



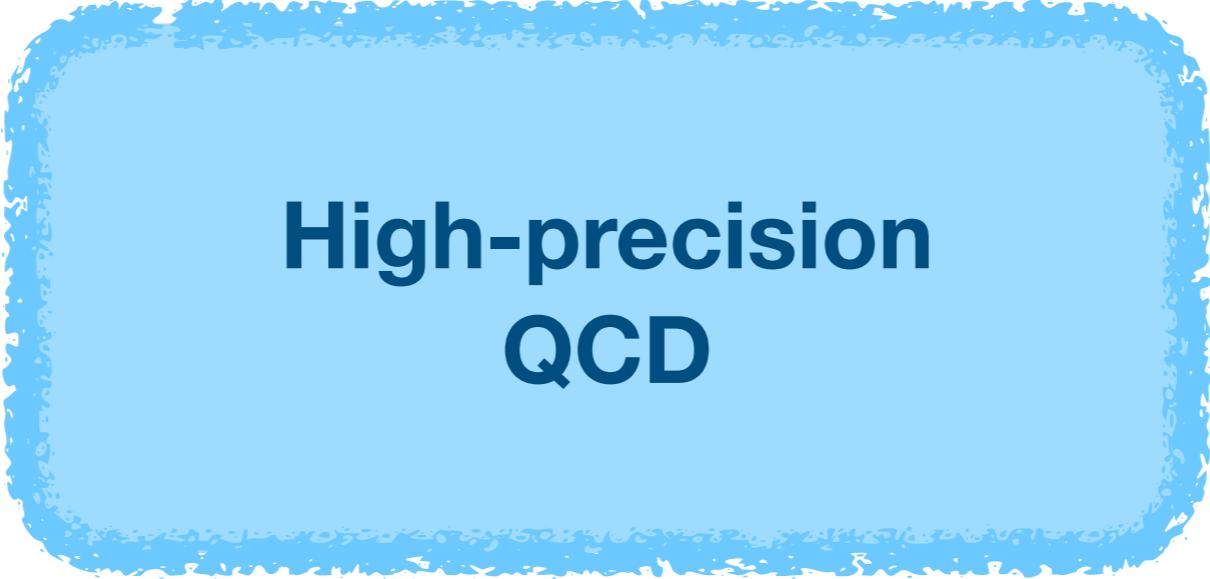
# Three-Loop scattering amplitudes in full-colour QCD

*Ph.D. supervisor:* Prof. Lorenzo Tancredi

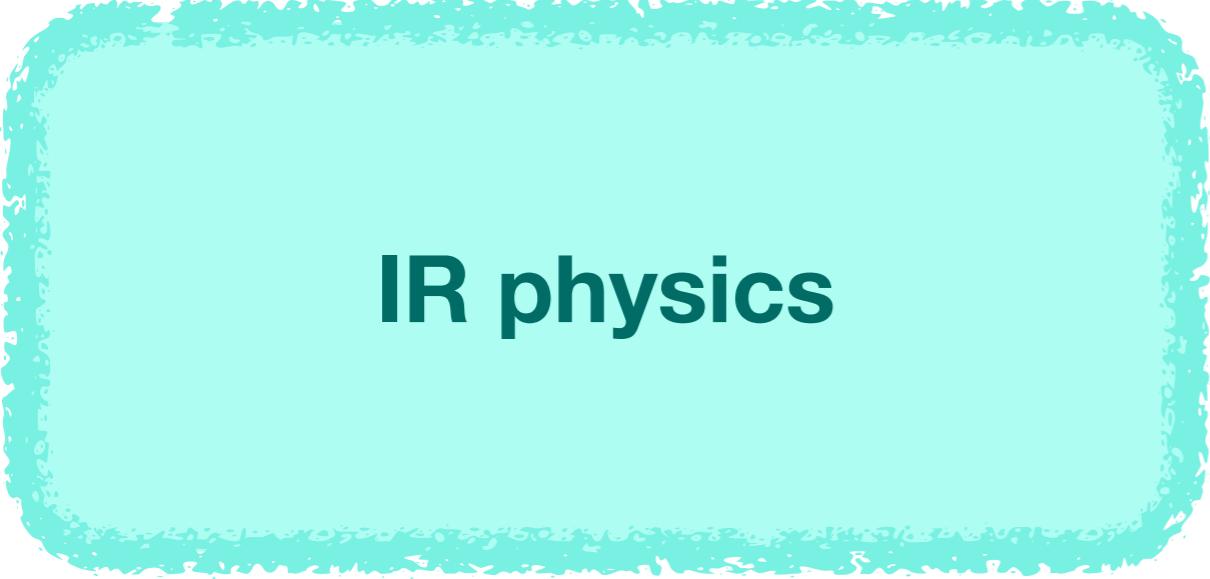
In collaboration with: Amlan Chakraborty,  
Fabrizio Caola,  
Andreas von Manteuffel,  
Lorenzo Tancredi.



# **High-precision QCD**



**High-precision  
QCD**



**IR physics**

**High-precision  
QCD**

**IR physics**

**High-Energy  
limit**

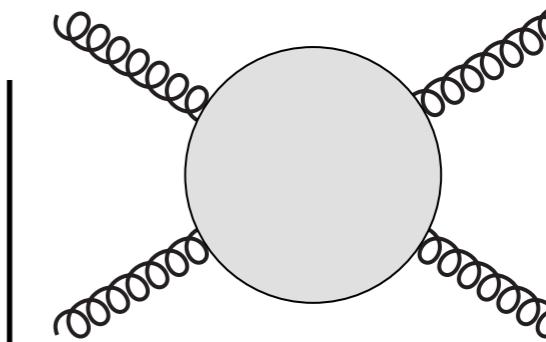
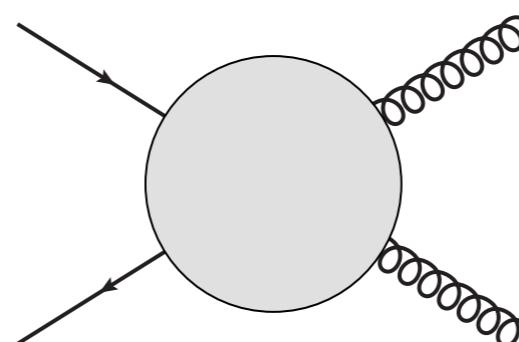
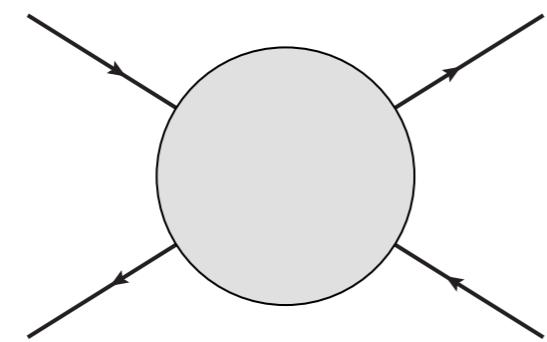
# of QCD  
Feynman  
diagrams

tree level

1-loop

2-loop

3-loop



# of QCD  
Feynman  
diagrams

tree level

1

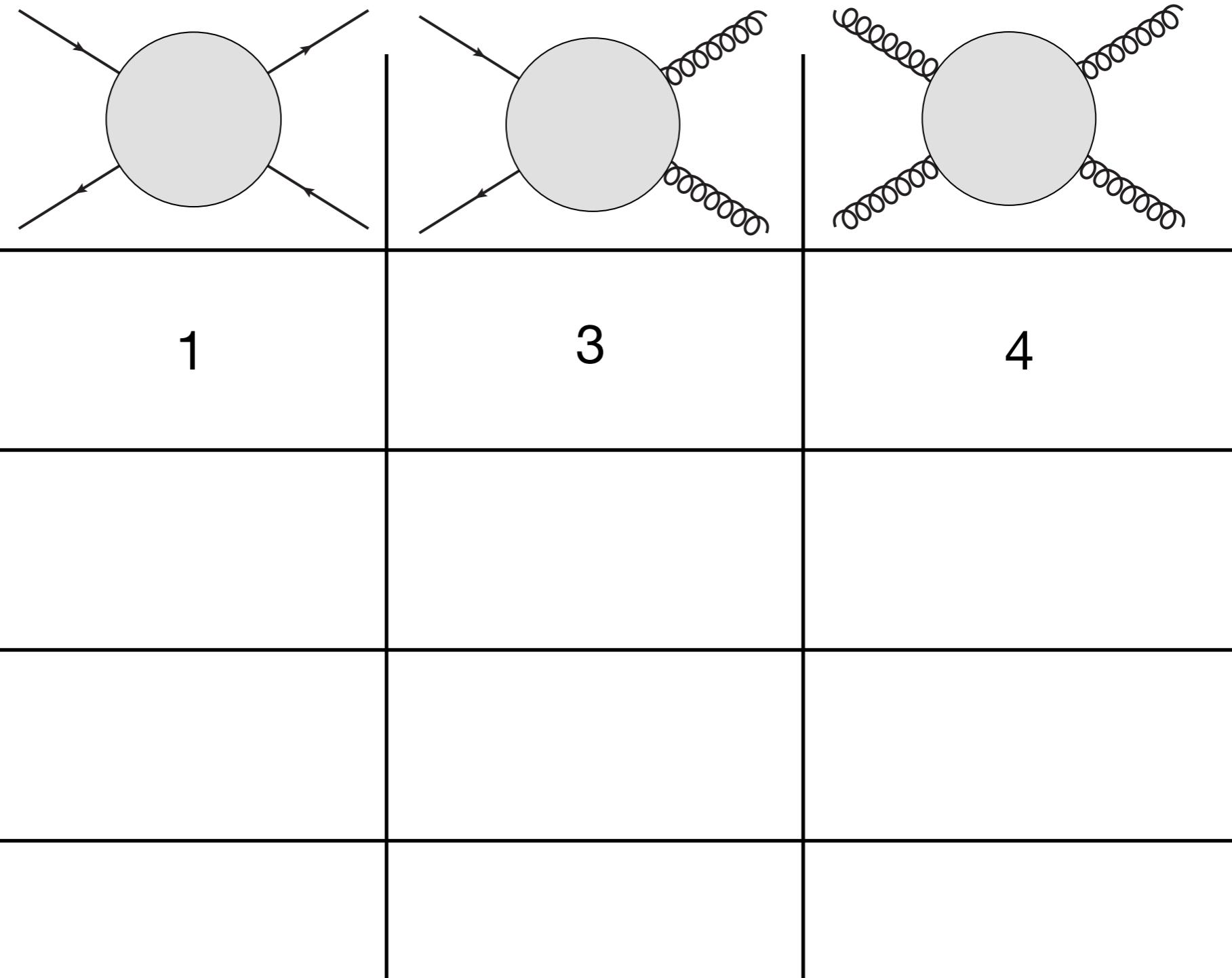
3

4

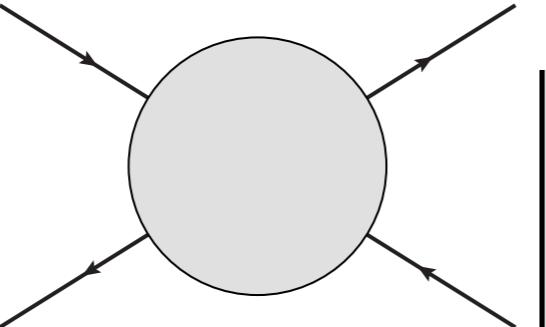
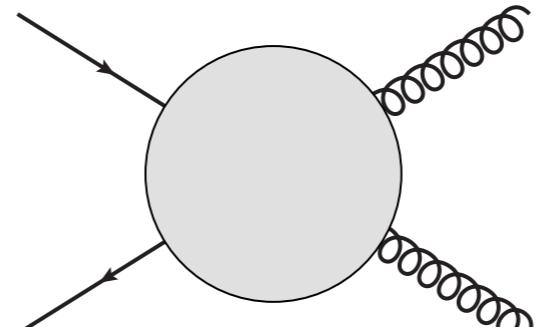
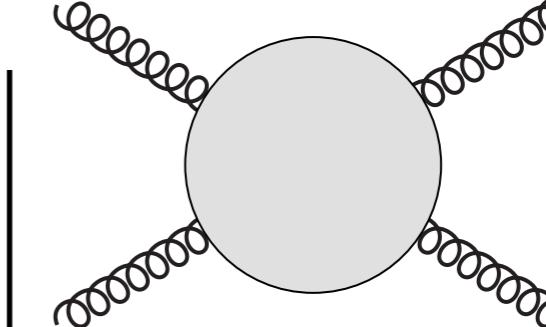
1-loop

2-loop

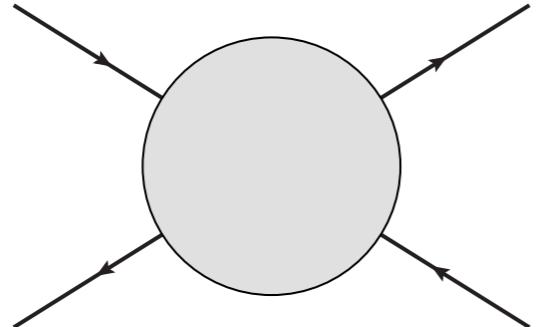
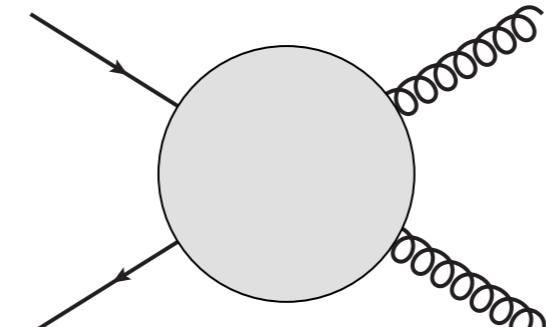
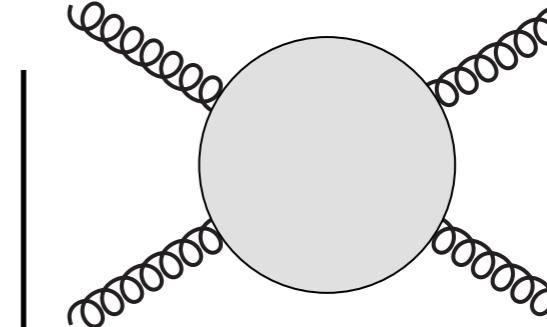
3-loop



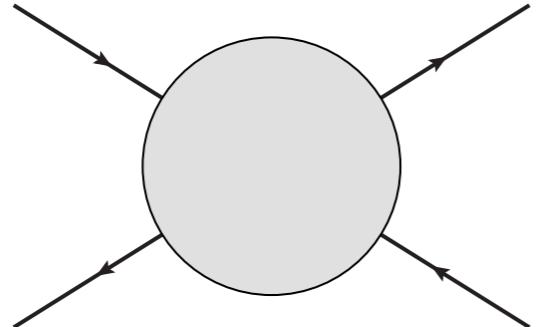
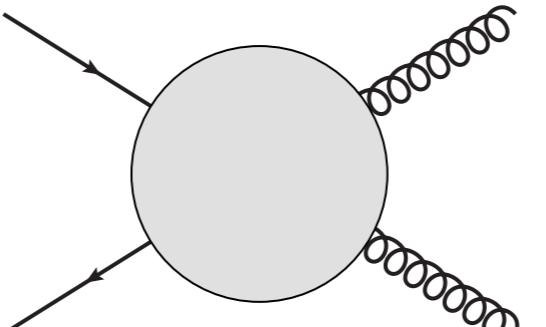
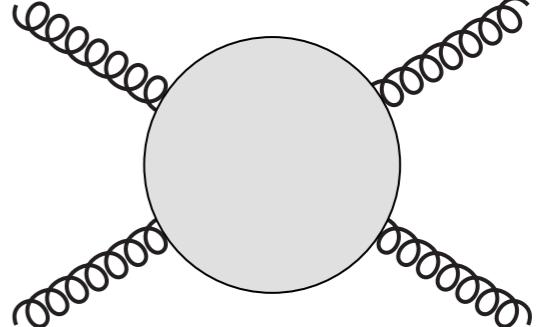
# of QCD  
Feynman  
diagrams

			
tree level	1	3	4
1-loop	9	30	81
2-loop			
3-loop			

# of QCD  
Feynman  
diagrams

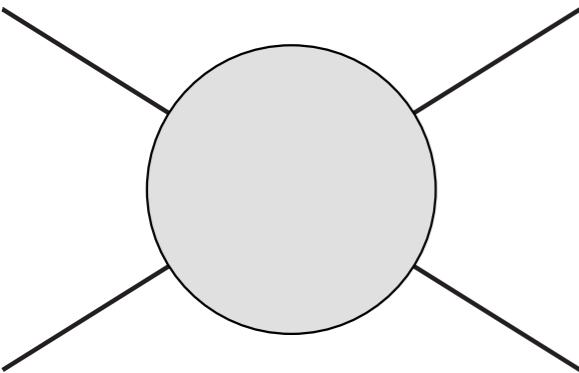
			
tree level	1	3	4
1-loop	9	30	81
2-loop	158	595	1771
3-loop			

# of QCD  
Feynman  
diagrams

			
tree level	1	3	4
1-loop	9	30	81
2-loop	158	595	1771
3-loop	3584	14971	48723 

# **State of the art**

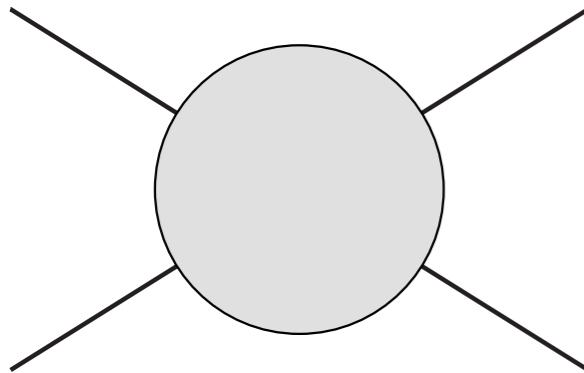
# State of the art



2-loop (~20 years ago)

Anastasiou, Glover, Oleari, Tejeda-Yeomans : [0101304](#), [0011094](#),  
Glover, Oleari, Tejeda-Yeomans : [0102201](#)

# State of the art

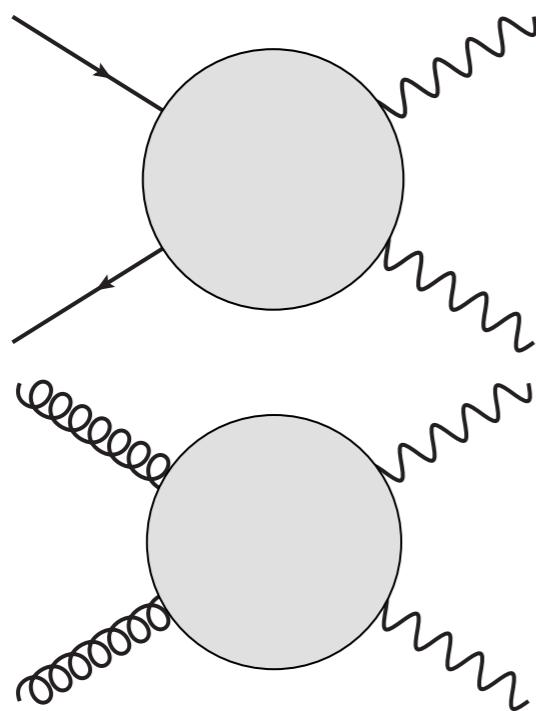


2-loop (~20 years ago)

Anastasiou, Glover, Oleari, Tejeda-Yeomans : [0101304](#), [0011094](#),  
Glover, Oleari, Tejeda-Yeomans : [0102201](#)

NEW

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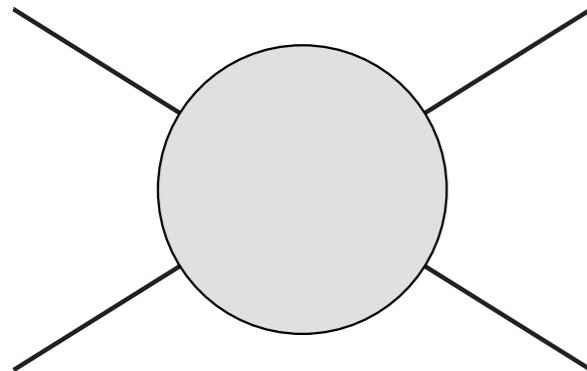


3-loop colour singlet

Caola, von Manteuffel, Tancredi:  
[2011.13946\(PRL\)](#)

Bargiela, Caola, von Manteuffel, Tancredi:  
[2111.13595\(JHEP\)](#)

# State of the art

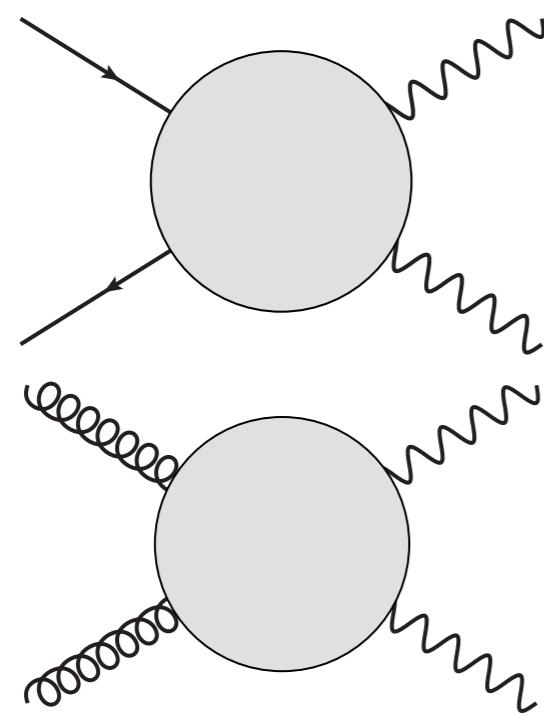


2-loop (~20 years ago)

Anastasiou, Glover, Oleari, Tejeda-Yeomans : [0101304](#), [0011094](#),  
Glover, Oleari, Tejeda-Yeomans : [0102201](#)

NEW

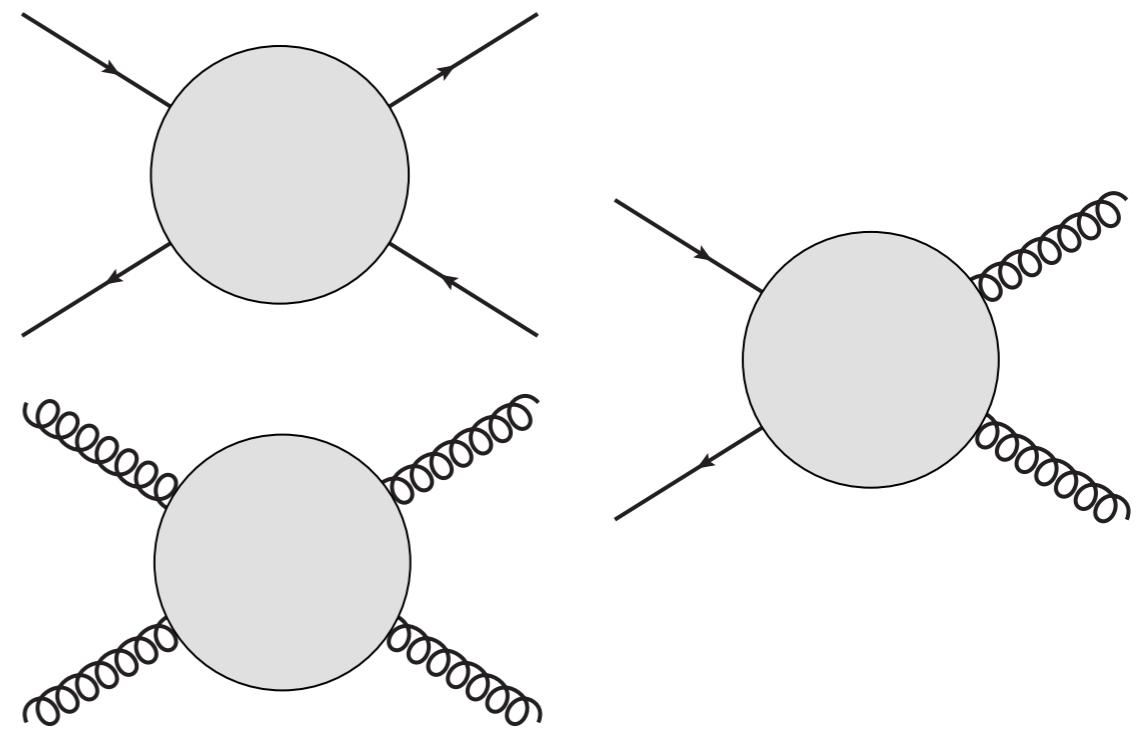
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3-loop colour singlet

Caola, von Manteuffel, Tancredi:  
[2011.13946\(PRL\)](#)

Bargiela, Caola, von Manteuffel, Tancredi:  
[2111.13595\(JHEP\)](#)

