathbf{x})} \quad \ell_k \in \text{pentagon alphabet}$$

- Determine e_k by reconstructing r on univariate slice

$$\mathbf{x} = \mathbf{c}_0 + \mathbf{c}_1 t \quad \rightsquigarrow \quad r(t)$$

and matching RHS

- Fix denominators

$$\{r_i\} \rightarrow \{n_i\}$$

[Abreu, Dormans, Febres Cordero, Ita, Page 2019]

Univariate partial fraction decomposition

- Example in y

$$\frac{N(x, y)}{x^2 y^2 (x^2 + y^2)} = \frac{q_1(x)}{y} + \frac{q_2(x) + q_3(x)y}{y^2} + \frac{q_4(x) + q_5(x)y}{x^2 + y^2}$$

- Only need to know y degree of N
- Choose y by studying one-loop

- Reconstruct r_i directly in decomposed form

$$\{r_i(\bar{x}, y)\} \rightarrow \{q_i(\bar{x})\}$$

- Reduce variables by one
- Lower degrees
- Simplifies reconstruction
 - $\times 10$ point numerical evaluation time (linear fit in $\{q_i\}$ over y)
 - $\times \frac{1}{100}$ samples required

Introduction
oo

Processes
oooo

Computation
ooo

Finite fields
oooo

Reconstruction
ooooo

Performance
●oooooo

Conclusion
oo

Outline

Two-loop five-point amplitudes in massless QCD with finite fields

Introduction

Processes

Computation

Finite fields

Reconstruction

Performance

Conclusion

Implementation

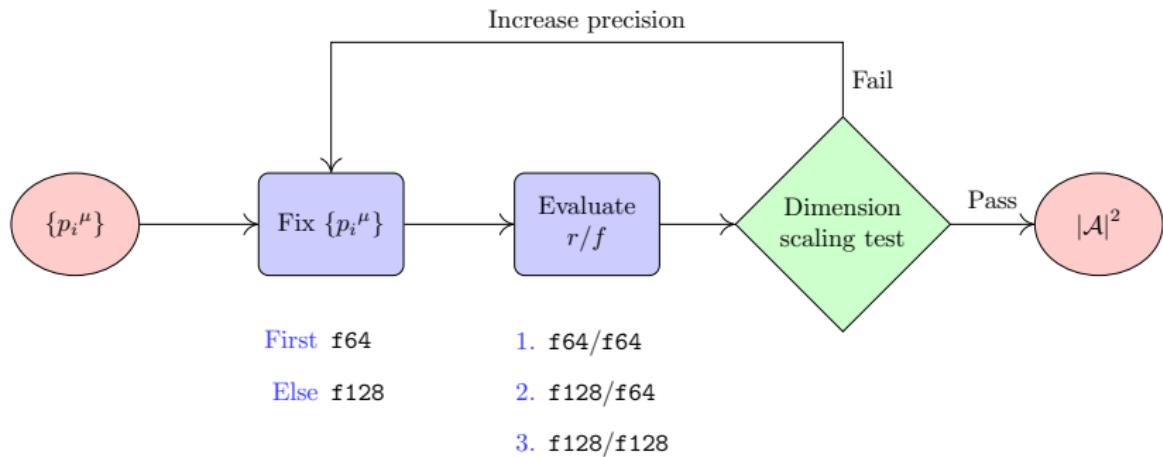
- Finite remainders coded up in C++ as analytic expressions
- Construct partial amplitudes as

$$F^h = r_i^h(\mathbf{x}) M_{ij}^h f_j^h$$

f_j^h	special function monomials	global
M_{ij}^h	rational sparse matrices	partials
r_i^h	independent rational coefficients	helicities
\mathbf{x}	momentum twistor variables	global

- Independent helicities permuted to all mostly-plus
 - Mostly-minus: $r_i^* M_{ij} P(f_j)$

