

Two-loop master integrals and form-factors for pseudo-scalar quarkonia

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based on arXiv: 2206.03848 & 22XX.XXXX with S. Abreu, M. Becchetti, C. Duhr

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Introduction: What is a Quarkonium?

- $\bullet\,$ similar to positronium bound state $\mathrm{e^+e^-}$ in QED
- bound state of heavy quark and its anti-quark in QCD, e.g. Charmonium (charm quark) and Bottomonium (bottom quark)



[Figure from Wikipedia 'Quarkonium']

- Toponium $(t\overline{t})$ bound state: high mass of top quark \rightarrow decays via weak interaction before formation of bound state
- for light quarks: mixing between (u,d,s) quarks due to low mass difference $\rightarrow \pi$ -meson, the ρ -meson and the η -meson

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Motivation: Why study Quarkonia?

- charmonium production allows us to probe QCD at its interplay between the perturbative and non-perturbative regimes
- deeper understanding of confinement (production mechanism)
- access to spin/momentum distribution of gluons in protons
 → use quarkonia to constrain the gluon PDFs in
 the proton
- it is interesting to assess the convergence of perturbative expansion in α_s where $\alpha_s(m_c) \sim 0.34$ and $\alpha_s(m_b) \sim 0.22$

the η_c - a good gluon probe

• η_c is a gluon probe at low scales at $M_{\eta_c} = 3 \text{ GeV}$

- is a pseudo-scalar particle and simplest of all quarkonia as far as computation of hadro-production
- η_c cross section computation known
 - at NLO since 1992 in collinear factorisation

[J. Kühn, E. Mirkes, Phys.Lett. B296 (1992) 425-429]

• at LO since 2012 and at NLO since 2013 in TMD factorisation

[D. Boer, C. Pisano, Phys.Rev. D86 (2012) 094007]

[J.P. Ma, J.X. Wang, S. Zhao, Phys.Rev. D88 (2013) no.1, 014027]



• large scale uncertainties

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- large scale uncertainties
- issue of negative cross-sections

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 - due to over-subtraction of initial-state collinear singularities into PDFs

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 - resolved with new scale prescription for μ_F (green curve)

[J.-P. Lansberg, Melih A. Ozcelik, Eur.Phys.J.C 81 (2021) 6, 497 (arXiv:2012.00702)]

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• for general scale reduction need NNLO calculation

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- for general scale reduction need NNLO calculation \rightarrow need two-loop form-factors

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Form-factors

compute two-loop form-factors analytically in different channels that contribute at NNLO accuracy

•
$$\gamma\gamma \leftrightarrow \eta_Q \left({}^1S_0^{[1]} \right) \rightarrow \text{exclusive/inclusive decay}$$

- $gg \leftrightarrow \eta_Q \left({}^1S_0^{[1]} \right) \to$ hadro-production and hadronic decay width
- $\gamma g \leftrightarrow {}^{1}S_{0}^{[8]} \rightarrow$ colour-octet contribution $gg \leftrightarrow {}^{1}S_{0}^{[8]} \rightarrow$ colour-octet contribution
- $\gamma\gamma \leftrightarrow para-Positronium$

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- $\gamma\gamma \leftrightarrow para-Positronium$
- form-factors applicable to both production and decay
- in the past form-factors have been computed only in numerical form
 - $\eta_O o \gamma\gamma$ [A. Czarnecki, K. Melnikov, Phys.Lett.B 519 (2001) 212-218] [F. Feng, Y. Jia, W.-L. Sang, Phys.Rev.Lett. 115 (2015) 22, 222001]
 - para-Positronium $\rightarrow \gamma \gamma$ [A. Czarnecki, K. Melnikov, A. Yelkhovsky, Phys.Rev.A 61 (2000) 052502]

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$$\gamma(k_1) + \gamma(k_2) \to Q(p_1)\overline{Q}(p_2) \tag{1}$$

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 (1)

•
$$p^2 = m_Q^2$$
 for final-state heavy quarks with $p = p_1 = p_2$

- $k_1^2 = k_2^2 = 0$ for initial-state photons
- threshold kinematics with $\hat{s}=M_{Q}^{2}=4m_{Q}^{2}$ where $M_{Q}=2m_{Q}$



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- threshold kinematics with $\left| \hat{s} = M_{Q}^2 = 4 m_{Q}^2
 ight|$ where $M_{Q} = 2 m_{Q}$
- generate Feynman diagram with FeynArts (\sim 450 diagrams for $gg \leftrightarrow \eta_Q$ case)