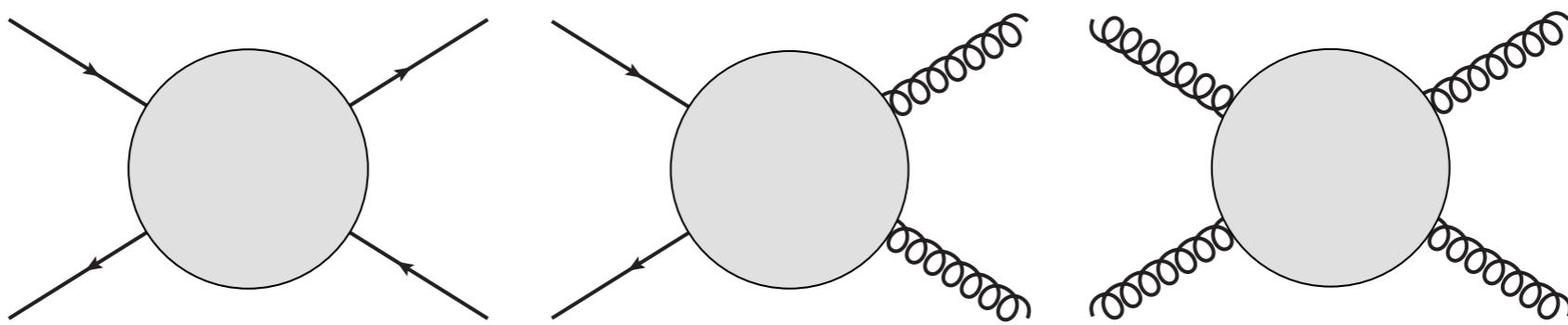


3-loop fully coloured

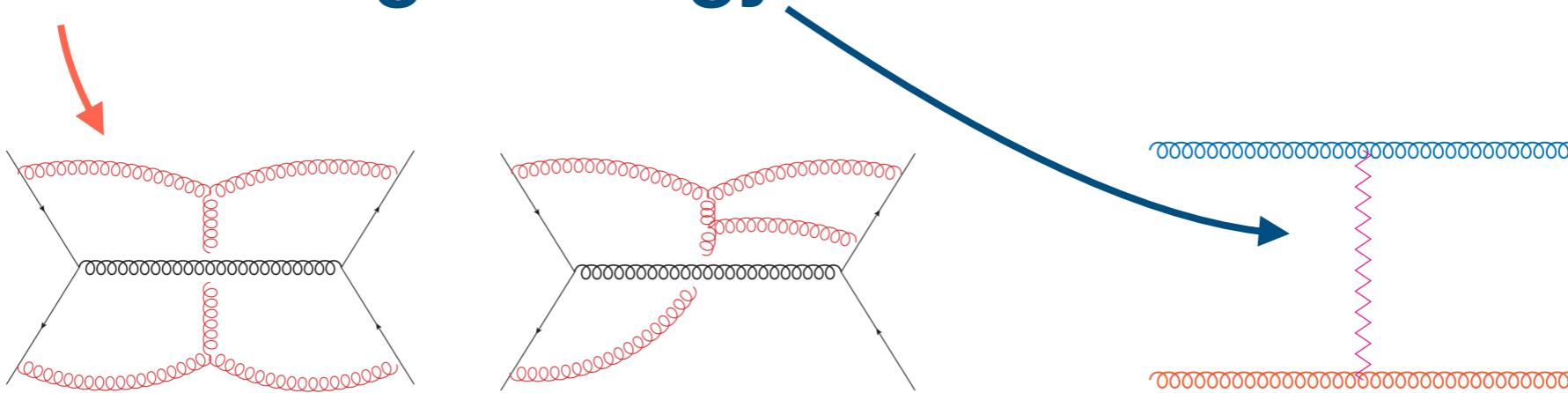
Chakraborty, Caola, Gambuti, Tancredi, von Manteuffel:  
[2108.00055\(JHEP\)](#), [2207.03503](#), [2112.11097\(PRL\)](#)

# Outline

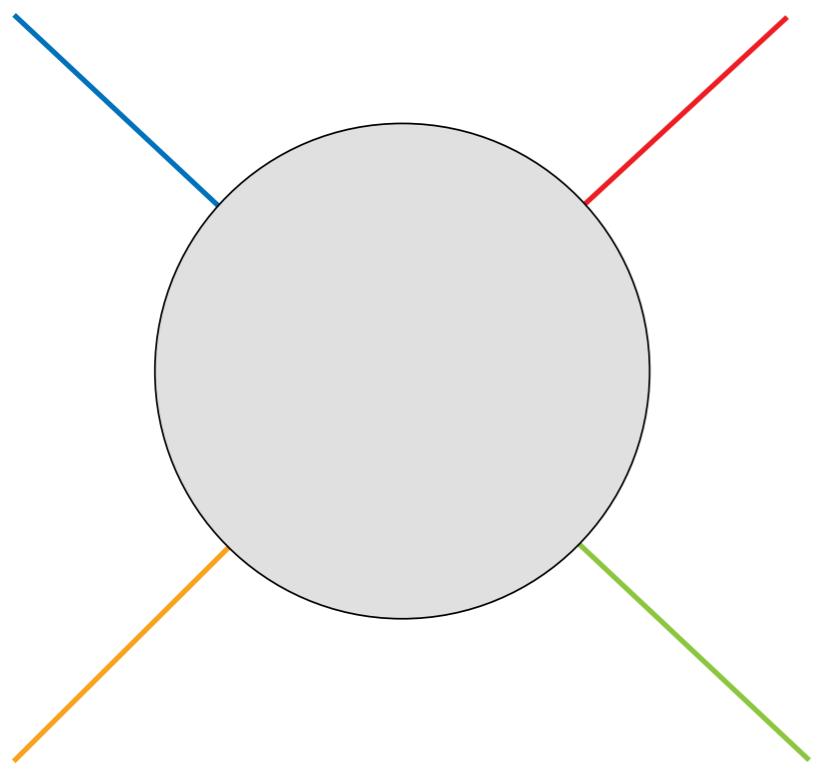
- **Amplitudes**



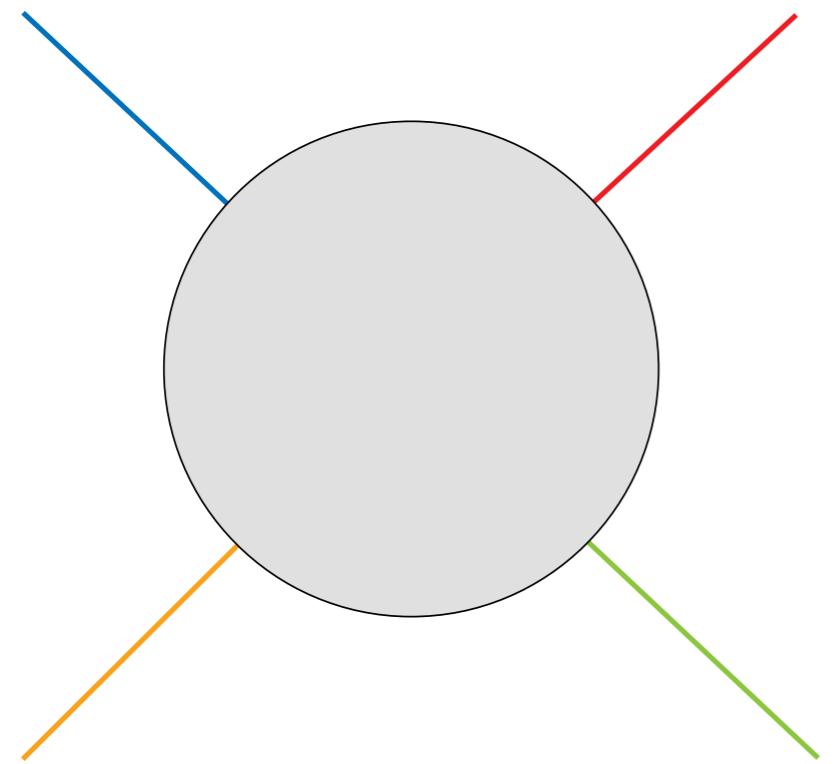
- **IR and high-energy behaviour**



# Amplitudes

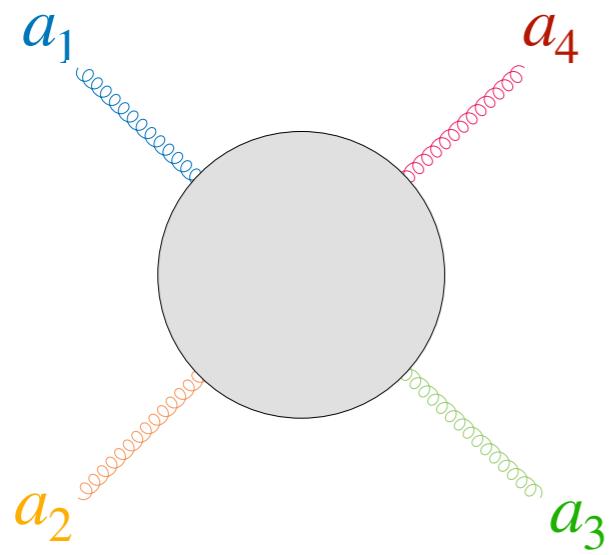


=  $\sum$  Feynman Diagrams

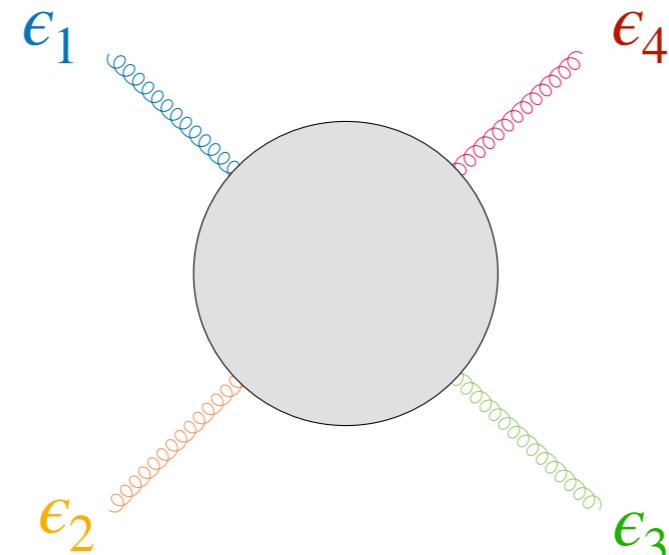


=  $\sum$  Feynman Diagrams

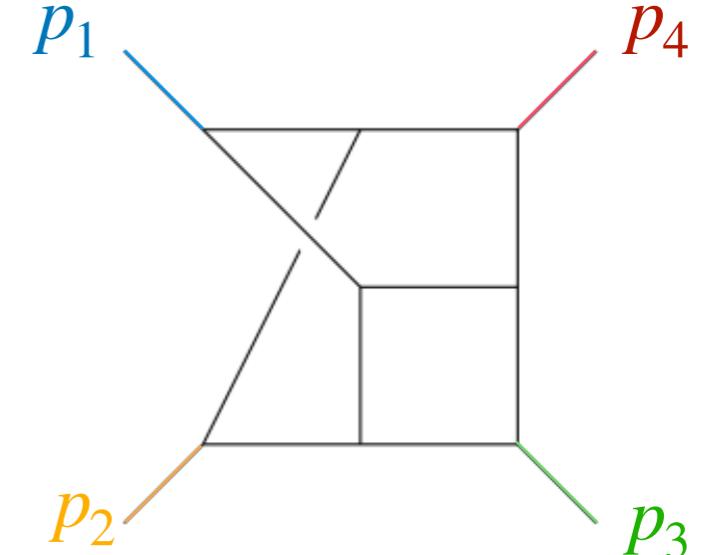
**Colour**



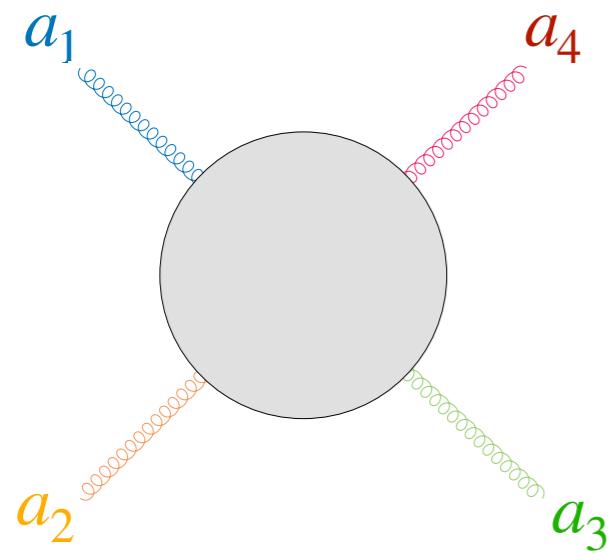
**Spin**



**Kinematics**

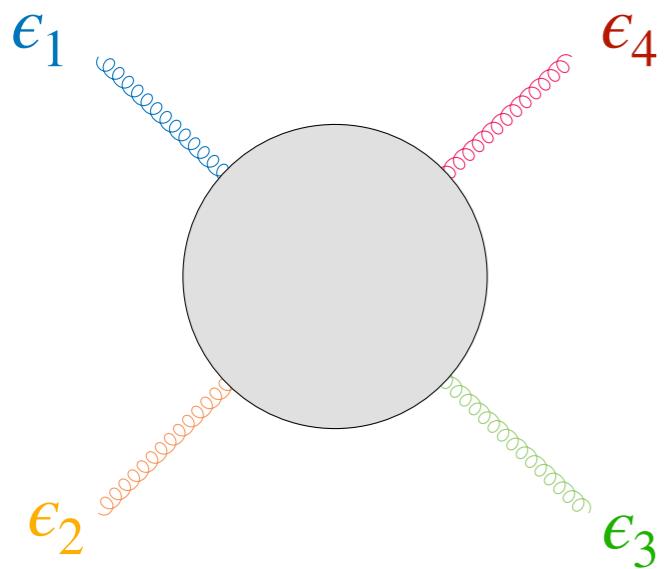


# Colour

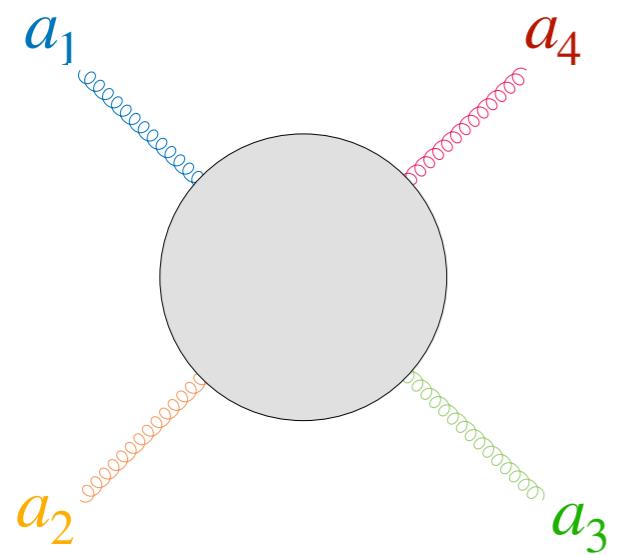


$$A^{a_1 a_2 a_3 a_4} = \sum_{c=1}^6 A_c \mathcal{C}_c^{a_1 a_2 a_3 a_4}$$

# Spin



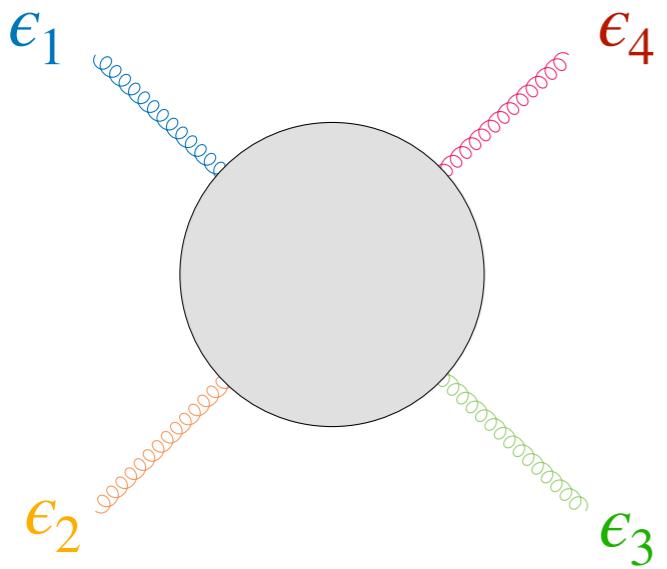
# Colour



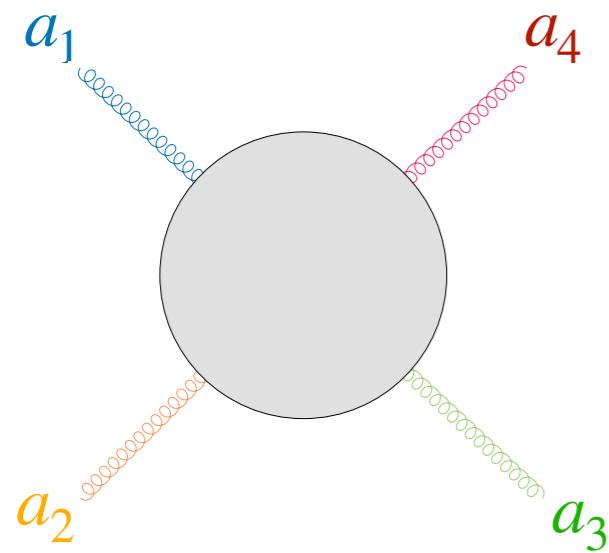
$$A^{a_1 a_2 a_3 a_4} = \sum_{c=1}^6 A_c \mathcal{C}_c^{a_1 a_2 a_3 a_4}$$

Partial Amplitudes  
(gauge invariant!)

# Spin



# Colour

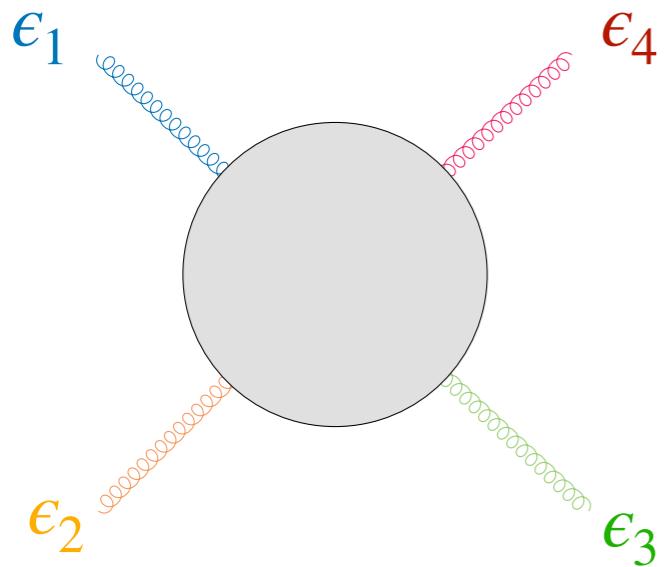


$$A^{a_1 a_2 a_3 a_4} = \sum_{c=1}^6 A_c C_c^{a_1 a_2 a_3 a_4}$$

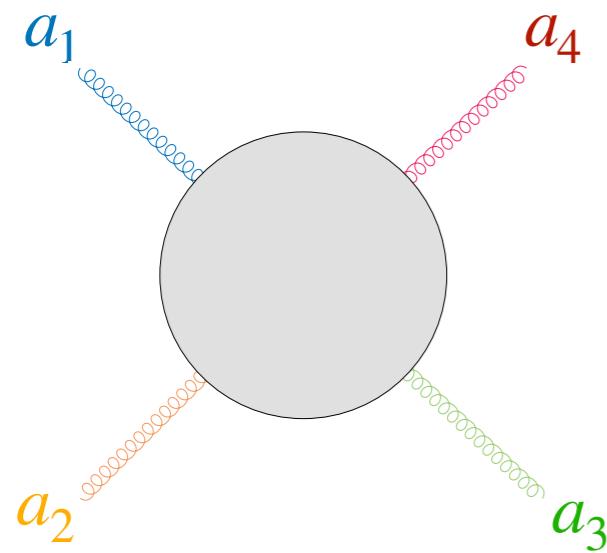
$Tr(\mathbf{T}^{a_1} \mathbf{T}^{a_2} \mathbf{T}^{a_3} \mathbf{T}^{a_4}) \dots$

Partial Amplitudes  
(gauge invariant!)

# Spin



# Colour

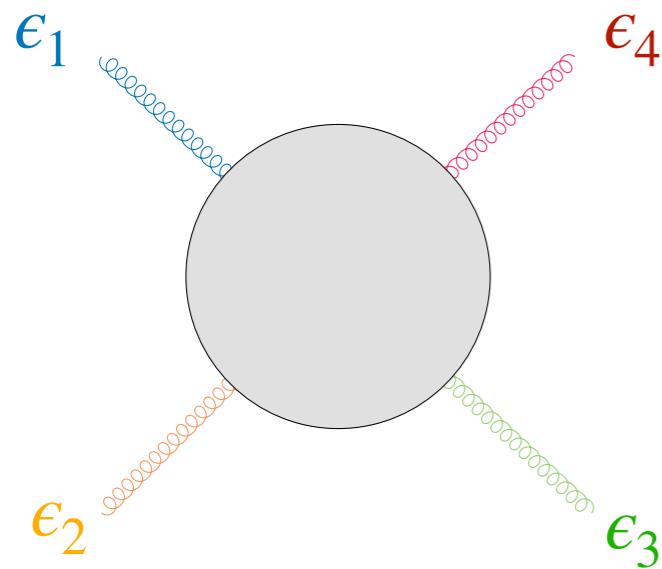


$$A^{a_1 a_2 a_3 a_4} = \sum_{c=1}^6 A_c C_c^{a_1 a_2 a_3 a_4}$$

$Tr(\mathbf{T}^{a_1} \mathbf{T}^{a_2} \mathbf{T}^{a_3} \mathbf{T}^{a_4}) \dots$

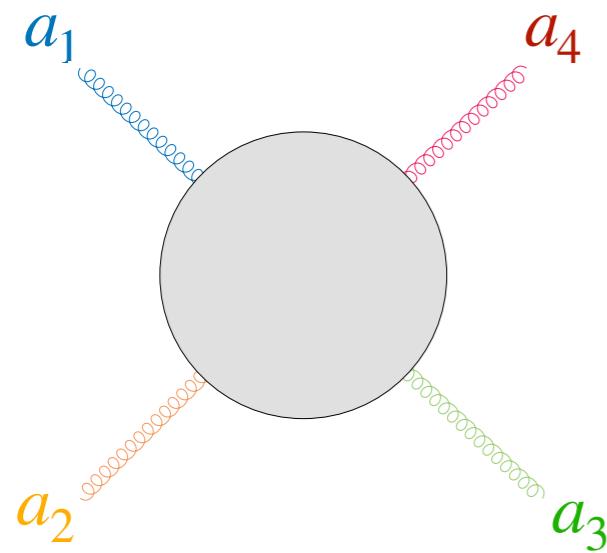
Partial Amplitudes  
(gauge invariant!)

# Spin



$$A_c = A_c^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\mu_1} \epsilon_{\mu_2} \epsilon_{\mu_3} \epsilon_{\mu_4}$$

# Colour

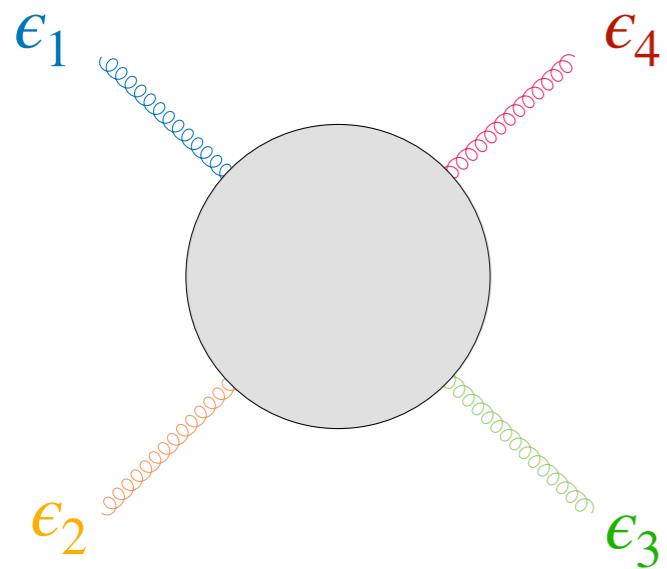


$$A^{a_1 a_2 a_3 a_4} = \sum_{c=1}^6 A_c C_c^{a_1 a_2 a_3 a_4}$$

$Tr(\mathbf{T}^{a_1} \mathbf{T}^{a_2} \mathbf{T}^{a_3} \mathbf{T}^{a_4}) \dots$

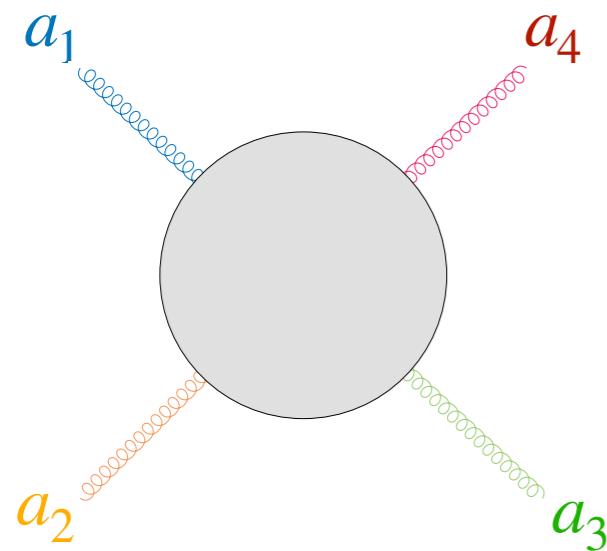
Partial Amplitudes  
(gauge invariant!)