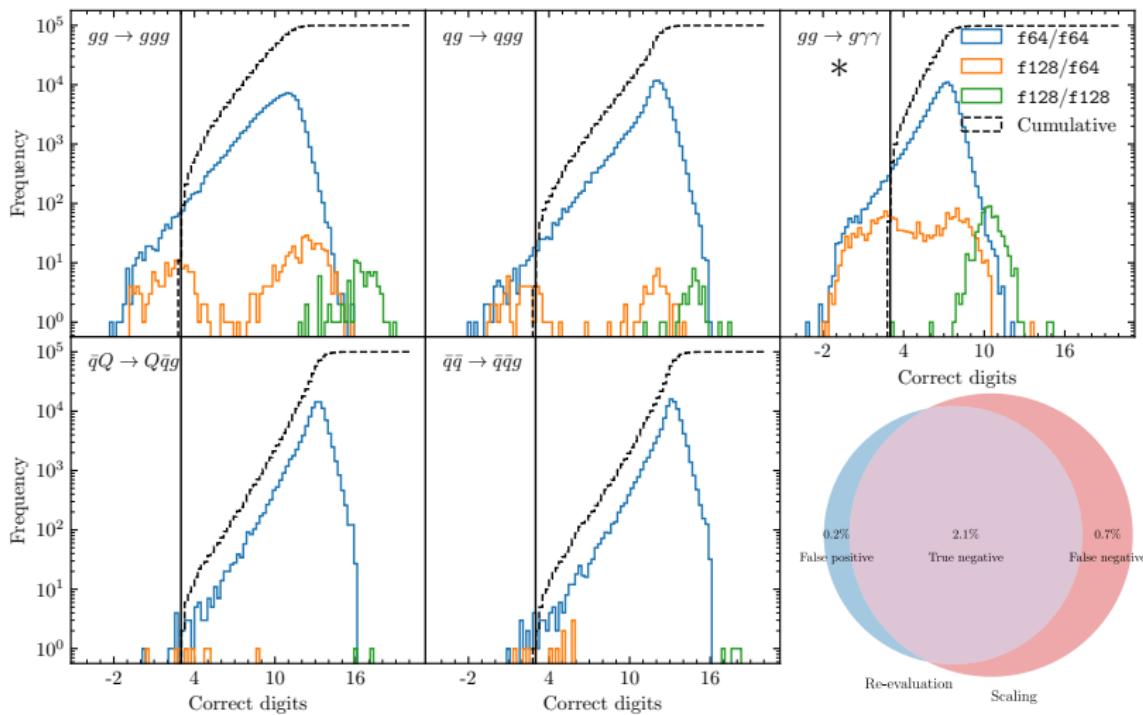
Evaluation strategy



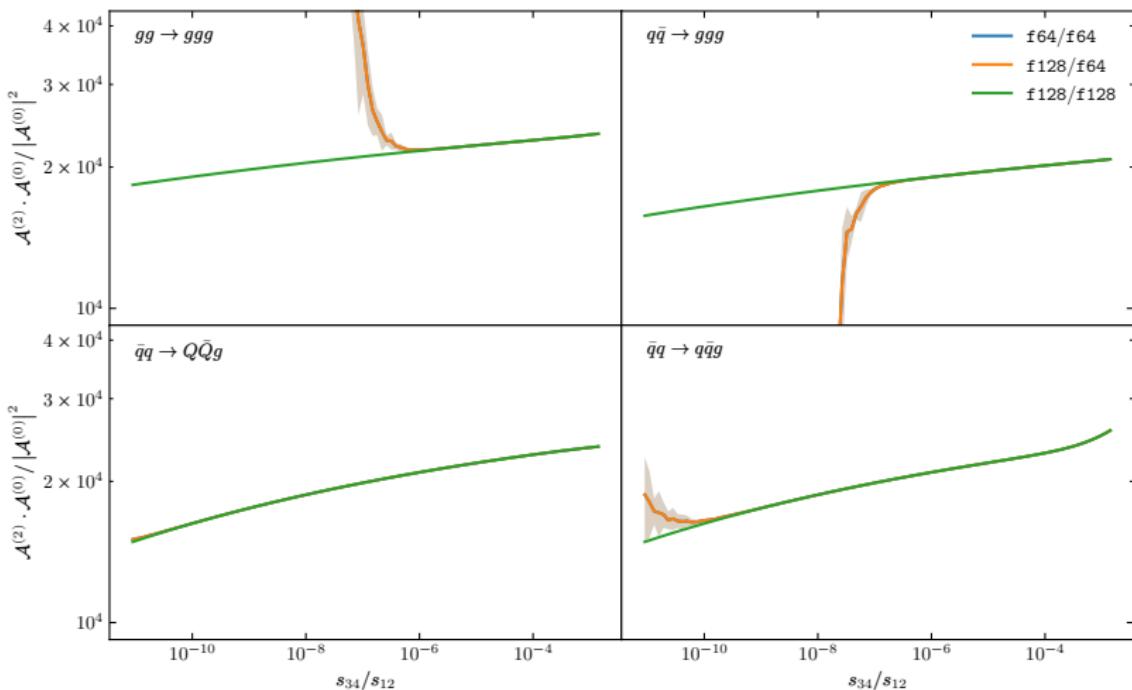
Timing

Channel	f64/f64		Evaluation strategy	
	Time (s)	f (%)	Time (s)	f (%)
$gg \rightarrow ggg$	1.39	69	1.89	77
$gg \rightarrow \bar{q}qg$	1.35	91	1.37	91
$qg \rightarrow qgg$	1.34	92	1.57	93
$q\bar{q} \rightarrow ggg$	1.34	93	1.38	93
$\bar{q}Q \rightarrow Q\bar{q}g$	1.14	99	1.16	99
$\bar{q}\bar{Q} \rightarrow \bar{q}\bar{Q}g$	1.36	99	1.39	99
$\bar{q}g \rightarrow \bar{q}Q\bar{Q}$	1.36	99	1.39	99
$\bar{q}q \rightarrow Q\bar{Q}g$	1.14	99	1.14	99
$\bar{q}g \rightarrow \bar{q}q\bar{q}$	1.84	99	1.90	99
$\bar{q}\bar{q} \rightarrow \bar{q}\bar{q}g$	1.82	99	1.94	99
$\bar{q}q \rightarrow q\bar{q}g$	1.71	99	1.77	99
$gg \rightarrow \gamma\gamma g$ *	9	99	26	99

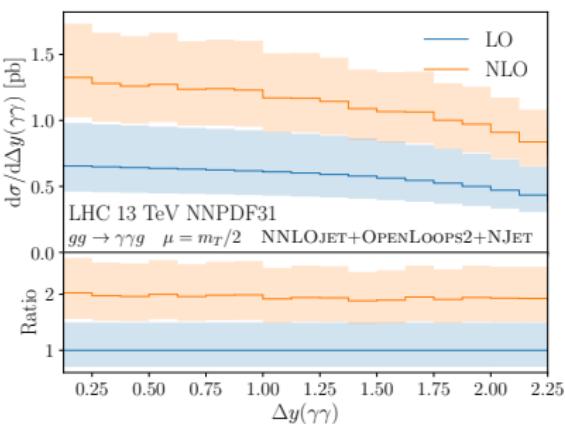