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- perform tensor integral decomposition in new basis
- reduce integrals to master integrals via IBP with FIRE

[A.V. Smirnov, Comput.Phys.Commun. 189 (2015) 182-191]

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Amplitude

• two-loop Amplitude $\mathcal{A}^{(2)}$:

$$\mathcal{A}^{(2)} = \mathcal{A}^{(0)} \sum_{i=1}^{n_{\text{master}}} c_i(\epsilon) \mathsf{MI}[i]$$

- tree-level Amplitude $\mathcal{A}^{(0)}$
- coefficient *c_i* contains information on:
 - rational factor depending on dimensional regulator ϵ
 - colour factor (C_A, C_F, T_F)
 - number of massive (n_h) and massless (n_l) closed fermion loops (vacuum & light-by-light)
- need to compute master integrals MI[i]

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(5)

Some examples of topologies:



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 - Partial Fraction Relations
 - Triangle Relations

Identity



relation at *integrand* level:

$$\underbrace{\frac{1}{\underbrace{\left[(q+p)^2-m^2\right]}_{D_1}\underbrace{\left[(q-p)^2-m^2\right]}_{D_3}} = \frac{1}{2}\underbrace{\frac{1}{\underbrace{\left[(q+p)^2-m^2\right]}_{D_1}\underbrace{q^2}_{D_2}} + \frac{1}{2}\underbrace{\frac{1}{\underbrace{q^2}_{D_2}\underbrace{\left[(q-p)^2-m^2\right]}_{D_3}}$$

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Identity



Example



• linear relations between integrals in different topology families

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Image: A math a math

- linear relations between integrals in different topology families
- not detected during IBP reduction (e.g. Kira, ...)

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$$m_{45} = \frac{2 \left(3 d-11\right) m^2}{\left(d-3\right) \left(3 d-10\right)} m_{53} - \frac{8 m^4}{\left(d-3\right) \left(3 d-10\right)} m_{54} + \frac{\left(d-2\right)^2}{4 \left(d-3\right) \left(3 d-10\right) m^4} m_{76}$$

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• question for future: can one systematically incorporate partial fraction relations into IBP reduction system (useful for phase-space integrations)?

Triangle Relations

Identity



relation at *integral* level:

$$\int d^{d}q \frac{1}{\left[q^{2}-m_{1}^{2}\right]^{2}\left[\left(q+p_{1}\right)^{2}-m_{2}^{2}\right]\left[\left(q-p_{2}\right)^{2}-m_{2}^{2}\right]} = \int d^{d}q \left(m_{1}\leftrightarrow m_{2}\right)$$

no constraint for p_1 and p_2 (can involve loop momenta), only constraint is that $k_1^2 = 0$

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Triangle Relations

Identity





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Triangle Relations

Identity



Example questions for future: can we systematically incorporate these relations into IBP? And are there more of these relations (box, pentagon integrals)?

Melih A. Ozcelik (TTP)

Two-loop master integrals

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- Multiple Polylogarithms points on the Riemann sphere
- elliptic Multiple Polylogarithms points on the torus
- iterated integrals of modular forms rational points on the torus

Multiple Polylogarithms (MPLs)

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Multiple Polylogarithms (MPLs)

[Goncharov, Remiddi, Vermaseren]

$$G(a_1, ..., a_n; z) = \int_0^z dt \frac{1}{t - a_1} G(a_2, ..., a_n; t)$$
(10)
$$G(0; t) = \log t$$
(11)

- weight of function corresponds to number of indices w = n
- *m*-loop amplitude usually exhibits functions up to weight of $w = 2m \rightarrow$ will be useful as cross-check of amplitude
- numerical evaluation can be achieved with GiNaC-interface

[Vollinga, Weinzierl]

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elliptic Multiple Polylogarithms (eMPLs)

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[Brown,Levin;Broedel,Duhr,Dulat,Tancredi;Weinzierl...]

$$E_{4}\begin{pmatrix} n_{1}\dots n_{m}\\ c_{1}\dots c_{m}\\ c_{m} \end{pmatrix}; x, \vec{q} = \int_{0}^{x} dt \,\psi_{n_{1}}\left(c_{1}, t, \vec{q}\right) E_{4}\begin{pmatrix} n_{2}\dots n_{m}\\ c_{2}\dots c_{m}\\ c_{m} \end{pmatrix}; t, \vec{q}$$
(12)
$$E_{4}\begin{pmatrix} \vec{1}\\ \vec{c} \end{pmatrix}; x, \vec{q} = G(\vec{c}; x)$$
(13)