# Spin



$$\begin{aligned} A_c &= A_c^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\mu_1} \epsilon_{\mu_2} \epsilon_{\mu_3} \epsilon_{\mu_4} \\ &= \sum_j F_c^j T_j^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\mu_1} \epsilon_{\mu_2} \epsilon_{\mu_3} \epsilon_{\mu_4} \end{aligned}$$

# Colour

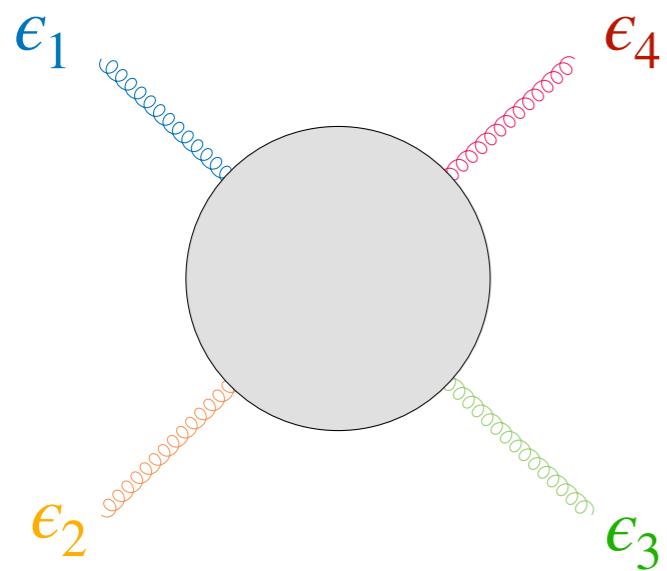


$$A^{a_1 a_2 a_3 a_4} = \sum_{c=1}^6 A_c C_c^{a_1 a_2 a_3 a_4}$$

$Tr(\mathbf{T}^{a_1} \mathbf{T}^{a_2} \mathbf{T}^{a_3} \mathbf{T}^{a_4}) \dots$

Partial Amplitudes  
(gauge invariant!)

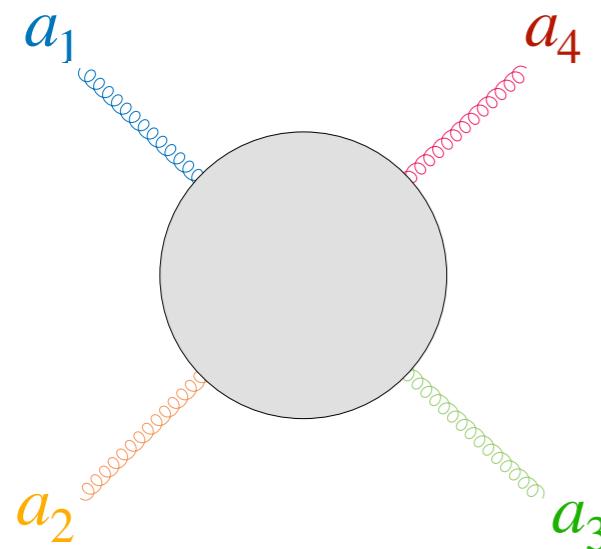
# Spin



$$A_c = A_c^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\mu_1} \epsilon_{\mu_2} \epsilon_{\mu_3} \epsilon_{\mu_4}$$
$$= \sum_j F_c^j T_j^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\mu_1} \epsilon_{\mu_2} \epsilon_{\mu_3} \epsilon_{\mu_4}$$

Form Factors

# Colour

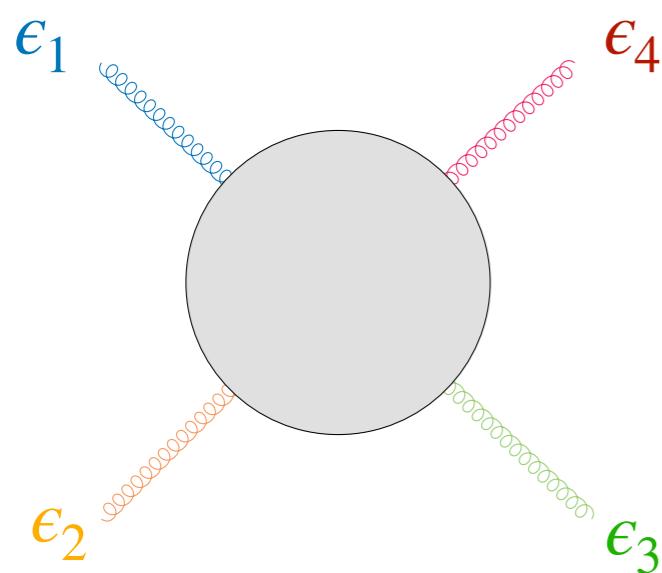


$$A^{a_1 a_2 a_3 a_4} = \sum_{c=1}^6 A_c C_c^{a_1 a_2 a_3 a_4}$$

$Tr(\mathbf{T}^{a_1} \mathbf{T}^{a_2} \mathbf{T}^{a_3} \mathbf{T}^{a_4}) \dots$

Partial Amplitudes  
(gauge invariant!)

# Spin



$$A_c = A_c^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\mu_1} \epsilon_{\mu_2} \epsilon_{\mu_3} \epsilon_{\mu_4}$$
$$= \sum_j F_c^j T_j^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\mu_1} \epsilon_{\mu_2} \epsilon_{\mu_3} \epsilon_{\mu_4}$$

Form Factors

- Transversality
- Gauge fixing
- 't Hooft-Veltman scheme

$$T_j = T_j^{\mu_1 \mu_2 \mu_3 \mu_4} \epsilon_{\mu_1} \epsilon_{\mu_2} \epsilon_{\mu_3} \epsilon_{\mu_4}$$

$$T_1 = \epsilon_1 \cdot p_3 \ \epsilon_2 \cdot p_1 \ \epsilon_3 \cdot p_1 \ \epsilon_4 \cdot p_2 ,$$

$$T_2 = \epsilon_1 \cdot p_3 \ \epsilon_2 \cdot p_1 \ \epsilon_3 \cdot \epsilon_4, \quad T_3 = \epsilon_1 \cdot p_3 \ \epsilon_3 \cdot p_1 \ \epsilon_2 \cdot \epsilon_4,$$

$$T_4 = \epsilon_1 \cdot p_3 \ \epsilon_4 \cdot p_2 \ \epsilon_2 \cdot \epsilon_3, \quad T_5 = \epsilon_2 \cdot p_1 \ \epsilon_3 \cdot p_1 \ \epsilon_1 \cdot \epsilon_4,$$

$$T_6 = \epsilon_2 \cdot p_1 \ \epsilon_4 \cdot p_2 \ \epsilon_1 \cdot \epsilon_3, \quad T_7 = \epsilon_3 \cdot p_1 \ \epsilon_4 \cdot p_2 \ \epsilon_1 \cdot \epsilon_2,$$

$$T_8 = \epsilon_1 \cdot \epsilon_2 \ \epsilon_3 \cdot \epsilon_4 + \epsilon_1 \cdot \epsilon_4 \ \epsilon_2 \cdot \epsilon_3 + \epsilon_1 \cdot \epsilon_3 \ \epsilon_2 \cdot \epsilon_4 .$$

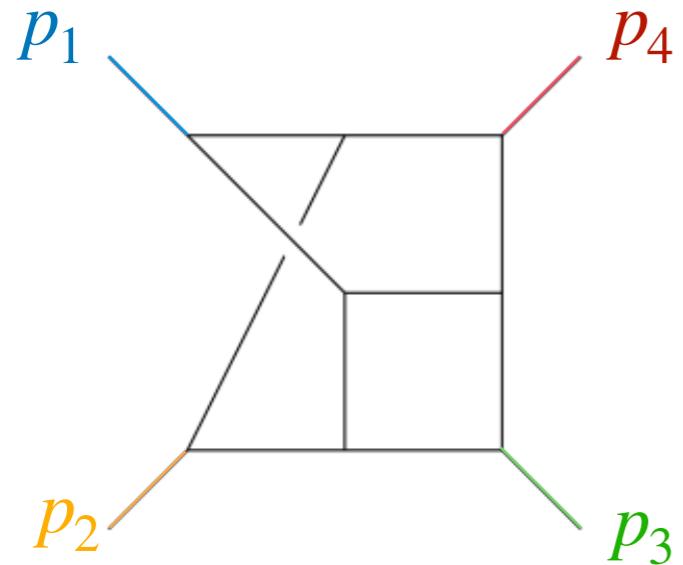
$$A^{a_1 a_2 a_3 a_4} = \sum_{c,i} F_c^i T_i \mathcal{C}_c^{a_1 a_2 a_3 a_4}$$

**Non-perturbative !!**

$$A^{a_1 a_2 a_3 a_4} = \sum_{c,i} F_c^i T_i \mathcal{C}_c^{a_1 a_2 a_3 a_4}$$

**Non-perturbative !!**

## Kinematics

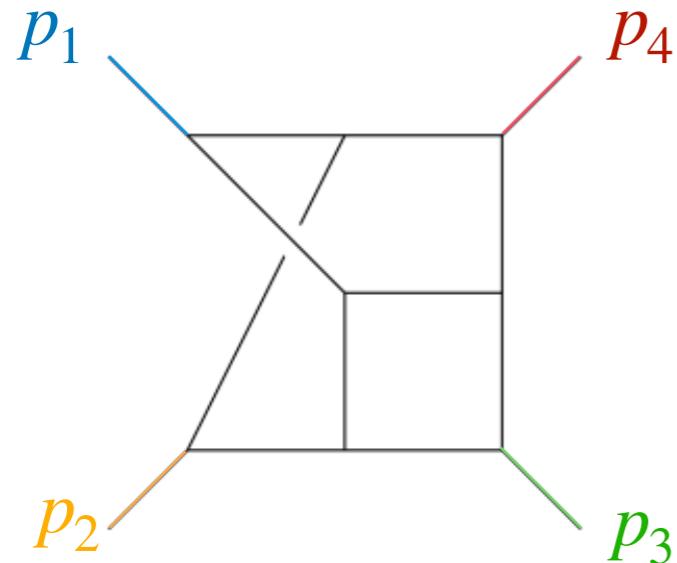


$$I_n \sim \int \frac{d^d \{k_j\}}{(2\pi)^{dl}} \frac{\mathcal{N}(\{p\}, \{k\})}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}}$$

$$A^{a_1 a_2 a_3 a_4} = \sum_{c,i} F_c^i T_i \mathcal{C}_c^{a_1 a_2 a_3 a_4}$$

**Non-perturbative !!**

## Kinematics

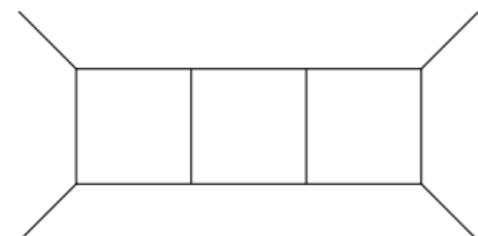


$$I_n \sim \int \frac{d^d \{k_j\}}{(2\pi)^{dl}} \frac{\mathcal{N}(\{p\}, \{k\})}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}}$$

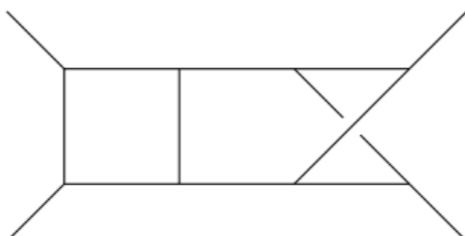
$(k_1 \cdot p_1)(k_2 \cdot k_3) + (k_1 \cdot p_2)(k_3 \cdot p_3) + \dots$

$$I_n \sim \int \frac{\mathrm{d}^d\{k_j\}}{(2\pi)^{dl}} \frac{\mathcal{N}(\{p\},\{k\})}{D_1^{n_1} D_2^{n_2} \ldots D_N^{n_N}}$$

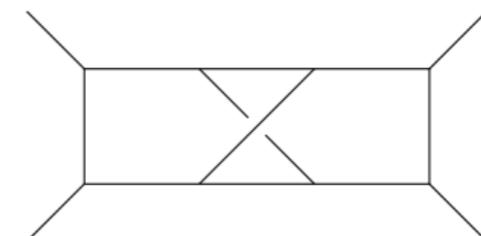
$$I_n \sim \int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{\mathcal{N}(\{p\}, \{k\})}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}}$$



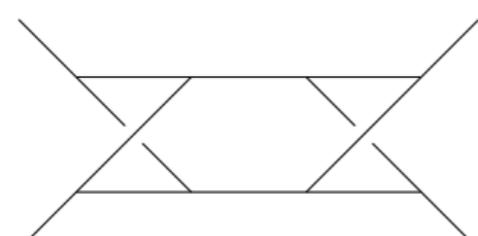
(a)



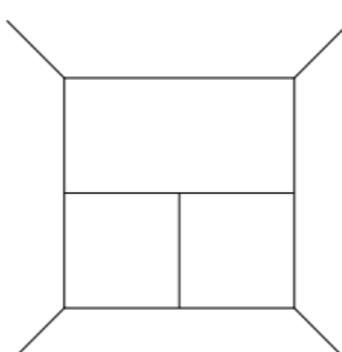
(b)



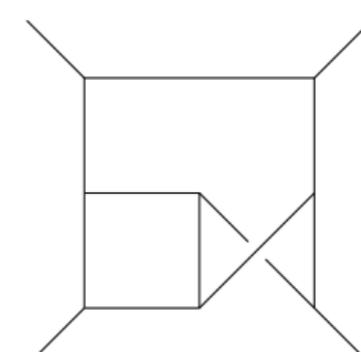
(c)



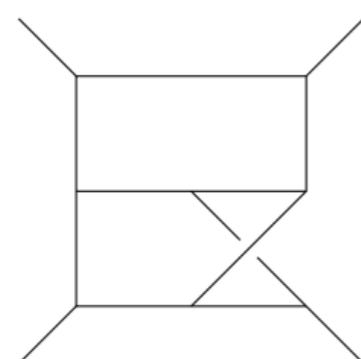
(d)



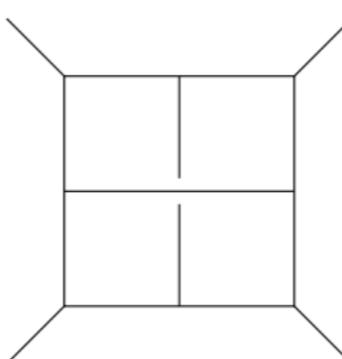
(e)



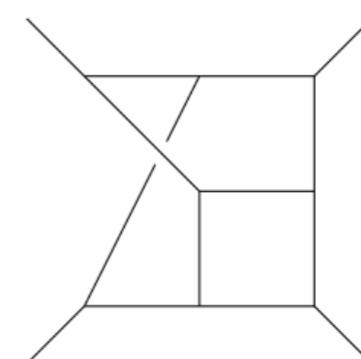
(f)



(g)

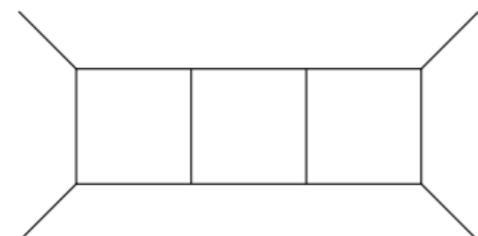


(h)

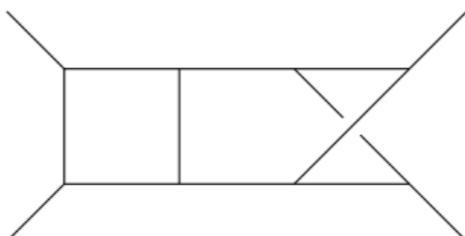


(i)

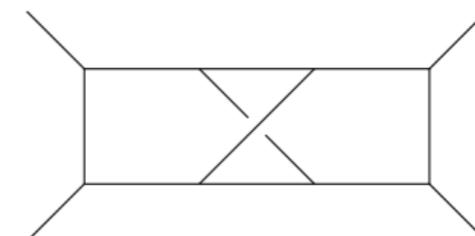
$$I_n \sim \int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{\mathcal{N}(\{p\}, \{k\})}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}}$$



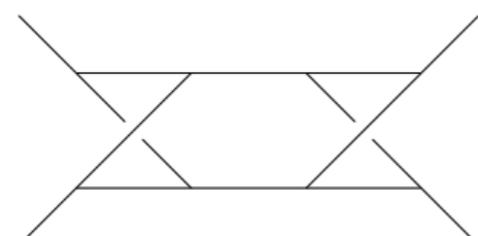
(a)



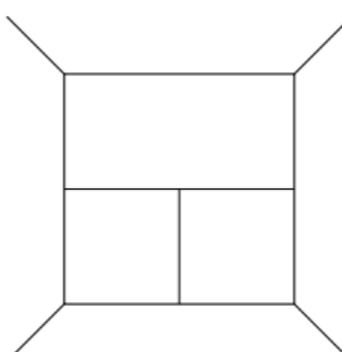
(b)



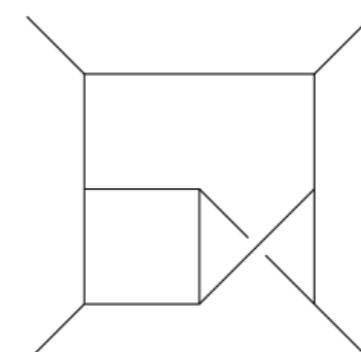
(c)



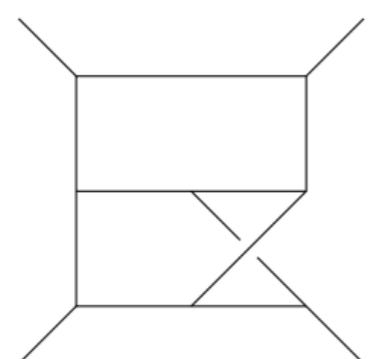
(d)



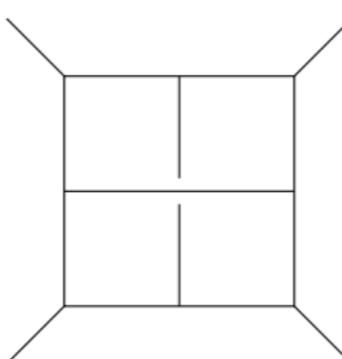
(e)



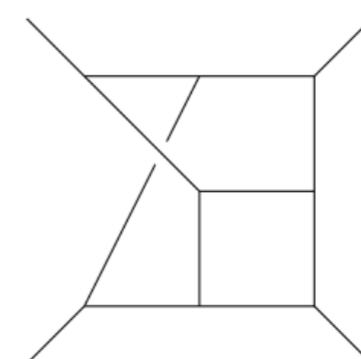
(f)



(g)



(h)

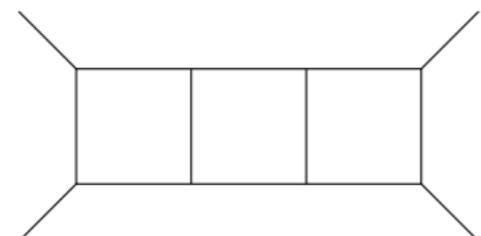


(i)

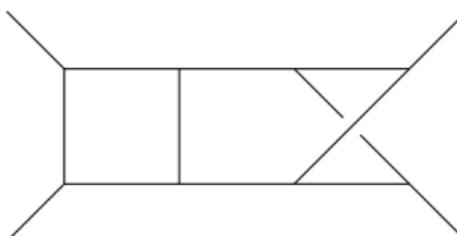
***N = 10***

Henn, Mistlberger, V.A. Smirnov, Wasser: [2002.09492](https://arxiv.org/abs/2002.09492)

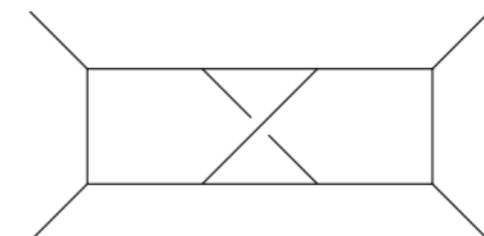
$$I_n \sim \int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{\mathcal{N}(\{p\}, \{k\})}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}}$$



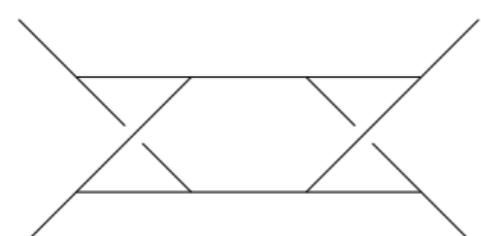
(a)



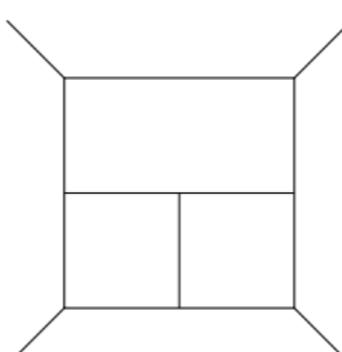
(b)



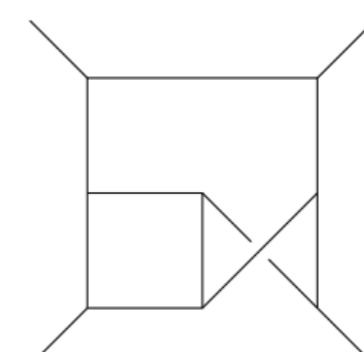
(c)



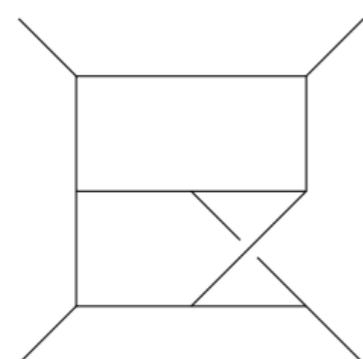
(d)



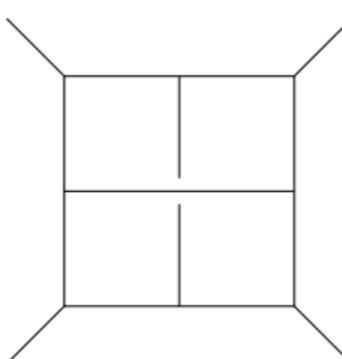
(e)



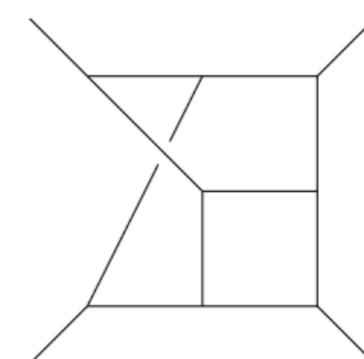
(f)



(g)



(h)

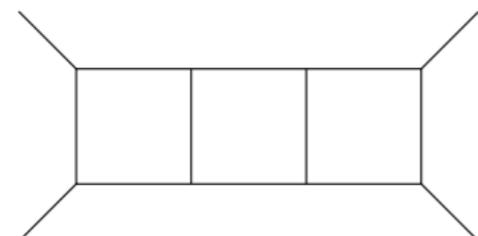


(i)

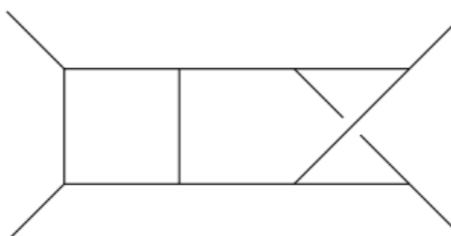
$N = 10 \rightarrow M = 15$

Henn, Mistlberger, V.A. Smirnov, Wasser: [2002.09492](#)

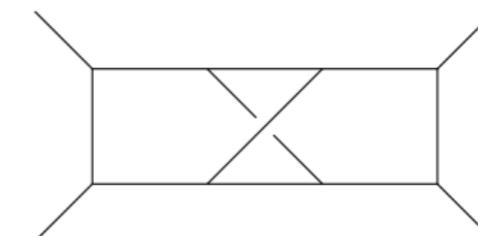
$$I_n \sim \int \frac{d^d \{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{\textcolor{red}{n_{N+1}}} \dots D_M^{\textcolor{red}{n_M}}} \\ n_i \in \mathbb{Z}$$



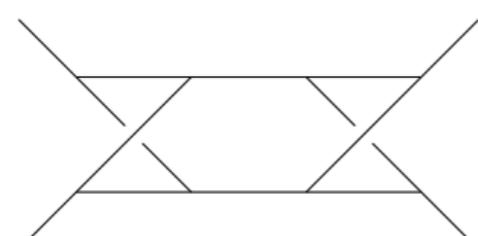
(a)



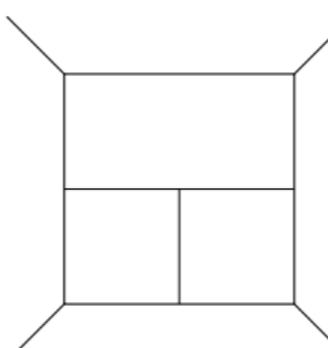
(b)



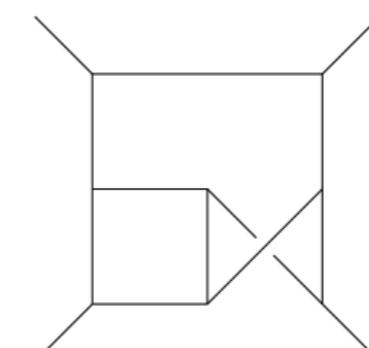
(c)