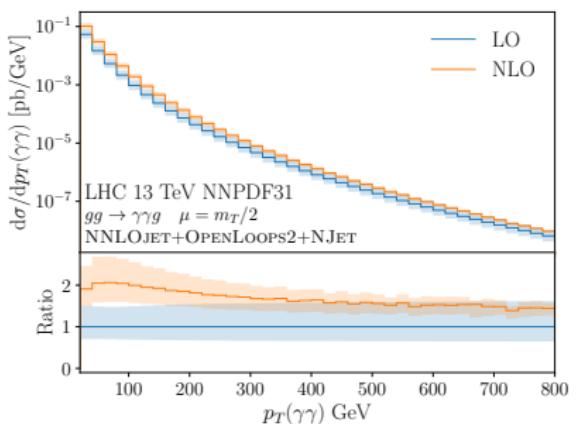
Stability



IR performance



Phenomenology



$$gg \rightarrow g\gamma\gamma$$

Introduction
oo

Processes
oooo

Computation
ooo

Finite fields
oooo

Reconstruction
ooooo

Performance
oooooooo

Conclusion
●○

Outline

Two-loop five-point amplitudes in massless QCD with finite fields

Introduction

Processes

Computation

Finite fields

Reconstruction

Performance

Conclusion

Conclusion

Two-loop five-point amplitudes in massless QCD with finite fields

- Huge success of technical advancements in precision QCD
 - Numerical evaluation over finite fields
 - Bypass intermediate complexity
 - Reconstruct compact analytic forms
- Efficient and stable evaluation over physical scattering region
 - $pp \rightarrow jjj$ (LC)
 - $gg \rightarrow g\gamma\gamma$
- Driving pheno predictions towards 1%