• \vec{q} are the roots of the elliptic curve defined by

$$y^{2} = (t - q_{1})(t - q_{2})(t - q_{3})(t - q_{4})$$
(14)

• $\psi_{n_1}(c_1, t, \vec{q})$ are the elliptic kernels

• e.g.
$$\psi_0(0, t, \vec{q}) = \frac{c_4}{y}$$
 where $c_4 = \frac{1}{2}\sqrt{(q_1 - q_3)(q_2 - q_4)}$

• e.g.
$$\psi_1(c, t, \vec{q_r}) = \frac{1}{t-c}$$

• define weight as $w = \sum_{i=1}^{m} |n_i|$ and length as l = m

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elliptic Multiple Polylogarithms (eMPLs)

eMPLs in torus representation

[Brown,Levin;Broedel,Duhr,Dulat,Tancredi;Weinzierl...]

$$\tilde{\Gamma}({}^{n_1...n_m}_{z_1...z_m}; z, \tau) = \int_0^z dz' g^{(n_1)} (z' - z_1, \tau) \,\tilde{\Gamma}({}^{n_2...n_m}_{z_2...z_m}; z', \tau)$$
(15)

• a torus is double-periodic and can be defined as a two-dimensional lattice

$$\Lambda_{\tau} = \mathbb{Z} + \mathbb{Z} \tau = \{ m + n \tau | m, n \in \mathbb{Z} \}$$
(16)

- au characterises the shape of the torus
- z are the points on the torus within Λ_{τ}

Iterated integrals of modular forms

if all z_i are rational points on the torus of the form

$$z_i = rac{r}{N} + rac{s}{N} au$$
 with $0 \le r, s < N$ and $r, s, N \in \mathbb{N}$ (17)

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 \rightarrow can rewrite them in terms of iterated integrals of modular forms

$$I(f_{1},...,f_{n};\tau) = \int_{i\infty}^{\tau} \frac{d\tau'}{2\pi i} f_{1} I(f_{2},...,f_{n};\tau)$$
(18)
$$f_{i} = h_{N,r,s}^{(n)}(\tau) = -\sum_{\substack{(a,b) \in \mathbb{Z}^{2} \\ (a,b) \neq (0,0)}} \frac{e^{2\pi i \frac{(bs-ar)}{N}}}{(a\tau+b)^{n}}$$
(19)

Feynman integral can be represented via two graph polynomials \mathcal{U} and \mathcal{F} which are the first and second Symanzik polynomial respectively.

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 - each massless propagator/edge contributes linearly to ${\cal F}$
 - each massive propagator/edge contributes quadratically to ${\cal F}$
- need to integrate out each single edge x_i ; one done via Cheng-Wu delta function $\delta(1 \Delta_H)$.

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Master Integrals - Elliptic Curves

We encounter two different types of elliptic curves,

• one is associated to the elliptic sunrise

$$\vec{q} = \left(\frac{1}{2}\left(1 - \sqrt{1+2i}\right), \frac{1}{2}\left(1 - \sqrt{1-2i}\right), \frac{1}{2}\left(1 + \sqrt{1+2i}\right), \frac{1}{2}\left(1 + \sqrt{1-2i}\right)\right)$$
(21)

• the other is associated to the master integral

$$ec{q}=\left(1-\sqrt{5},0,2,1+\sqrt{5}
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(22)

and appears only in light-by-light scattering contribution

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[Vollinga, Weinzierl]

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- PSLQ procedure: find additional relations between elliptic integrals beyond equivalence relations shown earlier

Melih A. Ozcelik (TTP)

[Vollinga, Weinzierl]

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Form-factors

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- QED corrections to para-Positronium result, agreement with existing numerical results in literature
 [A. Czarnecki, K. Melnikov, A. Yelkhovsky, Phys.Rev.A 61 (2000) 052502]

Summary: Form-factors

- computed all two-loop master integrals analytically
- produced high-precision numerics (> 1000 digits)
- find some interesting equivalence relations
- have complete analytical results for form-factors available
- form-factors are finite after UV and IR renormalisation
 - \rightarrow ready for phenomenological applications

Thank you for attention!

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