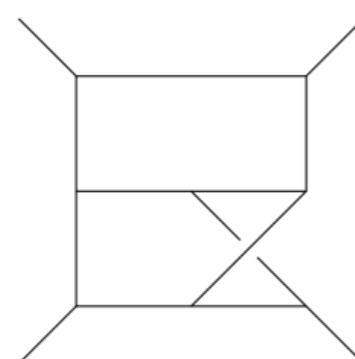
(d)



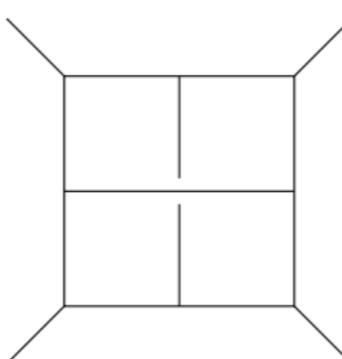
(e)



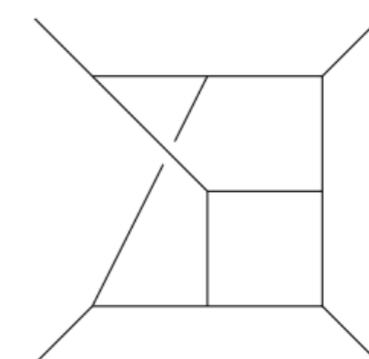
(f)



(g)



(h)



(i)

$N = 10 \rightarrow M = 15$

Henn, Mistlberger, V.A. Smirnov, Wasser: [2002.09492](#)

# The integrals

$$\int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{\textcolor{red}{n_{N+1}}} \dots D_M^{\textcolor{red}{n_M}}}$$

## 3 integral families

Family	PL	NPL1	NPL2
$D_1$	$k_1^2$	$k_1^2$	$k_1^2$
$D_2$	$k_2^2$	$k_1^2$	$k_1^2$
$D_3$	$k_3^2$	$k_3^2$	$k_3^2$
$D_4$	$(k_1 - p_1)^2$	$(k_1 - p_1)^2$	$(k_1 - p_1)^2$
$D_5$	$(k_2 - p_1)^2$	$(k_2 - p_1)^2$	$(k_2 - p_1)^2$
$D_6$	$(k_3 - p_1)^2$	$(k_3 - p_1)^2$	$(k_3 - p_1)^2$
$D_7$	$(k_1 - p_1 - p_2)^2$	$(k_1 - p_1 - p_2)^2$	$(k_1 - p_1 - p_2)^2$
$D_8$	$(k_2 - p_1 - p_2)^2$	$(k_2 - p_1 - p_2)^2$	$(k_3 - p_1 - p_2)^2$
$D_9$	$(k_3 - p_1 - p_2)^2$	$(k_3 - p_1 - p_2)^2$	$(k_1 - k_2)^2$
$D_{10}$	$(k_1 - p_1 - p_2 - p_3)^2$	$(k_1 - p_1 - p_2 - p_3)^2$	$(k_2 - k_3)^2$
$D_{11}$	$(k_2 - p_1 - p_2 - p_3)^2$	$(k_2 - p_1 - p_2 - p_3)^2$	$(k_1 - k_2 - p_3)^2$
$D_{12}$	$(k_3 - p_1 - p_2 - p_3)^2$	$(k_3 - p_1 - p_2 - p_3)^2$	$(k_2 - k_3 + p_1 + p_2 + p_3)^2$
$D_{13}$	$(k_1 - k_2)^2$	$(k_1 - k_2)^2$	$(k_2 + p_3)^2$
$D_{14}$	$(k_1 - k_3)^2$	$(k_2 - k_3)^2$	$(k_1 - k_3)^2$
$D_{15}$	$(k_2 - k_3)^2$	$(k_1 - k_2 + k_3)^2$	$(k_2 - p_1 - p_2)^2$

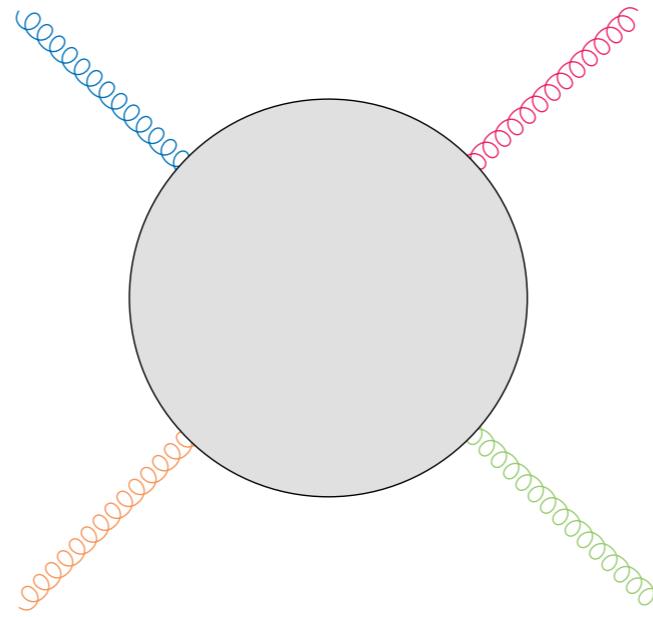
# The integrals

$$\int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{\textcolor{red}{n_{N+1}}} \dots D_M^{n_M}}$$

# The integrals

$$\int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{n_{N+1}} \dots D_M^{n_M}}$$

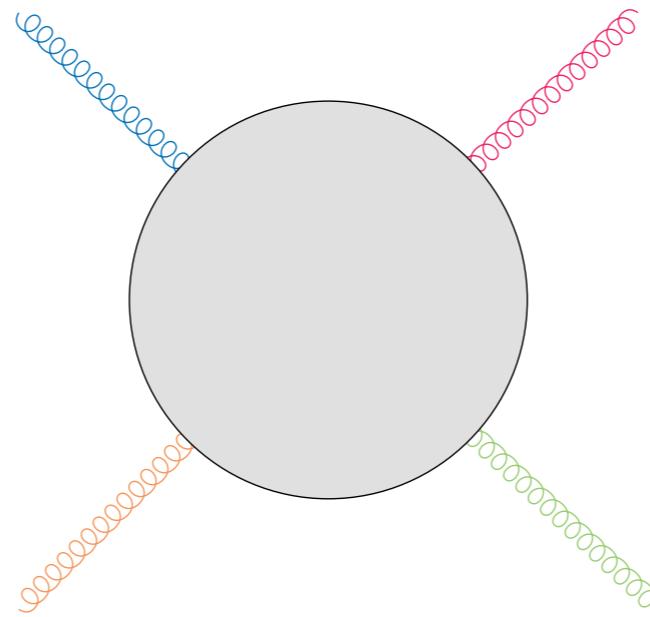
gg → gg:



# The integrals

$$\int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{n_{N+1}} \dots D_M^{n_M}}$$

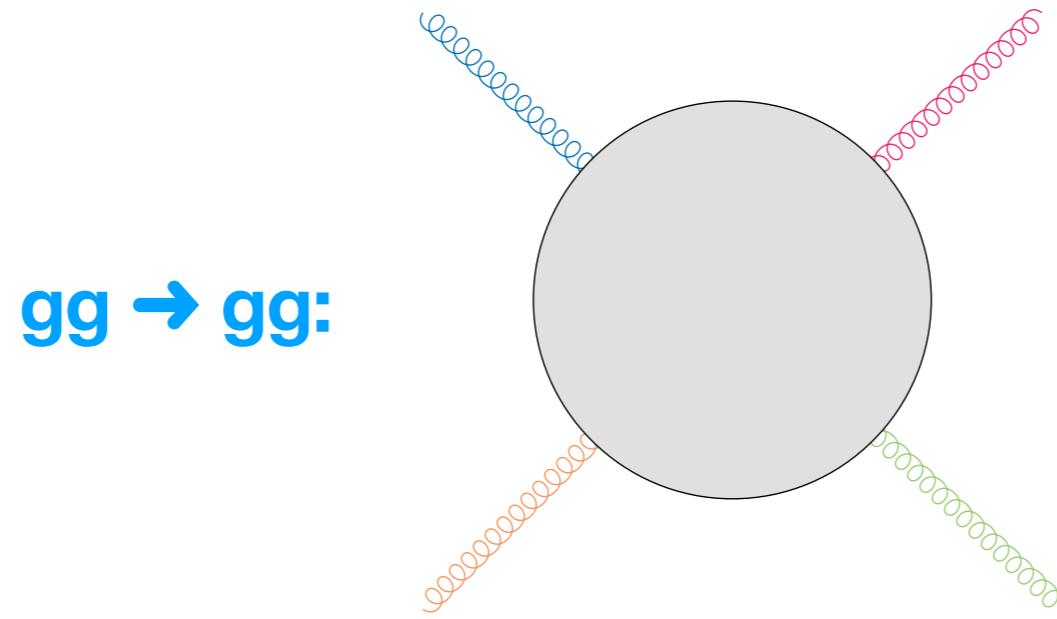
**gg → gg:**



**Feynman Rules**

# The integrals

$$\int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{n_{N+1}} \dots D_M^{n_M}}$$



**gg → gg:**

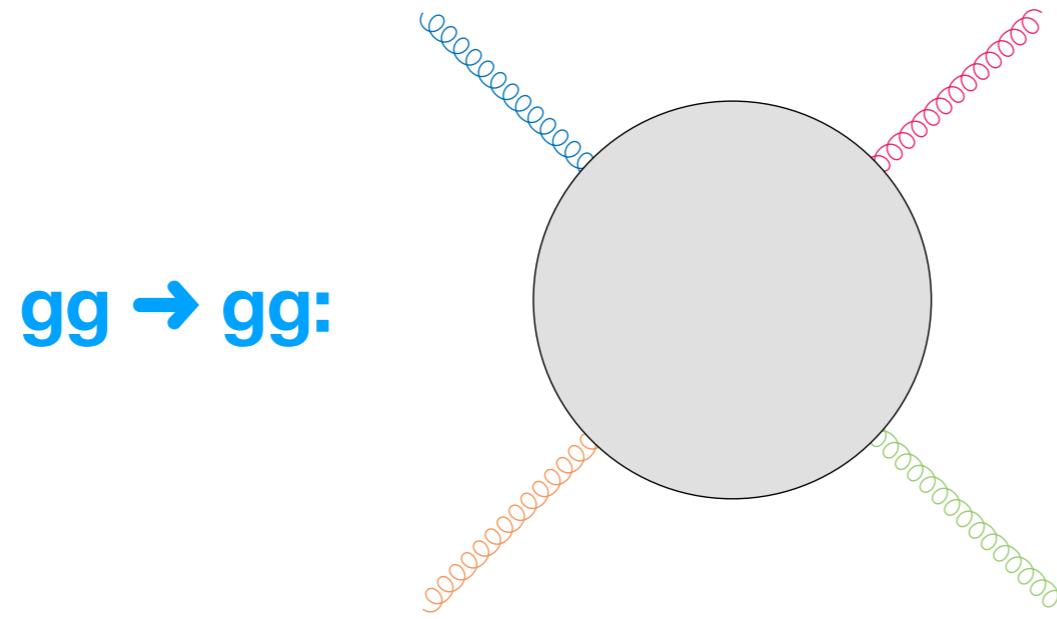
Feynman Rules

10 million integrals

200 GB !!

# The integrals

$$\int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{n_{N+1}} \dots D_M^{n_M}}$$



**gg → gg:**

Feynman Rules

10 million integrals

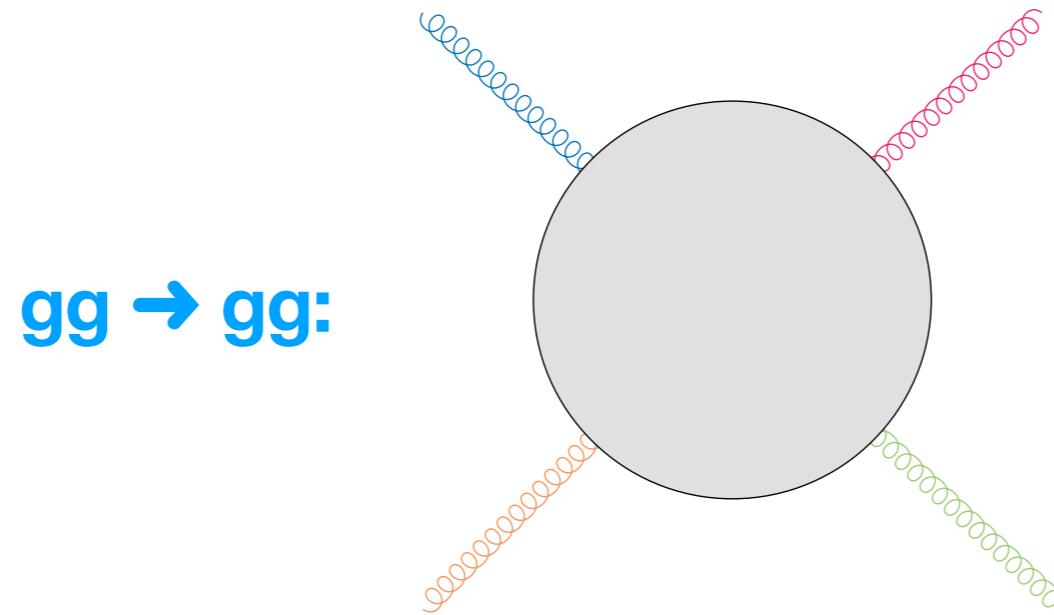
IBPs



**200 GB !!**

# The integrals

$$\int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{n_{N+1}} \dots D_M^{n_M}}$$



**gg → gg:**

Feynman Rules

10 million integrals

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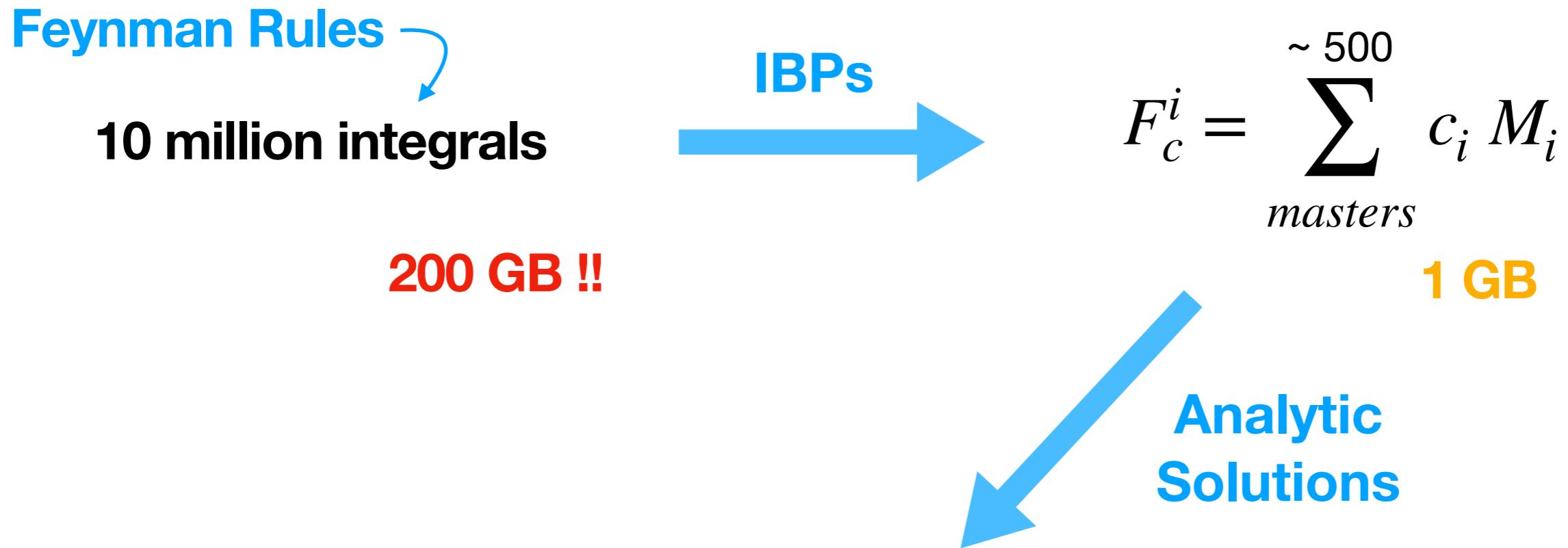
IBPs

$$F_c^i = \sum_{masters}^{\sim 500} c_i M_i$$

**1 GB**

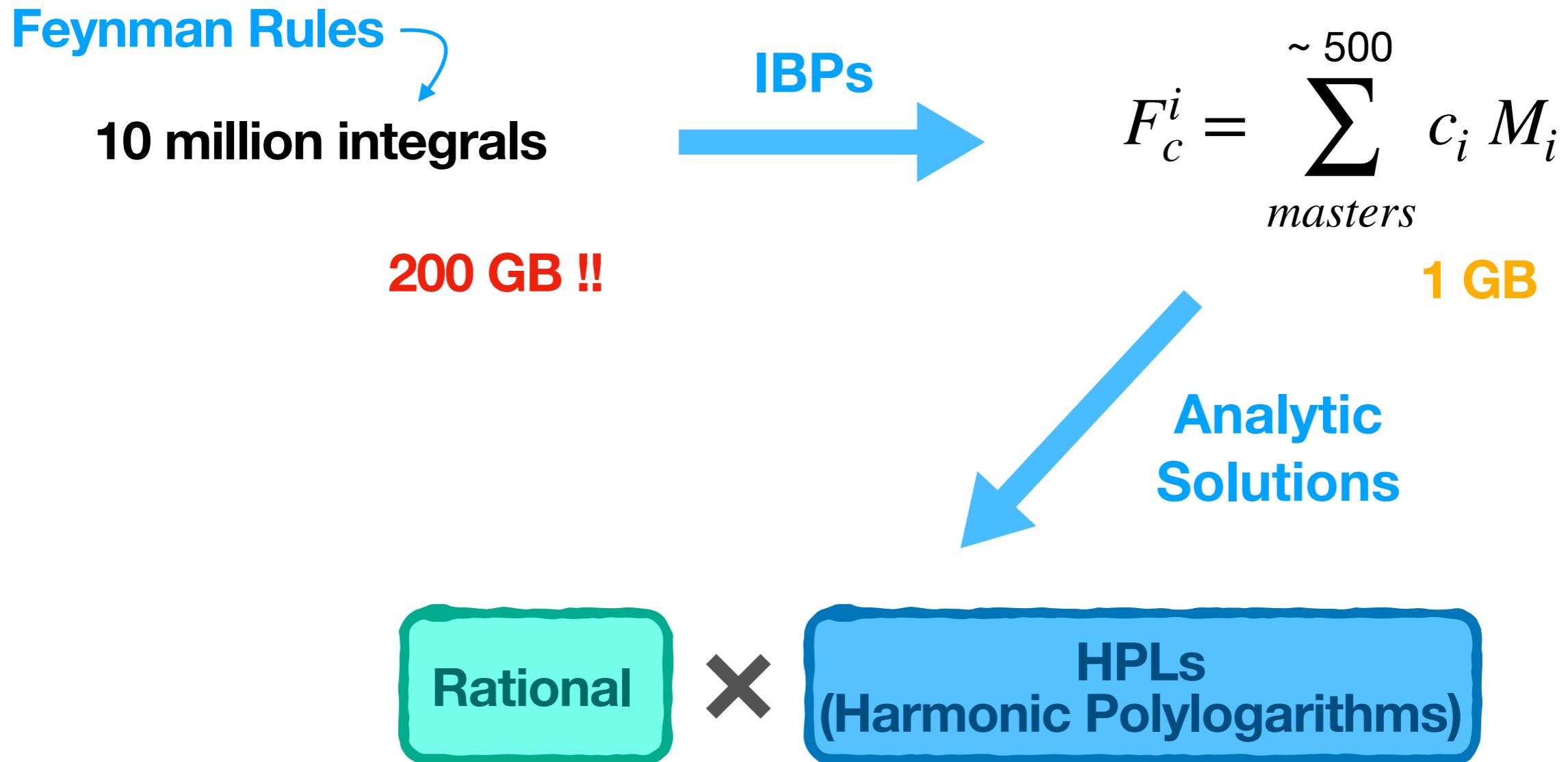
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$$\int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{n_{N+1}} \dots D_M^{n_M}}$$



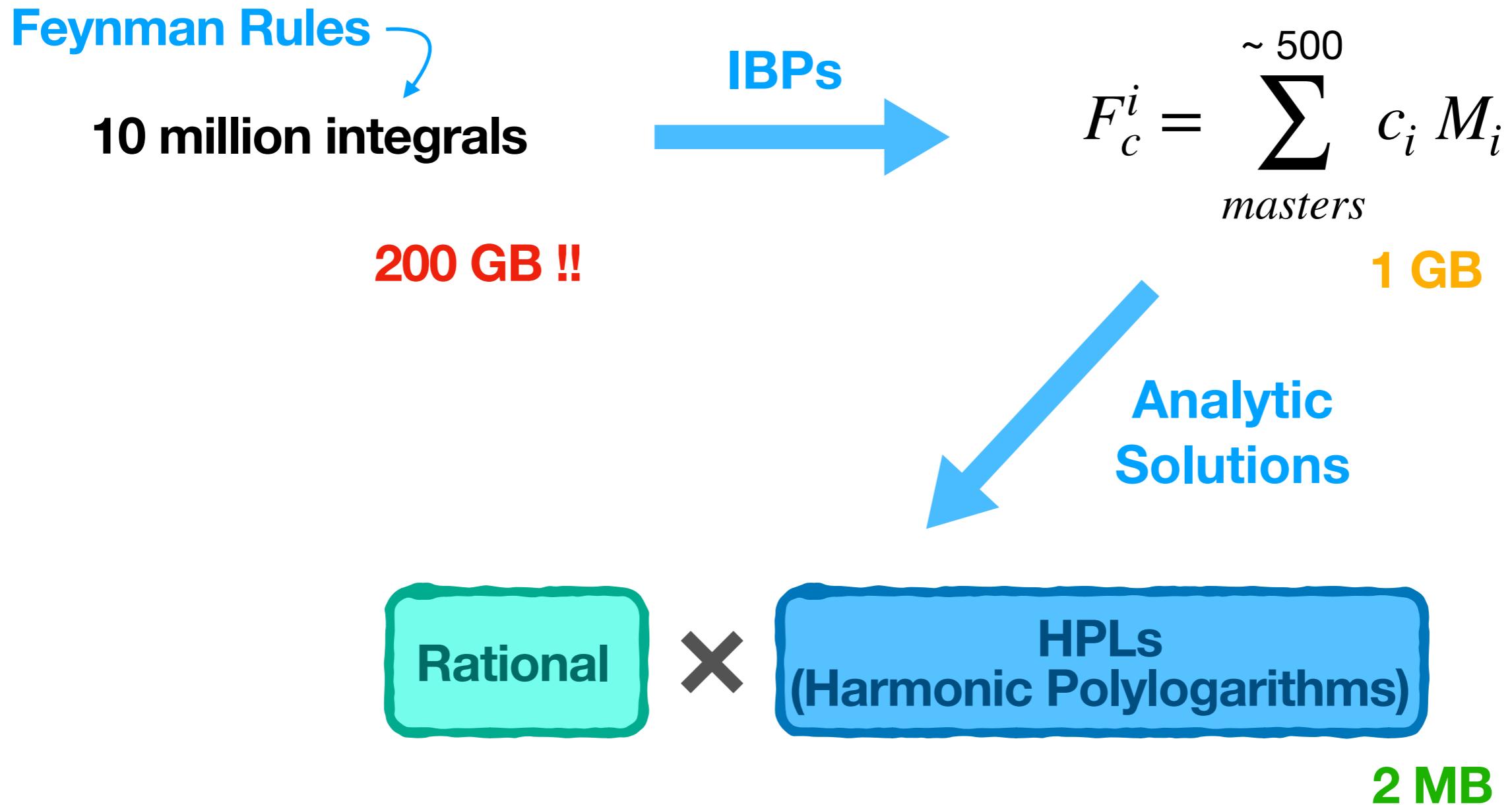
# The integrals

$$\int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{n_{N+1}} \dots D_M^{n_M}}$$



# The integrals

$$\int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{n_{N+1}} \dots D_M^{n_M}}$$



# The integrals

$$\int \frac{d^d\{k_j\}}{(2\pi)^{dl}} \frac{1}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N} D_{N+1}^{n_{N+1}} \dots D_M^{n_M}}$$