References I

-  McCauley, Thomas et al. (Dec. 2015). "Multi-jet event recorded by the CMS detector (Run 2, 13 TeV)". CMS Collection. URL: <https://cds.cern.ch/record/2114784>.
-  Czakon, Michal, Alexander Mitov, Rene Poncelet (2021). "Next-to-Next-to-Leading Order Study of Three-Jet Production at the LHC". In: *Phys. Rev. Lett.* 127.15. [Erratum: *Phys. Rev. Lett.* 129, 119901 (2022)], p. 152001. DOI: [10.1103/PhysRevLett.127.152001](https://doi.org/10.1103/PhysRevLett.127.152001). arXiv: [2106.05331 \[hep-ph\]](https://arxiv.org/abs/2106.05331).
-  Abreu, S., F. Febres Cordero, H. Ita, B. Page, V. Sotnikov (2021). "Leading-color two-loop QCD corrections for three-jet production at hadron colliders". In: *JHEP* 07, p. 095. DOI: [10.1007/JHEP07\(2021\)095](https://doi.org/10.1007/JHEP07(2021)095). arXiv: [2102.13609 \[hep-ph\]](https://arxiv.org/abs/2102.13609).
-  Badger, Simon, Christian Brønnum-Hansen, Dmitry Chicherin, Thomas Gehrmann, Heribertus Bayu Hartanto, Johannes Henn, Matteo Marcoli, RM, Tiziano Peraro, Simone Zoia (2021). "Virtual QCD corrections to gluon-initiated diphoton plus jet production at hadron colliders". In: *JHEP* 11, p. 083. DOI: [10.1007/JHEP11\(2021\)083](https://doi.org/10.1007/JHEP11(2021)083). arXiv: [2106.08664 \[hep-ph\]](https://arxiv.org/abs/2106.08664).
-  Badger, Simon, Thomas Gehrmann, Matteo Marcoli, RM (2022). "Next-to-leading order QCD corrections to diphoton-plus-jet production through gluon fusion at the LHC". In: *Phys. Lett. B* 824, p. 136802. DOI: [10.1016/j.physletb.2021.136802](https://doi.org/10.1016/j.physletb.2021.136802). arXiv: [2109.12003 \[hep-ph\]](https://arxiv.org/abs/2109.12003).
-  Chawdhry, Herschel A., Michal Czakon, Alexander Mitov, Rene Poncelet (2021). "NNLO QCD corrections to diphoton production with an additional jet at the LHC". In: *JHEP* 09, p. 093. DOI: [10.1007/JHEP09\(2021\)093](https://doi.org/10.1007/JHEP09(2021)093). arXiv: [2105.06940 \[hep-ph\]](https://arxiv.org/abs/2105.06940).
-  Peraro, Tiziano (2019). "FiniteFlow: multivariate functional reconstruction using finite fields and dataflow graphs". In: *JHEP* 07, p. 031. DOI: [10.1007/JHEP07\(2019\)031](https://doi.org/10.1007/JHEP07(2019)031). arXiv: [1905.08019 \[hep-ph\]](https://arxiv.org/abs/1905.08019).
-  Gehrmann, T., J. M. Henn, N. A. Lo Presti (2018). "Pentagon functions for massless planar scattering amplitudes". In: *JHEP* 10, p. 103. DOI: [10.1007/JHEP10\(2018\)103](https://doi.org/10.1007/JHEP10(2018)103). arXiv: [1807.09812 \[hep-ph\]](https://arxiv.org/abs/1807.09812).

References II

-  Chicherin, Dmitry, Vasily Sotnikov (2020). "Pentagon Functions for Scattering of Five Massless Particles". In: *JHEP* 20, p. 167. DOI: 10.1007/JHEP12(2020)167. arXiv: 2009.07803 [hep-ph].
-  Chicherin, Dmitry, Vasily Sotnikov, Simone Zoia (2022). "Pentagon functions for one-mass planar scattering amplitudes". In: *JHEP* 01, p. 096. DOI: 10.1007/JHEP01(2022)096. arXiv: 2110.10111 [hep-ph].
-  Badger, Simon, Benedikt Biedermann, RM, Peter Uwer, Valery Yundin (2021). *NJet v3*. URL: <https://bitbucket.org/njet/njet>.
-  Nogueira, Paulo (1993). "Automatic Feynman graph generation". In: *J. Comput. Phys.* 105, pp. 279–289. DOI: 10.1006/jcph.1993.1074.
-  Ruijl, Ben, Takahiro Ueda, Jos Vermaseren (July 2017). "FORM version 4.2". In: arXiv: 1707.06453 [hep-ph].
-  Lee, R. N. (Dec. 2012). "Presenting LiteRed: a tool for the Loop InTEgrals REDuction". In: arXiv: 1212.2685 [hep-ph].
-  Laporta, S. (2000). "High precision calculation of multiloop Feynman integrals by difference equations". In: *Int. J. Mod. Phys. A* 15, pp. 5087–5159. DOI: 10.1142/S0217751X00002159. arXiv: hep-ph/0102033.
-  Catani, S., M. H. Seymour (1997). "A General algorithm for calculating jet cross-sections in NLO QCD". In: *Nucl. Phys. B* 485. [Erratum: Nucl.Phys.B 510, 503–504 (1998)], pp. 291–419. DOI: 10.1016/S0550-3213(96)00589-5. arXiv: hep-ph/9605323.
-  Abreu, S., J. Dormans, F. Febres Cordero, H. Ita, B. Page (2019). "Analytic Form of Planar Two-Loop Five-Gluon Scattering Amplitudes in QCD". In: *Phys. Rev. Lett.* 122.8, p. 082002. DOI: 10.1103/PhysRevLett.122.082002. arXiv: 1812.04586 [hep-ph].