Feynman Rules

10 million integrals

200 GB !!

IBPs

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1 GB

Analytic  
Solutions

Rational

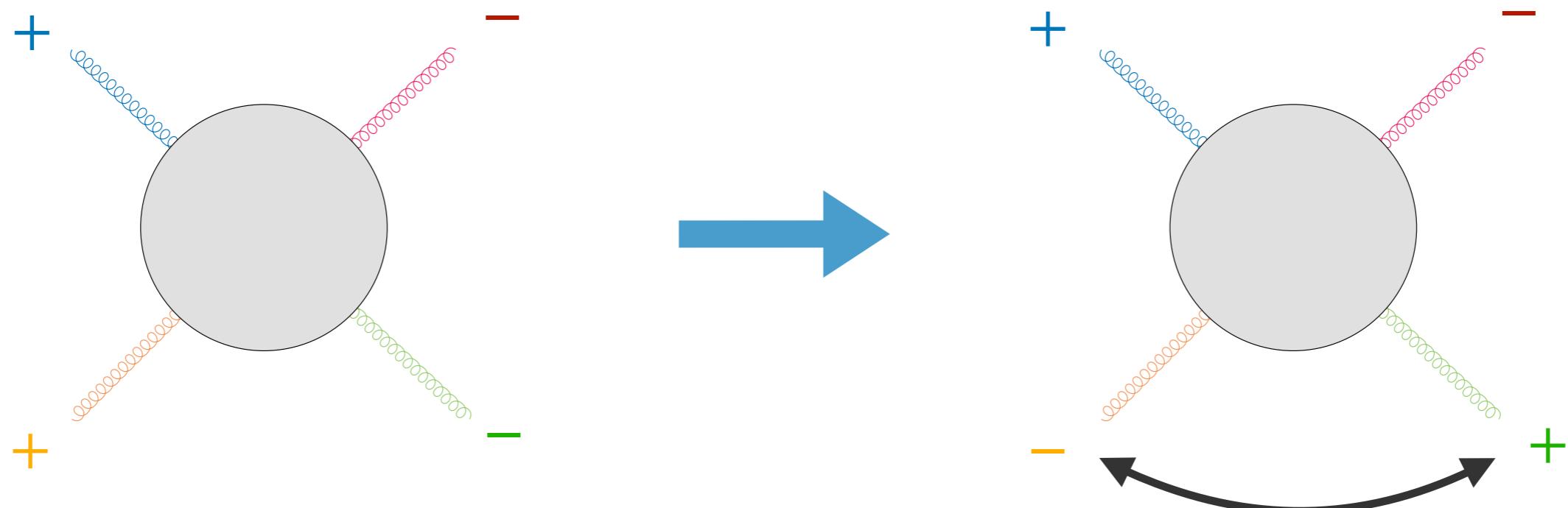
×

HPLs  
(Harmonic Polylogarithms)

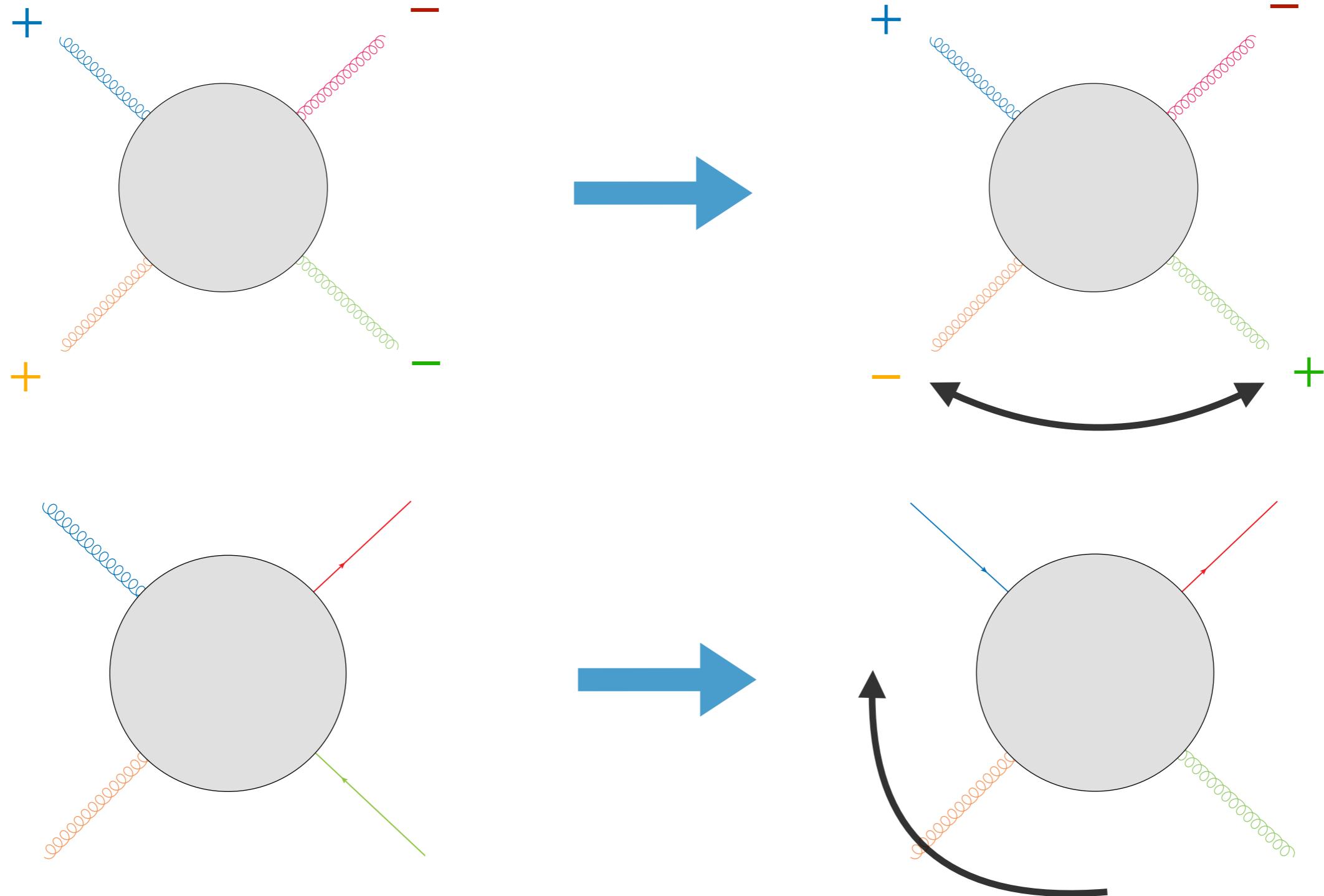
2 MB

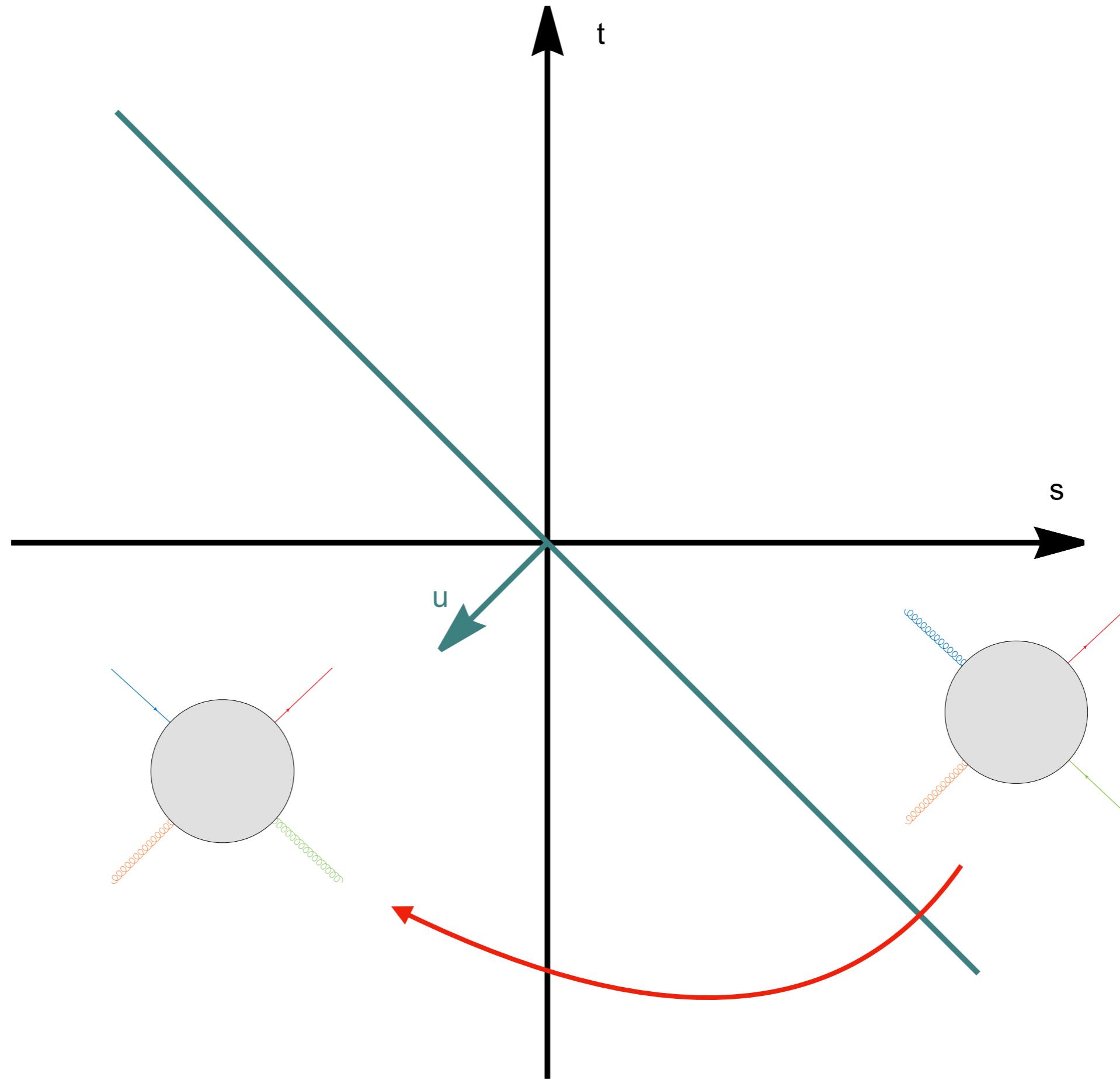
Analytically well understood !

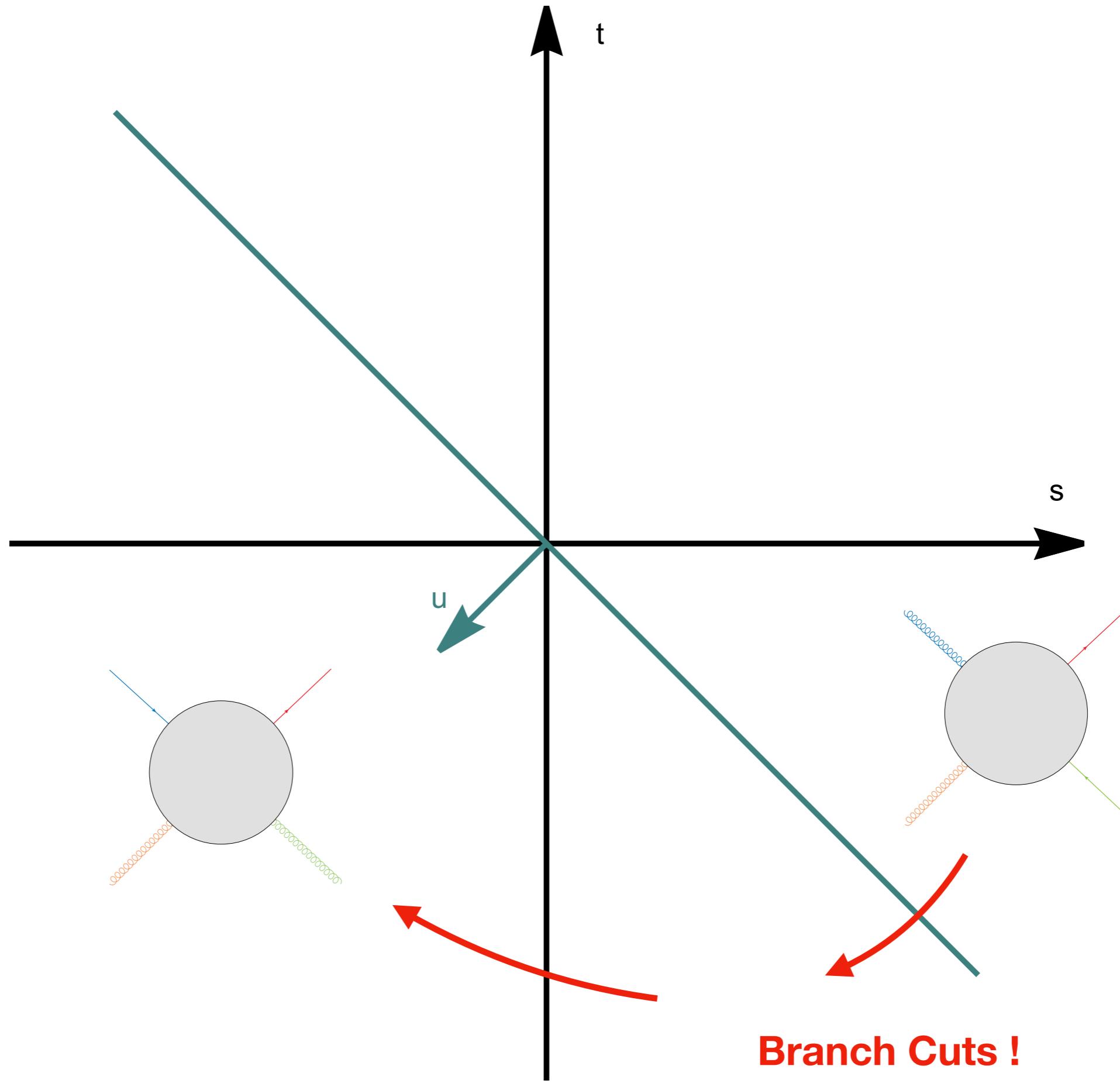
# Crossing the Amplitudes



# Crossing the Amplitudes







# Infrared Structure

# Infrared & Colour

$$\mathcal{H}_{\text{ren}}(\epsilon, \{p\}) = \mathcal{Z}(\epsilon, \{p\}, \mu) \mathcal{H}_{\text{fin}}(\mu, \{p\})$$

# Infrared & Colour

$$\underline{\mathcal{H}_{\text{ren}}(\epsilon, \{p\})} = \mathcal{Z}(\epsilon, \{p\}, \mu) \mathcal{H}_{\text{fin}}(\mu, \{p\})$$

IR-divergent

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IR-divergent

IR-finite

# Infrared & Colour

IR “renormalisation”



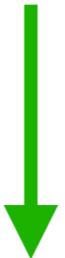
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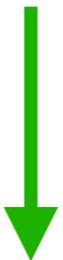
IR-divergent

IR-finite

$$\mathcal{Z}(\epsilon, \{p\}, \mu) = \mathbb{P} \exp \left[ \int_{\mu}^{\infty} \frac{d\mu'}{\mu'} \Gamma(\{p\}, \mu') \right]$$

# Infrared & Colour

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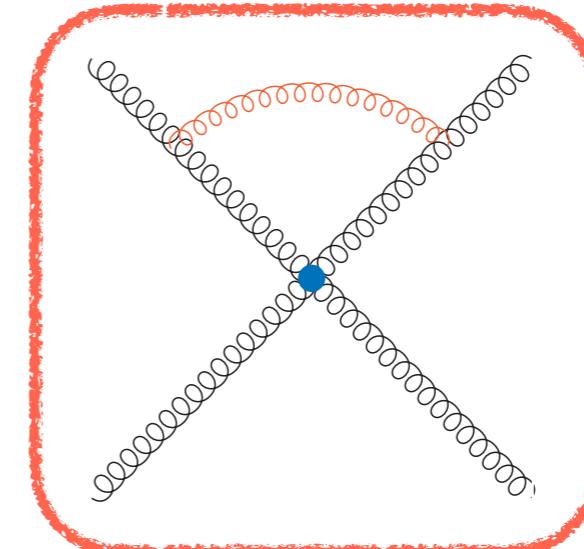
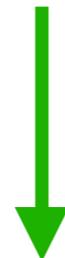
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# Infrared & Colour

IR “renormalisation”



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IR-divergent

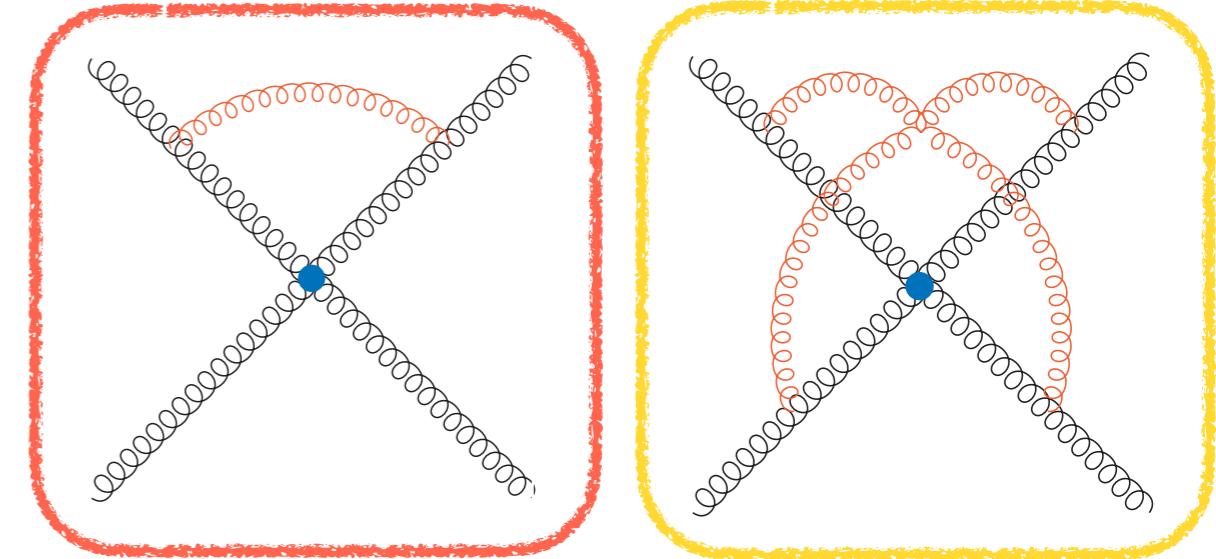
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# Infrared & Colour

IR “renormalisation”



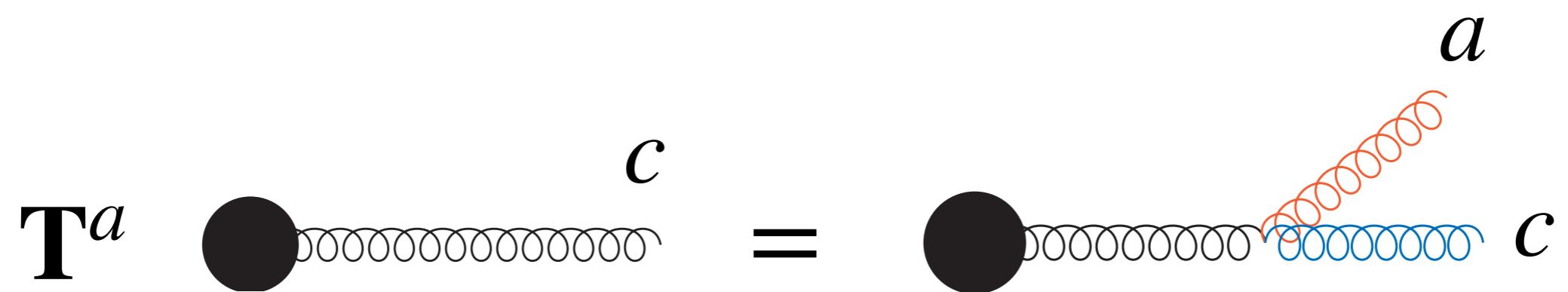
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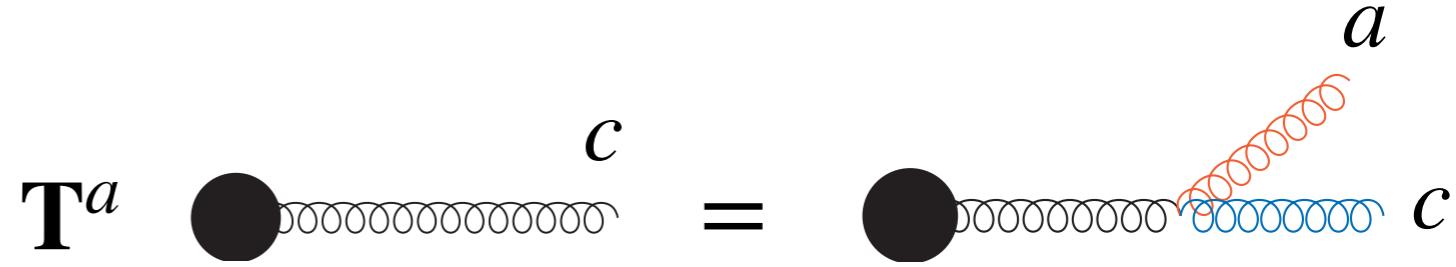
$$\mathcal{Z}(\epsilon, \{p\}, \mu) = \mathbb{P} \exp \left[ \int_{\mu}^{\infty} \frac{d\mu'}{\mu'} \Gamma(\{p\}, \mu') \right]$$

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$$\mathbf{T}^a X^c = - i f^a_{cc'} X^{c'}$$

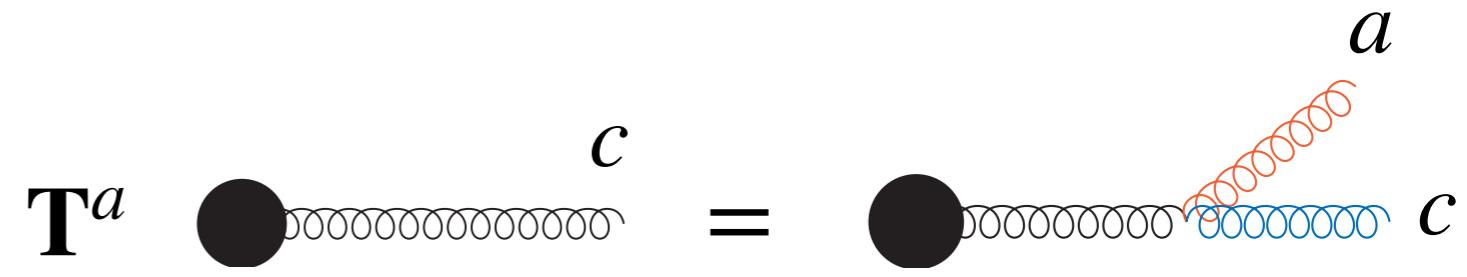
# Dipole



$$\mathbf{T}^a \ X^c = -if_{cc'}^a X^{c'}$$

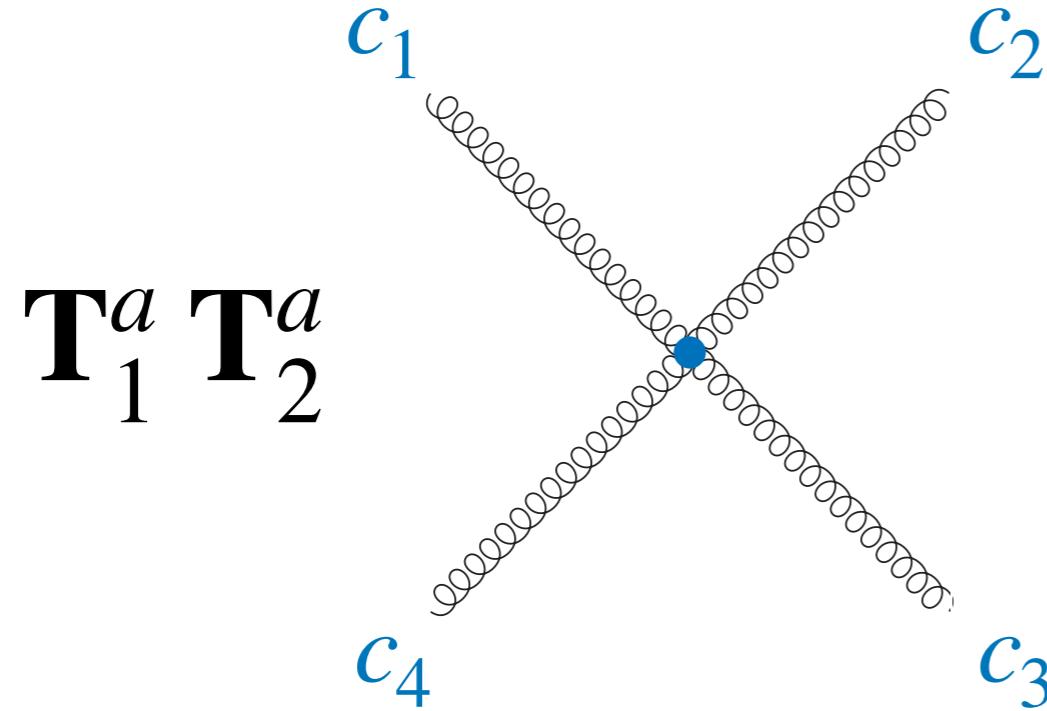
$$\Gamma_{\text{dipole}}(\{p\},\mu) = \sum_{1\leq i < j \leq 4} \mathbf{T}_i^a \ \mathbf{T}_j^a \ \gamma^{\text{cusp}}(\alpha_s) \ \log\left(\frac{\mu^2}{-s_{ij}-i\delta}\right) + \sum_{i=1}^4 \gamma^i(\alpha_s)$$

# Dipole

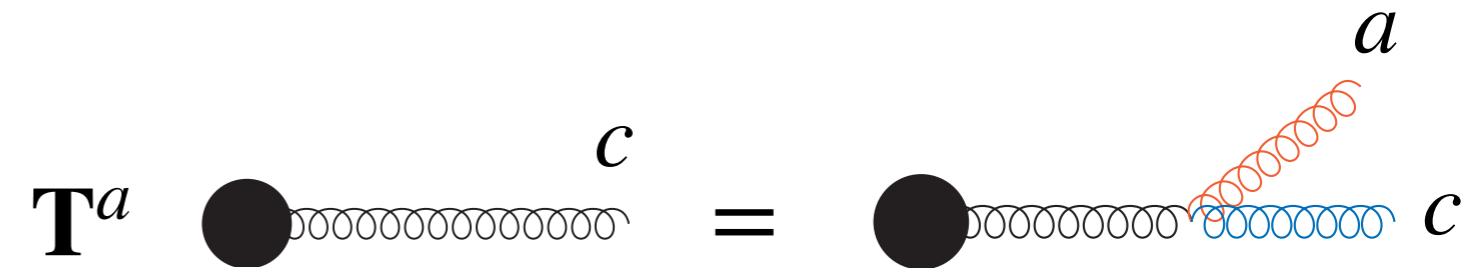


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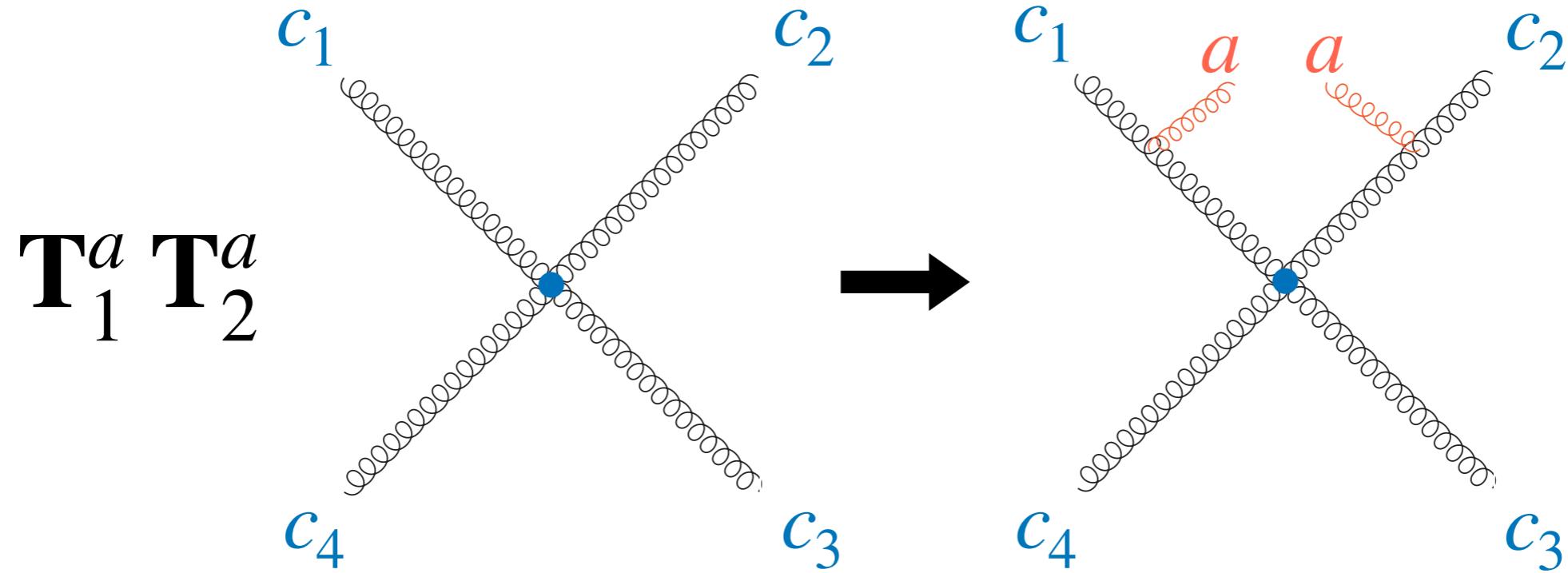


# Dipole

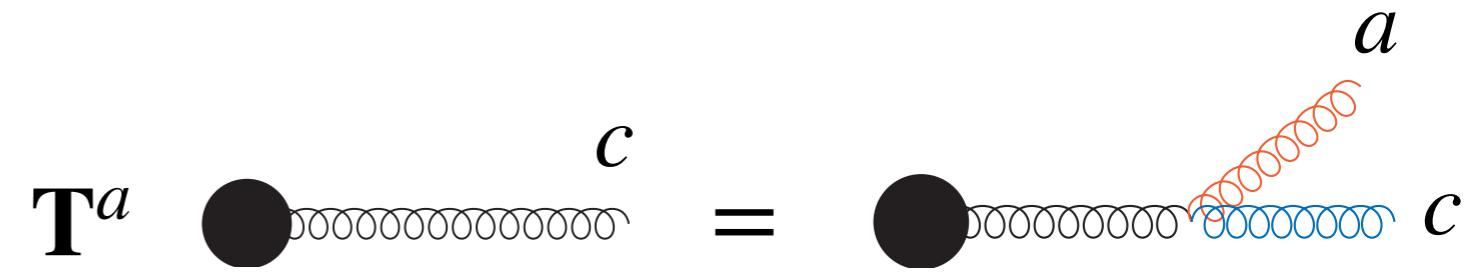


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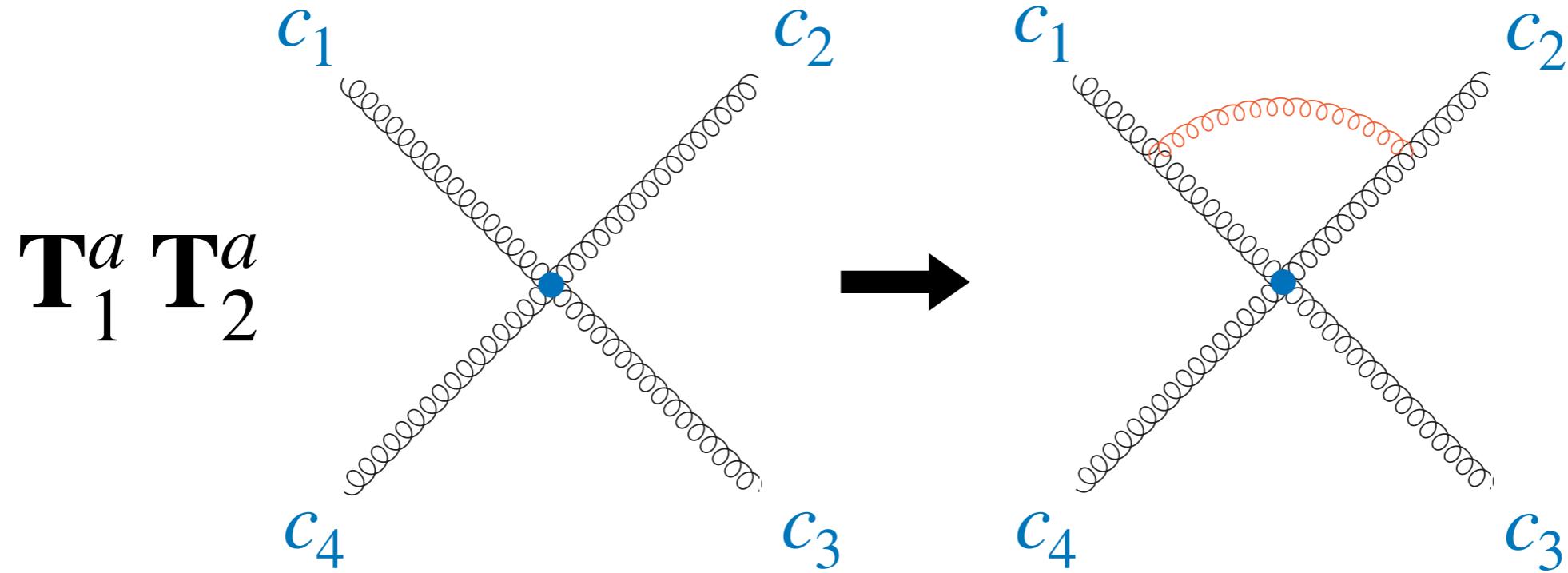


# Dipole



$$\mathbf{T}^a X^c = - i f_{cc'}^a X^{c'}$$

$$\Gamma_{\text{dipole}}(\{p\}, \mu) = \sum_{1 \leq i < j \leq 4} \mathbf{T}_i^a \mathbf{T}_j^a \gamma^{\text{cusp}}(\alpha_s) \log \left( \frac{\mu^2}{-s_{ij} - i\delta} \right) + \sum_{i=1}^4 \gamma^i(\alpha_s)$$



# Quadrupole

$$\begin{aligned}\Delta_4^{(3)} = & \ 128 \ f_{abe} \ f_{cde} \left[ \mathbf{T}_1^a \ \mathbf{T}_2^c \ \mathbf{T}_3^b \ \mathbf{T}_4^d \ D_1(x) - \mathbf{T}_4^a \ \mathbf{T}_1^b \ \mathbf{T}_2^c \ \mathbf{T}_3^d \ D_2(x) \right] \\ & - 16 \ C \ f_{abe} \ f_{cde} \sum_{i=1}^4 \sum_{\substack{1 \leq j < k \leq 4 \\ j, k \neq i}} \left\{ \mathbf{T}_i^a, \mathbf{T}_i^d \right\} \ \mathbf{T}_j^b \ \mathbf{T}_k^c ,\end{aligned}$$

Almelid, Duhr, Gardi: [1507.00047](#)

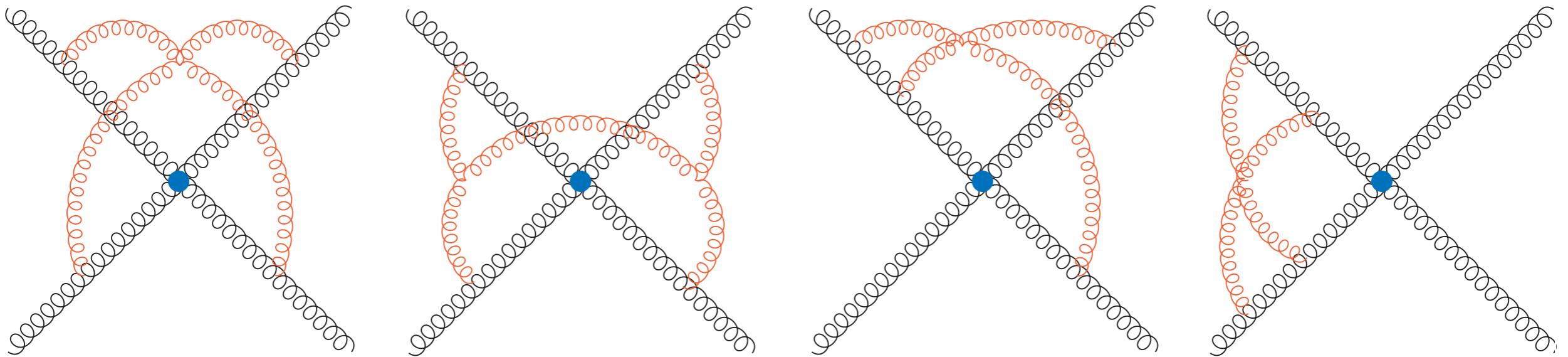
Henn, Mistlberger: [1608.00850](#)

# Quadrupole

$$\Delta_4^{(3)} = 128 f_{abe} f_{cde} \left[ \mathbf{T}_1^a \mathbf{T}_2^c \mathbf{T}_3^b \mathbf{T}_4^d D_1(x) - \mathbf{T}_4^a \mathbf{T}_1^b \mathbf{T}_2^c \mathbf{T}_3^d D_2(x) \right]$$
$$- 16 C f_{abe} f_{cde} \sum_{i=1}^4 \sum_{\substack{1 \leq j < k \leq 4 \\ j, k \neq i}} \left\{ \mathbf{T}_i^a, \mathbf{T}_i^d \right\} \mathbf{T}_j^b \mathbf{T}_k^c ,$$

Almelid, Duhr, Gardi: [1507.00047](#)

Henn, Mistlberger: [1608.00850](#)

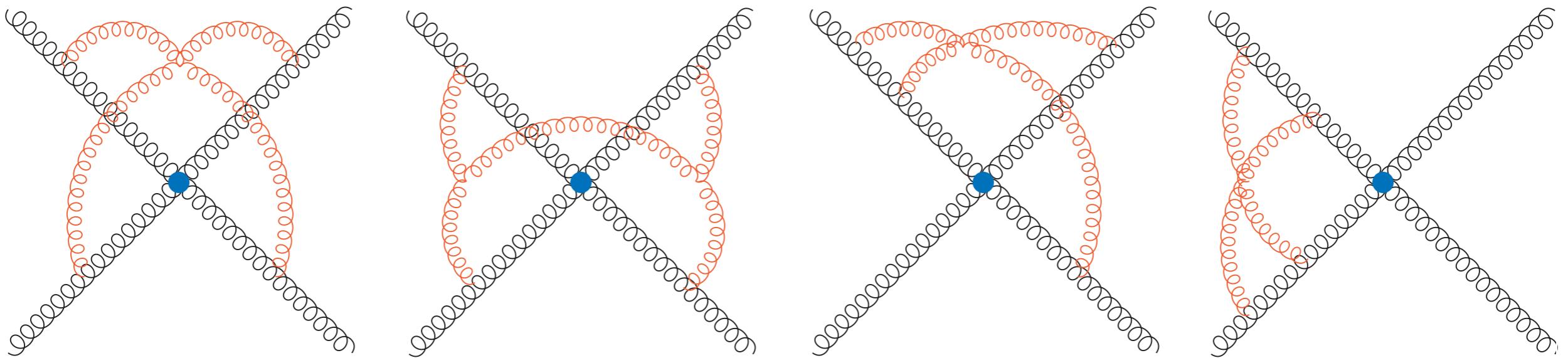


# Quadrupole

$$\Delta_4^{(3)} = 128 f_{abe} f_{cde} \left[ \mathbf{T}_1^a \mathbf{T}_2^c \mathbf{T}_3^b \mathbf{T}_4^d D_1(x) - \mathbf{T}_4^a \mathbf{T}_1^b \mathbf{T}_2^c \mathbf{T}_3^d D_2(x) \right]$$
$$- 16 C f_{abe} f_{cde} \sum_{i=1}^4 \sum_{\substack{1 \leq j < k \leq 4 \\ j, k \neq i}} \left\{ \mathbf{T}_i^a, \mathbf{T}_i^d \right\} \mathbf{T}_j^b \mathbf{T}_k^c ,$$

Almelid, Duhr, Gardi: [1507.00047](#)

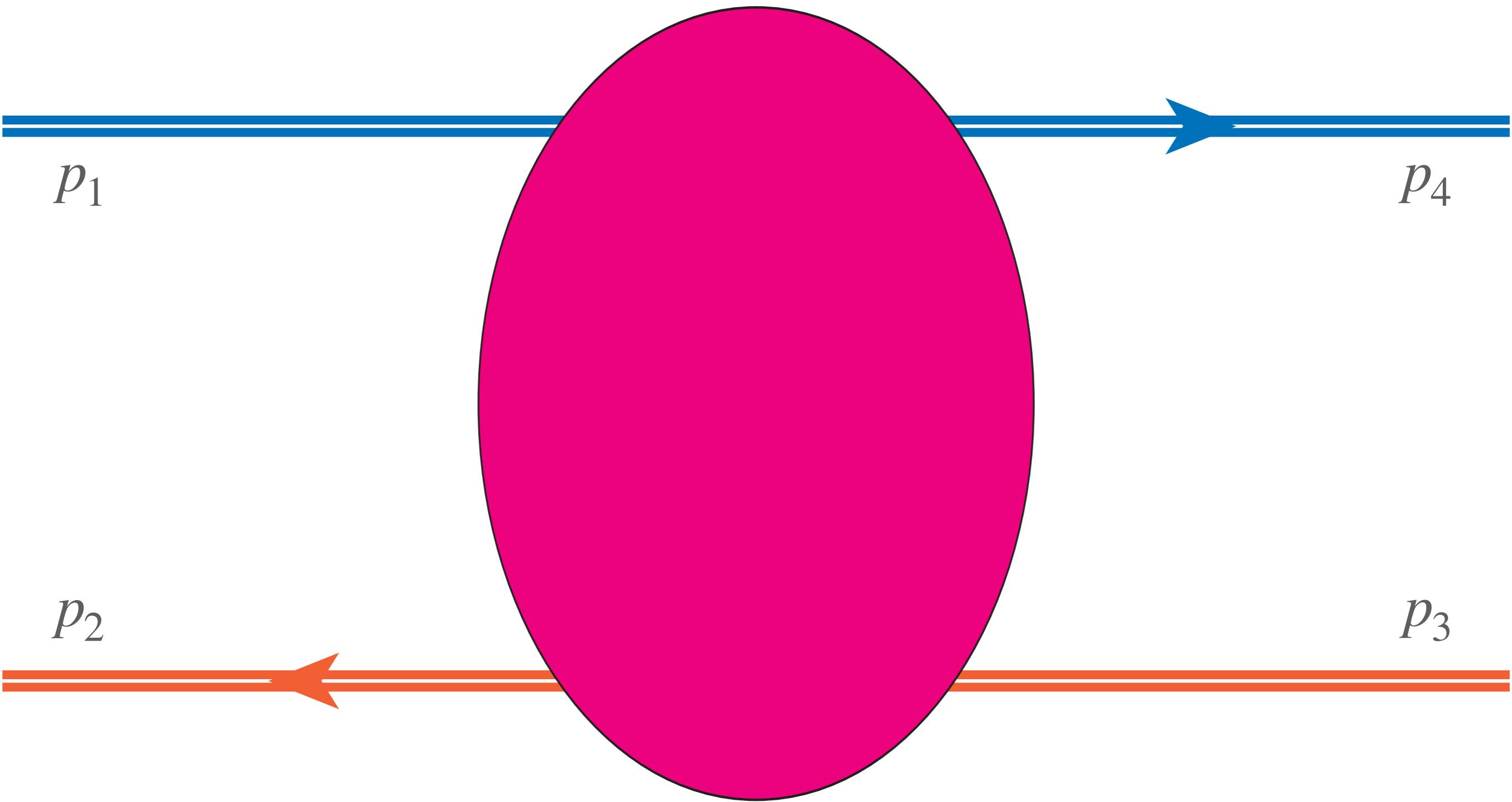
Henn, Mistlberger: [1608.00850](#)



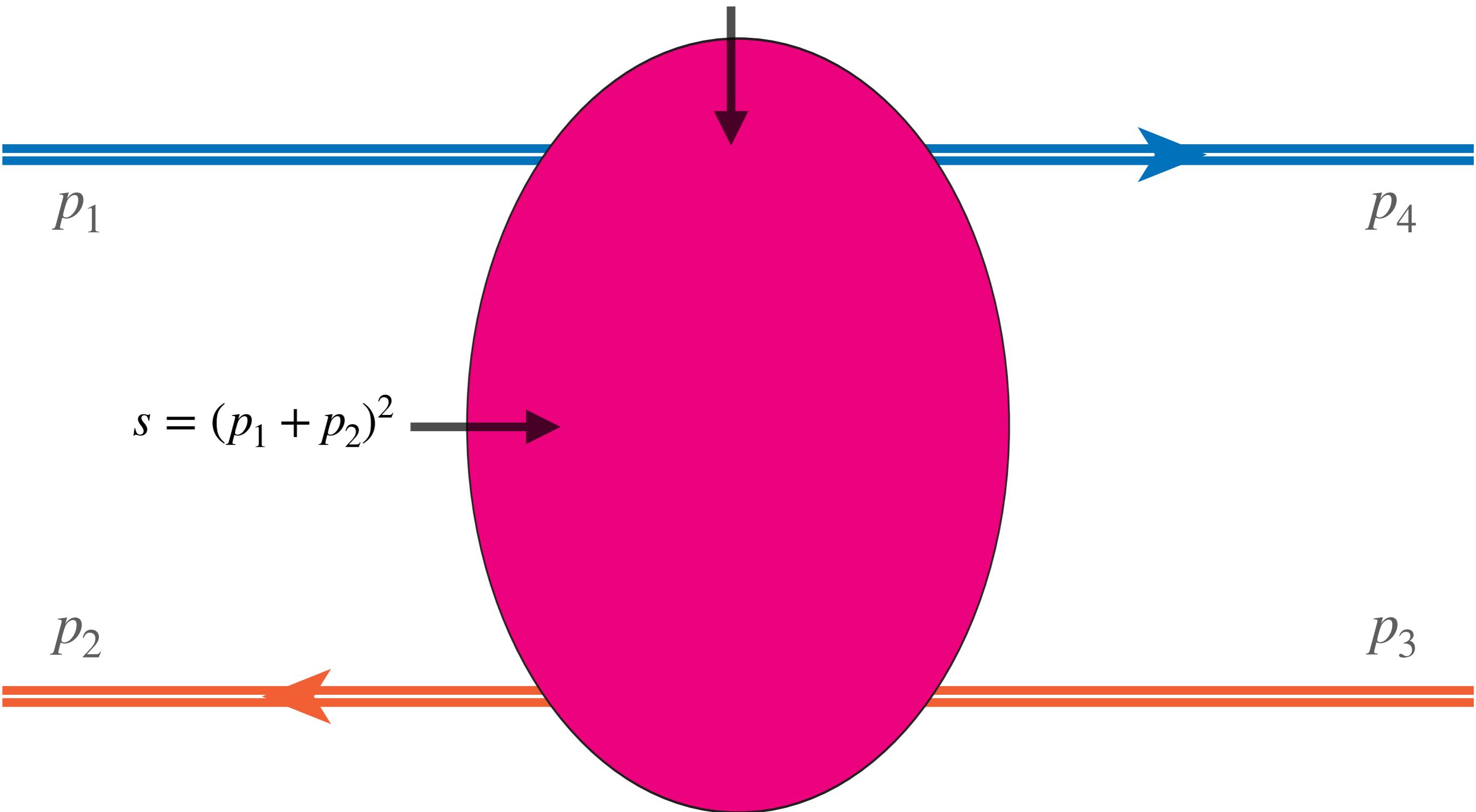
**Now confirmed in QCD:**

Chakraborty, Caola, Gambuti, Tancredi, von Manteuffel:  
[2207.03503](#), [2112.11097](#)

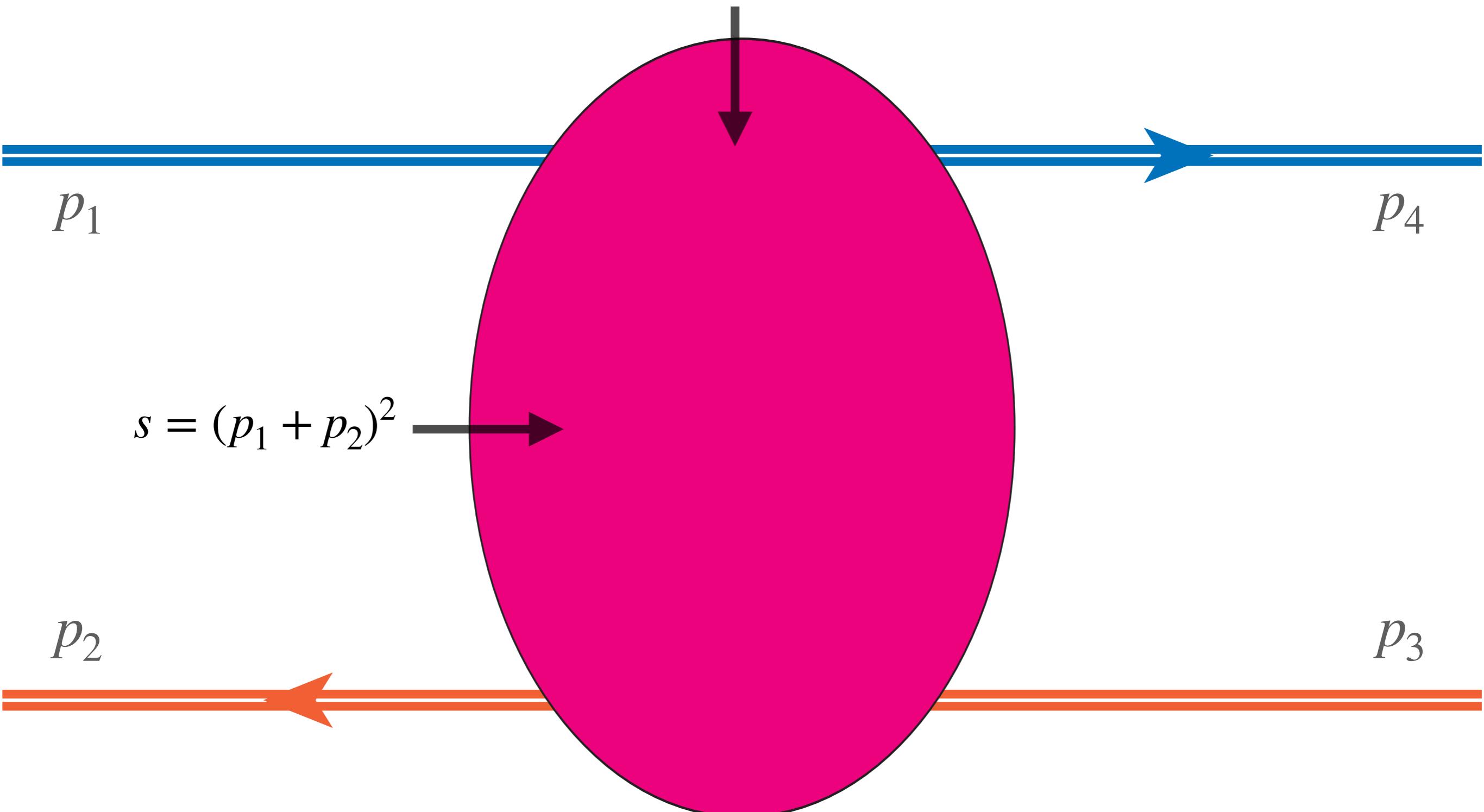
# High Energy Limit



$$u = (p_1 + p_4)^2$$

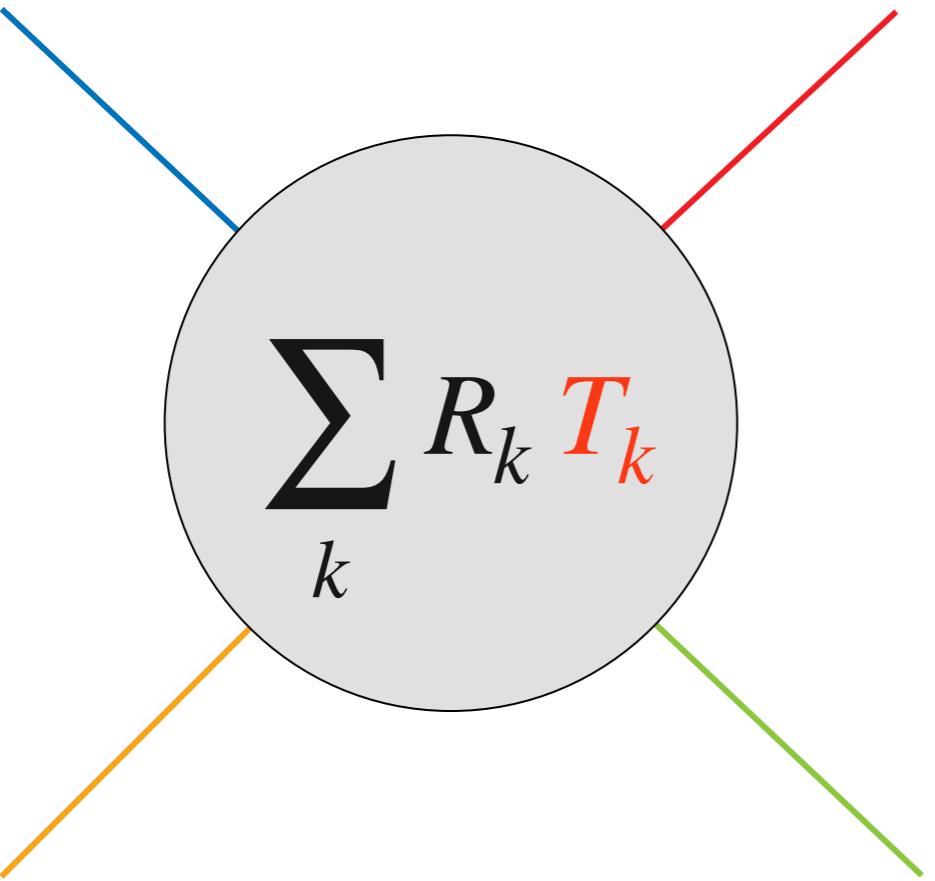


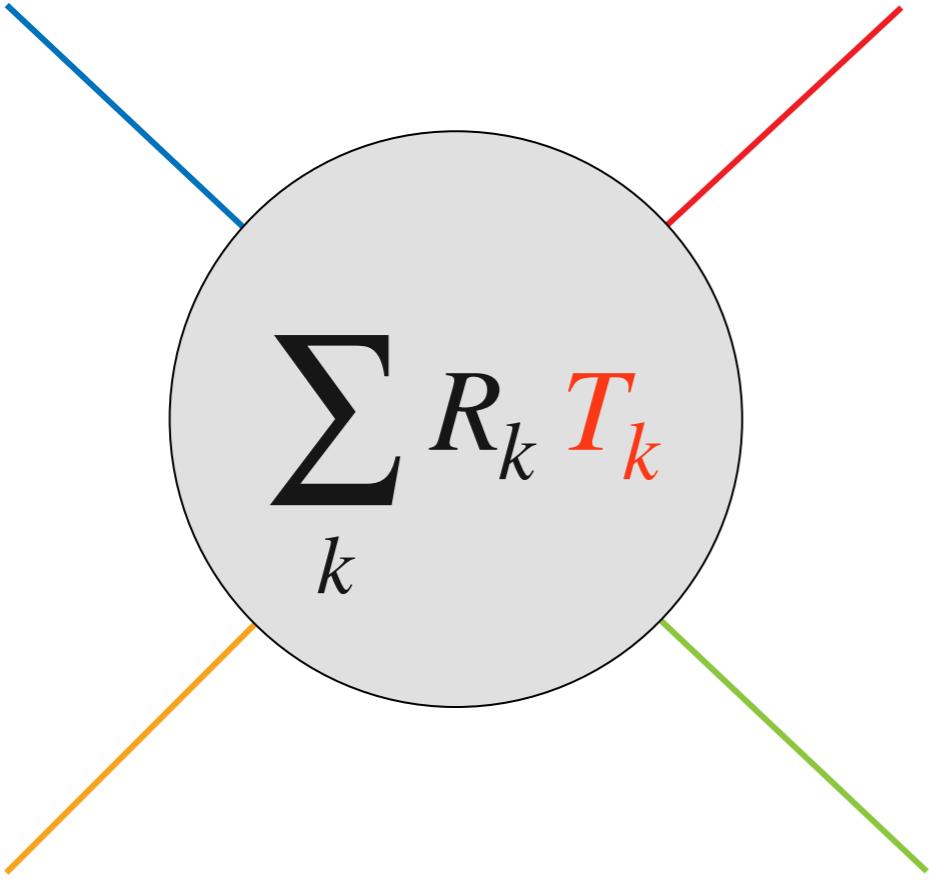
$$u = (p_1 + p_4)^2$$



**Regge Limit:**  $s \gg |u|$

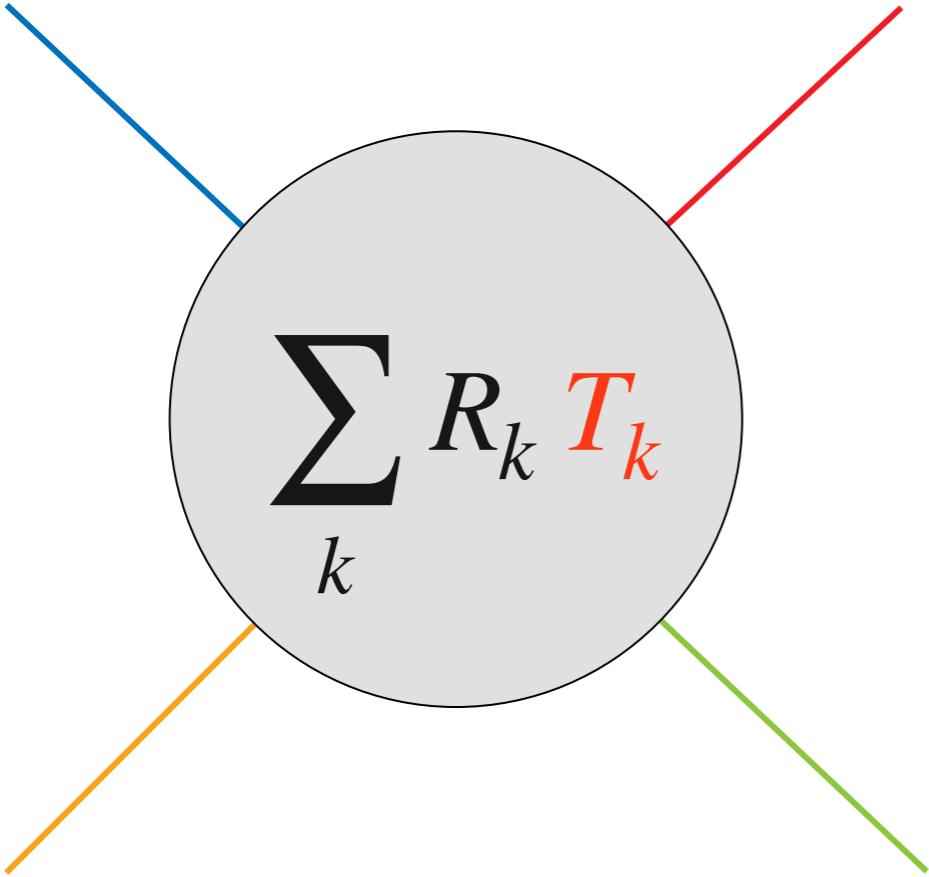
$$\sum_k R_k T_k$$





HPLs are singular!

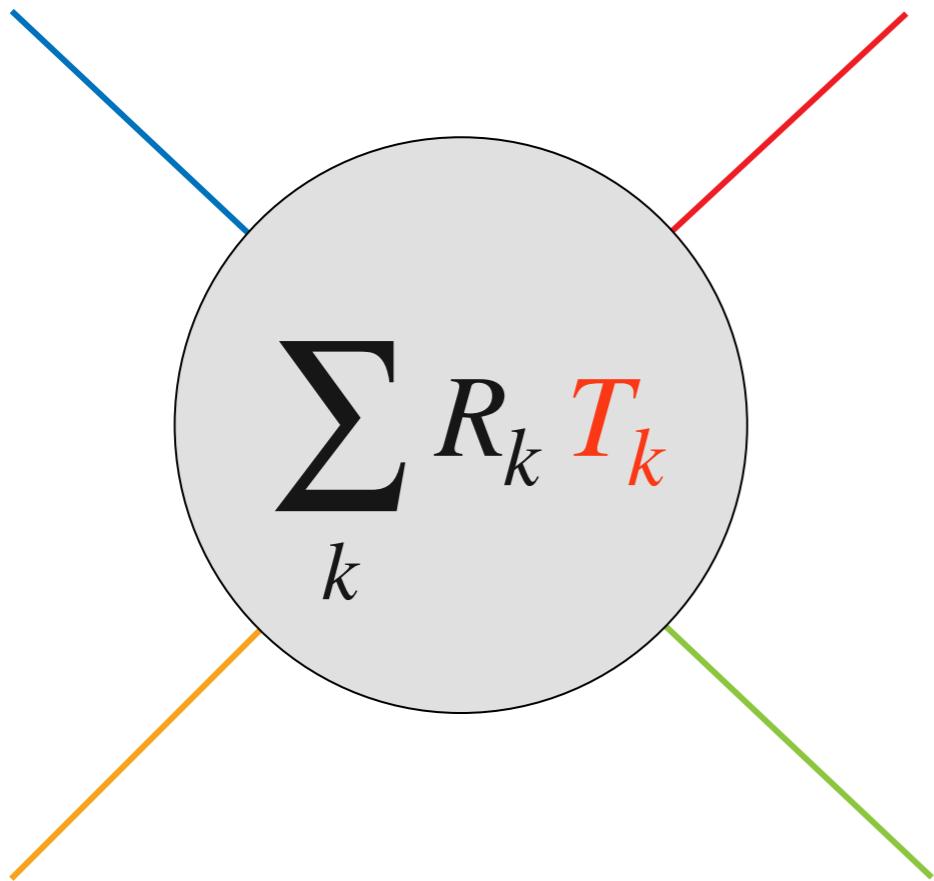
$$\sim \log^{\#} \left( \frac{-u}{s} \right)$$



HPLs are singular!

$$\sim \log^{\#} \left( \frac{-u}{s} \right)$$

$$s + t + u = 0 \implies |s| \approx |t|$$



HPLs are singular!

$$\sim \log^{\#} \left( \frac{-u}{s} \right)$$

$$s + t + u = 0 \implies |s| \approx |t|$$

$$A^\pm = \frac{1}{2} (A(s, t) \pm A(t, s))$$

$$L = \frac{1}{2} \left( \log \frac{-s}{-u} + \log \frac{-t}{-u} \right)$$

$$A^{\pm,(0)} = \#$$

$$A^{\pm,(1)} = \# L_- + \#$$

$$A^{\pm,(2)} = \# L_-^2 + \# L_- + \#$$

$$A^{\pm,(3)} = \# L_-^3 + \# L_-^2 + \# L_- + \#$$

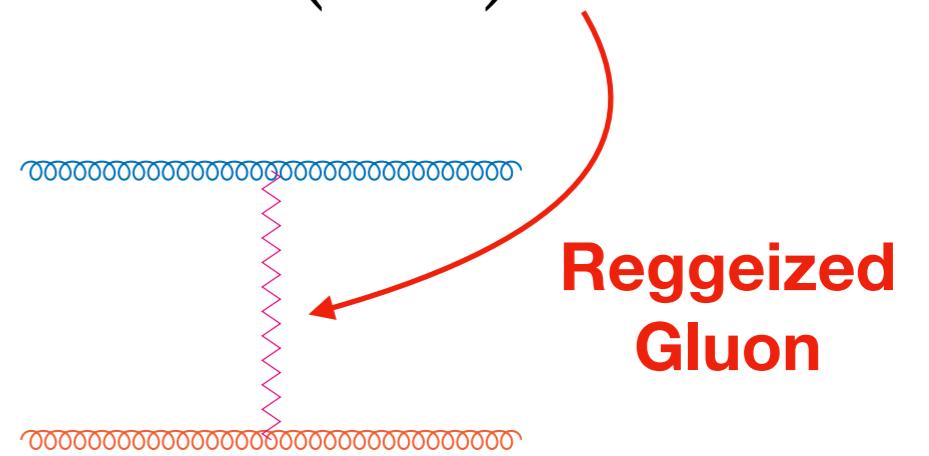
$$A^{\pm,(0)} = \boxed{\#}^{LL}$$

$$A^{\pm,(1)} = \boxed{\# L} + \#$$

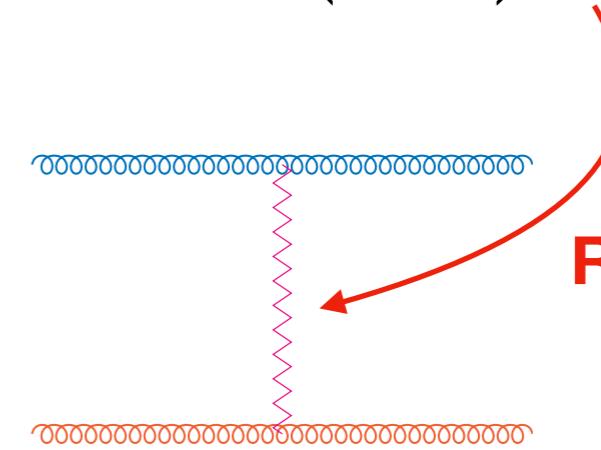
$$A^{\pm,(2)} = \boxed{\# L^2} + \# L + \#$$

$$A^{\pm,(3)} = \boxed{\# L^3} + \# L^2 + \# L + \#$$

$$A_{LL}^\pm = \left( \frac{s}{-u} \right)^{C_A \tau_g} A^{\pm,tree}$$



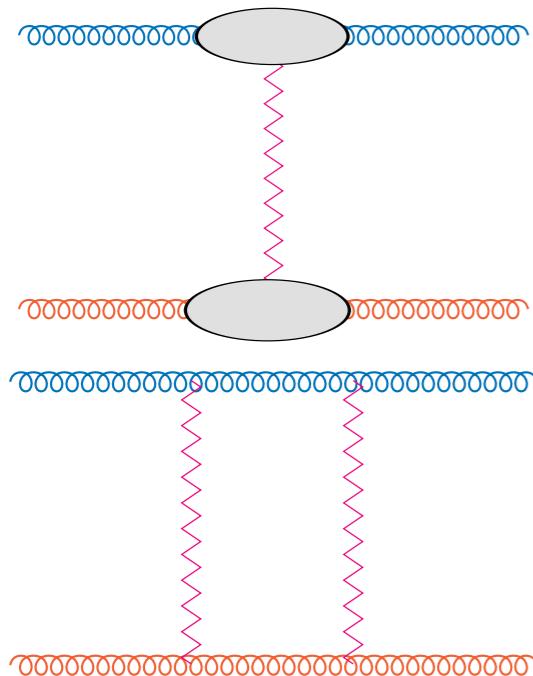
$$\begin{aligned}
 A^{\pm,(0)} &= \# \boxed{\text{LL}} \\
 A^{\pm,(1)} &= \# L + \# \boxed{\text{NLL}} \\
 A^{\pm,(2)} &= \# L^2 + \# L + \# \\
 A^{\pm,(3)} &= \# L^3 + \# L^2 + \# L + \#
 \end{aligned}$$

$$A_{LL}^\pm = \left( \frac{s}{-u} \right)^{C_A \tau_g} A^{\pm,tree}$$


**Reggeized Gluon**

Up to 2 loops in

$\tau_g(\alpha_s)$   
Regge Trajectory



$C_g, C_q$   
Impact Factors

Regge  
Cut

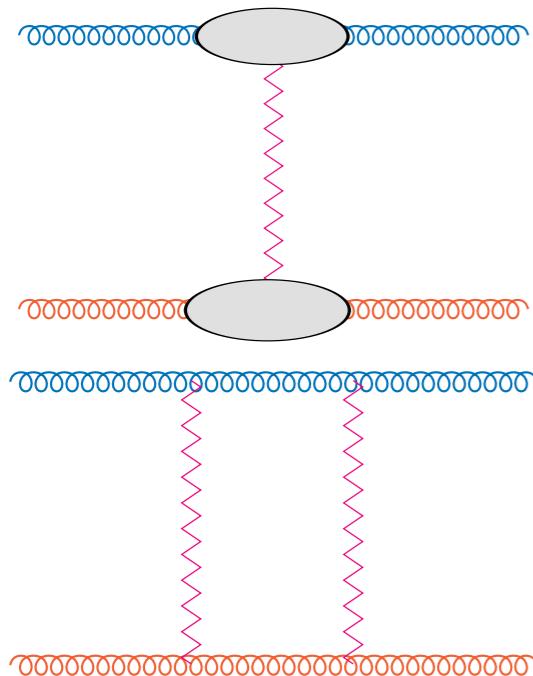
$$\begin{aligned}
 A^{\pm,(0)} &= \# \boxed{\text{LL}} \\
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 \end{aligned}$$

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**Reggeized Gluon**

Up to 2 loops in

$\tau_g(\alpha_s)$   
Regge Trajectory

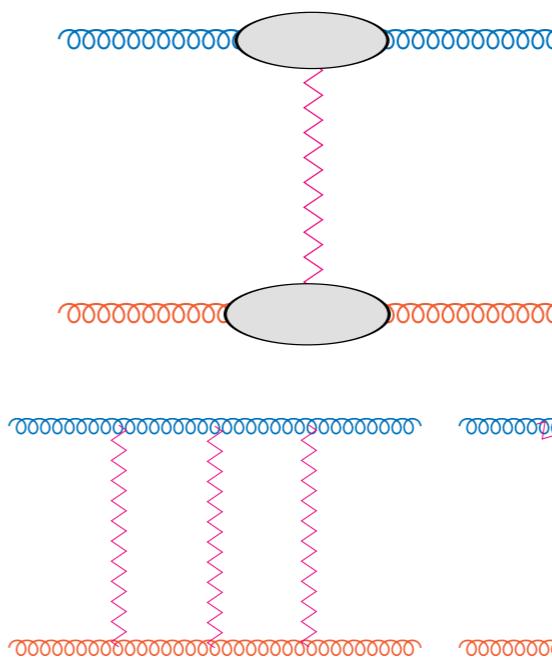


$C_g, C_q$   
Impact Factors

Regge Cut

Up to 3 loops in

$\tau_g(\alpha_s)$   
Regge Trajectory



$C_g, C_q$   
Impact Factors

Regge Cuts

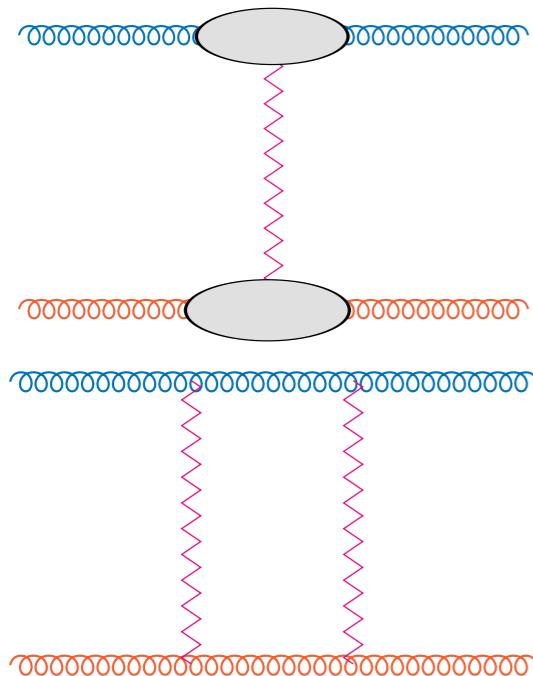
$$\begin{aligned}
 A^{\pm,(0)} &= \# \boxed{\text{LL}} \\
 A^{\pm,(1)} &= \# L + \# \boxed{\text{NLL}} \\
 A^{\pm,(2)} &= \# L^2 + \# L + \# \boxed{\text{NNLL}} \\
 A^{\pm,(3)} &= \# L^3 + \# L^2 + \# L + \#
 \end{aligned}$$

$$A_{LL}^\pm = \left( \frac{s}{-u} \right)^{C_A \tau_g} A^{\pm,tree}$$

**Reggeized Gluon**

Up to 2 loops in

$\tau_g(\alpha_s)$   
Regge Trajectory

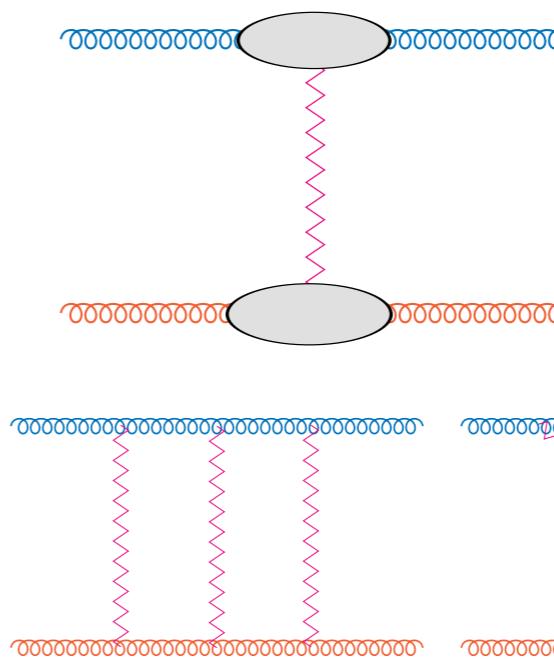


$C_g, C_q$   
Impact Factors

Regge Cut

Up to 3 loops in

$\tau_g(\alpha_s)$   
Regge Trajectory

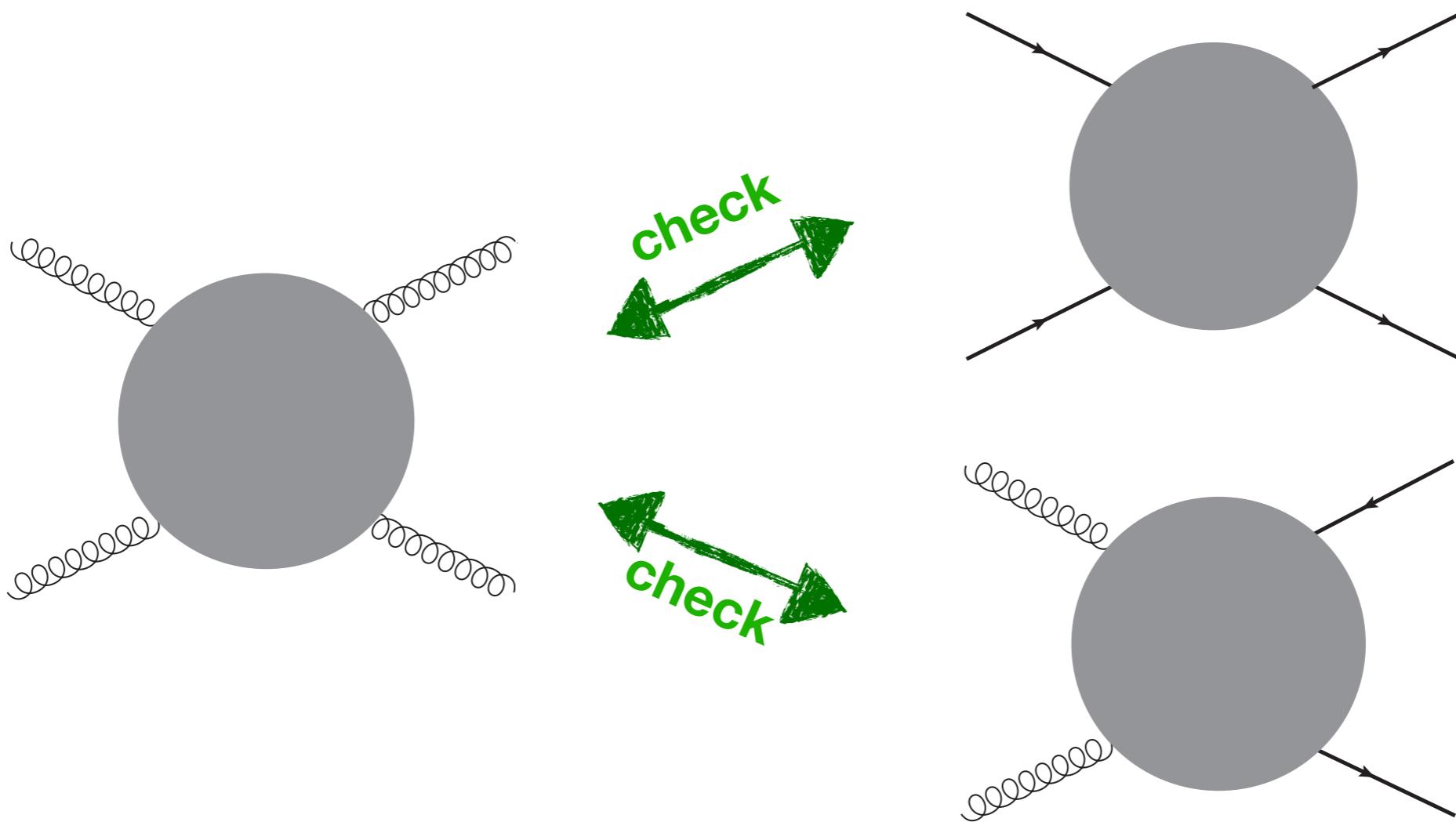


$C_g, C_q$   
Impact Factors

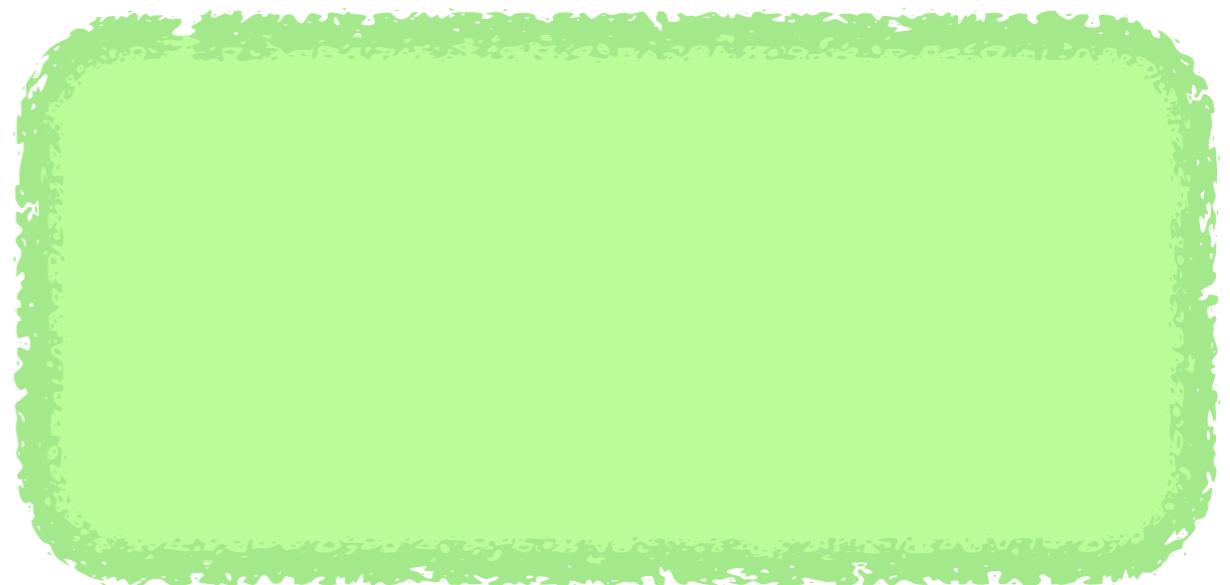
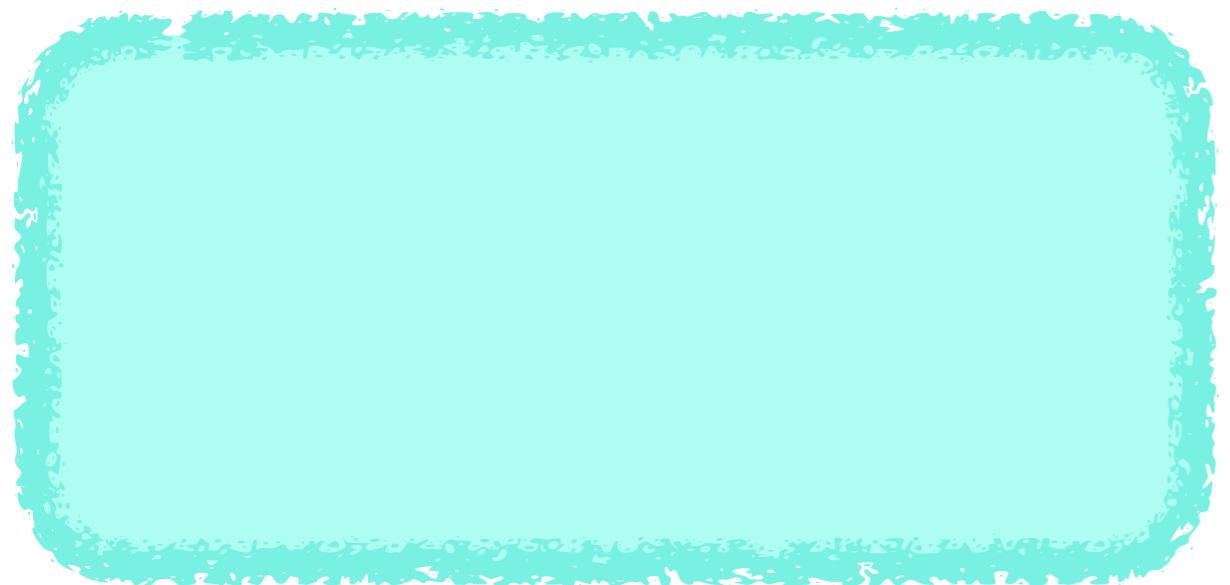
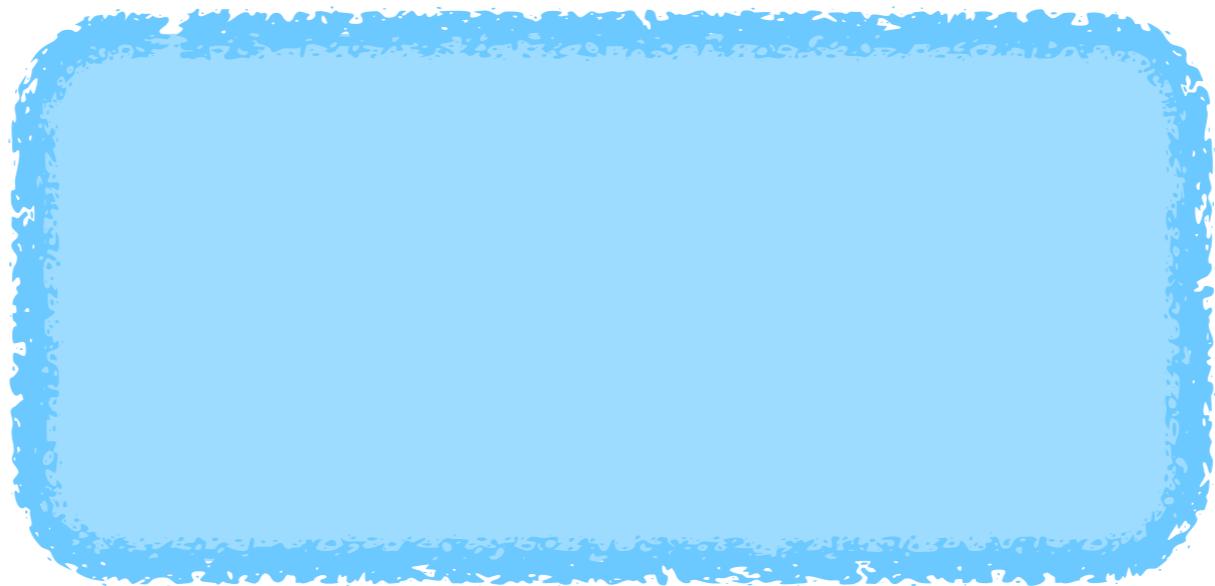
Regge Cuts

# Regge Limit as a check

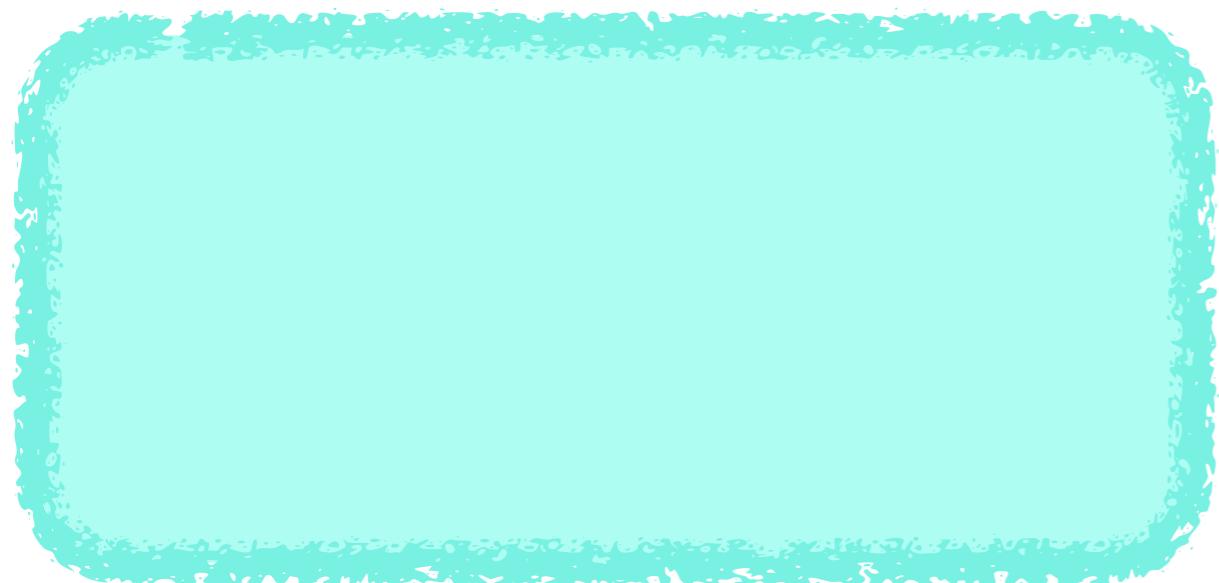
- 2-loop data
- A single 3-loop amplitude



# Conclusion



**contribution to  $N^3LO$**



**contribution to  $N^3LO$**

**confirmation of QCD  
quadrupole radiation**

**contribution to  $N^3LO$**

**confirmation of QCD  
quadrupole radiation**

**3-loop gluon Regge  
Trajectory**



# Thanks !!



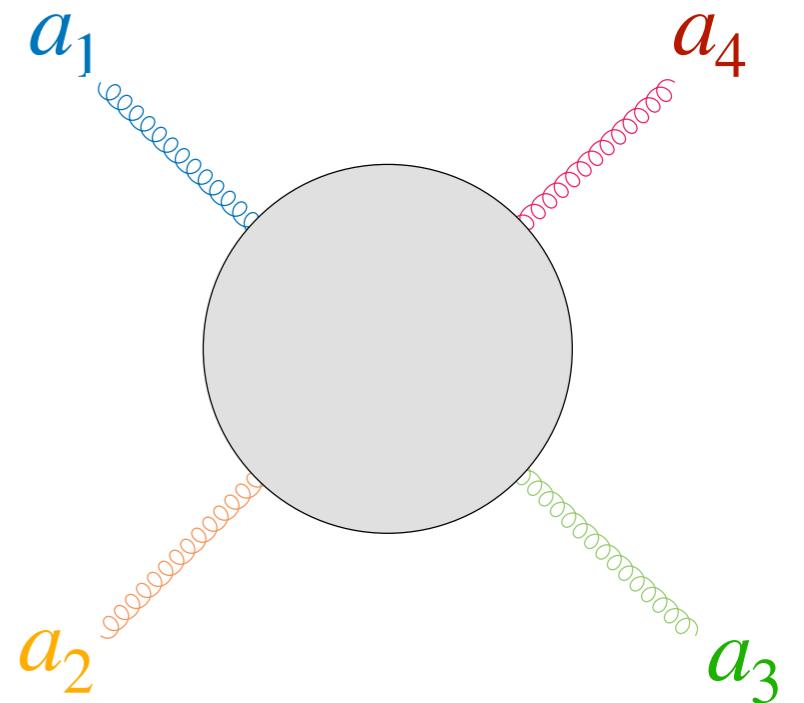


# Backup Slides

# Regge

# Colour Decomposition

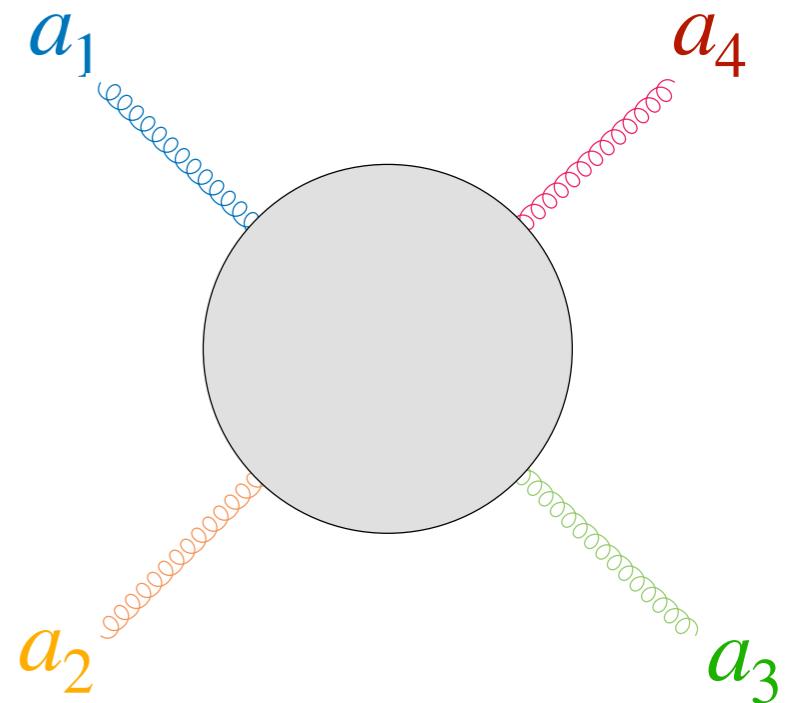
$$A = \sum_{i=1}^6 A_i \mathcal{C}_i$$



# Colour Decomposition

$$A = \sum_{i=1}^6 A_i \mathcal{C}_i$$

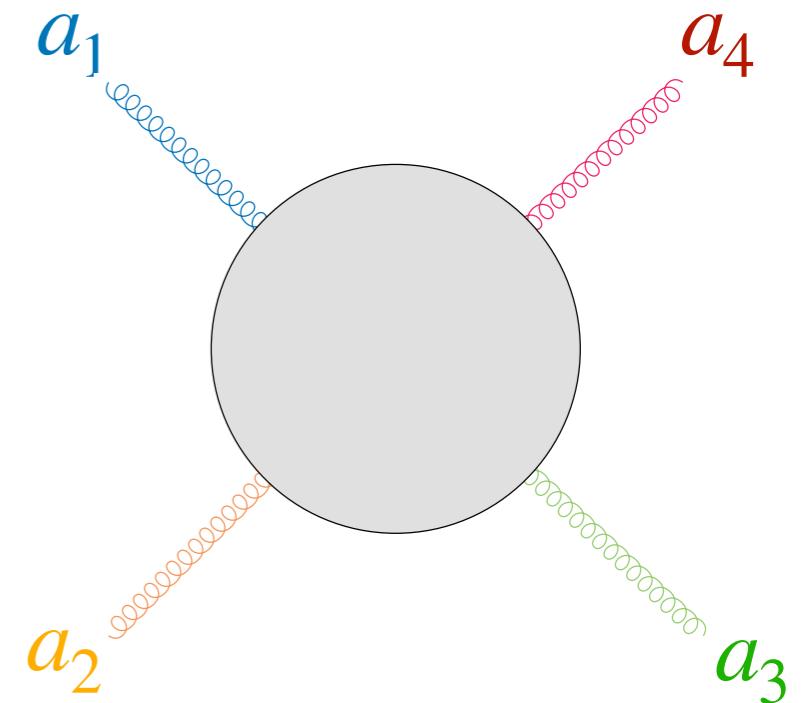
$$\mathcal{C}_1 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4})$$



# Colour Decomposition

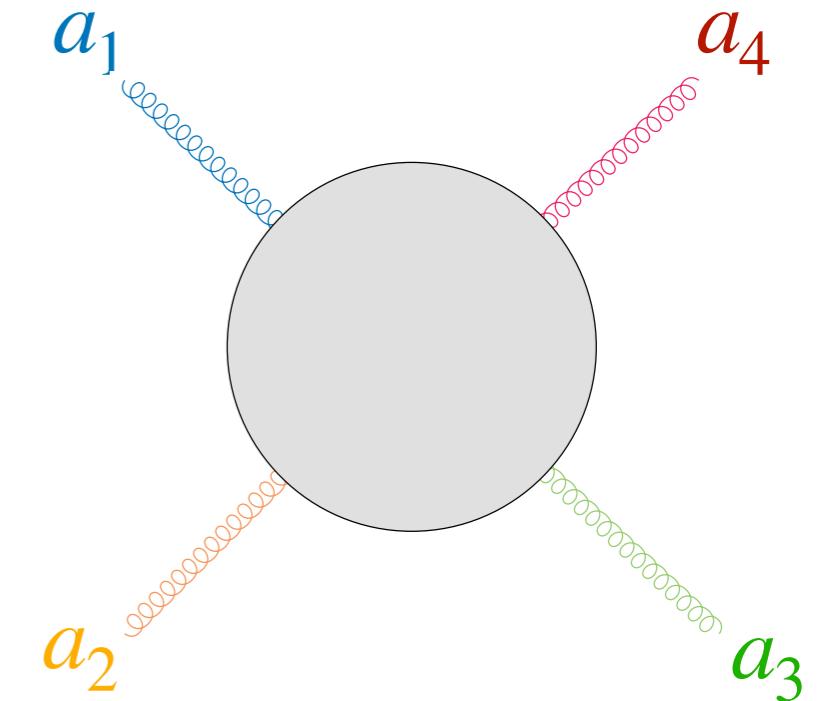
$$A = \sum_{i=1}^6 A_i \mathcal{C}_i$$

$$\mathcal{C}_1 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}) + Tr(\mathbf{T}^{a_4}\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$



# Colour Decomposition

$$A = \sum_{i=1}^6 A_i \mathcal{C}_i$$



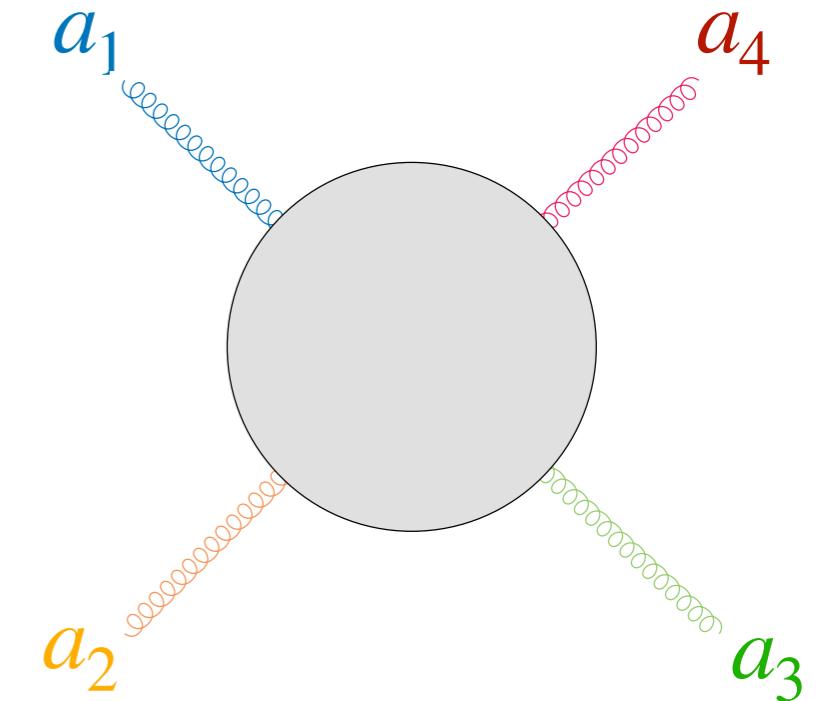
$$\mathcal{C}_1 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}) + Tr(\mathbf{T}^{a_4}\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$

$$\mathcal{C}_2 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_4}\mathbf{T}^{a_3}) + Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_4}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$

$$\mathcal{C}_3 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_4}\mathbf{T}^{a_2}\mathbf{T}^{a_3}) + Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_4}\mathbf{T}^{a_1})$$

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$$A = \sum_{i=1}^6 A_i \mathcal{C}_i$$



$$\mathcal{C}_1 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}) + Tr(\mathbf{T}^{a_4}\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$

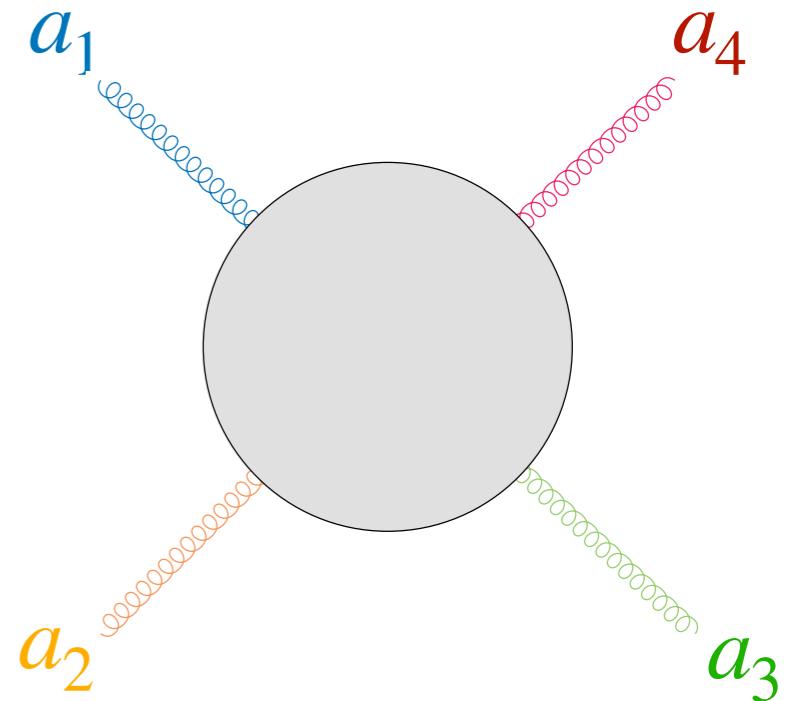
$$\mathcal{C}_2 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_4}\mathbf{T}^{a_3}) + Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_4}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$

$$\mathcal{C}_3 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_4}\mathbf{T}^{a_2}\mathbf{T}^{a_3}) + Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_4}\mathbf{T}^{a_1})$$

$$\mathcal{C}_4 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}) Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_4})$$

# Colour Decomposition

$$A = \sum_{i=1}^6 A_i \mathcal{C}_i$$



$$\mathcal{C}_1 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}) + Tr(\mathbf{T}^{a_4}\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$

$$\mathcal{C}_2 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_4}\mathbf{T}^{a_3}) + Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_4}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$

$$\mathcal{C}_3 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_4}\mathbf{T}^{a_2}\mathbf{T}^{a_3}) + Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_4}\mathbf{T}^{a_1})$$

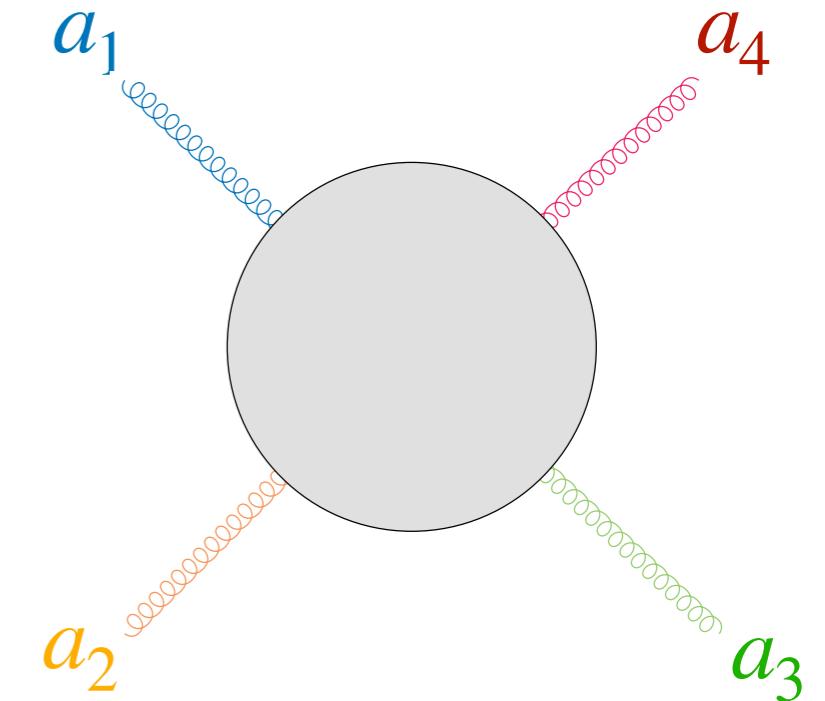
$$\mathcal{C}_4 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}) Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_4})$$

$$\mathcal{C}_5 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_3}) Tr(\mathbf{T}^{a_2}\mathbf{T}^{a_4})$$

$$\mathcal{C}_6 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_4}) Tr(\mathbf{T}^{a_2}\mathbf{T}^{a_3})$$

# Colour Decomposition

$$A = \sum_{i=1}^6 A_i \mathcal{C}_i$$



$$\mathcal{C}_1 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}) + Tr(\mathbf{T}^{a_4}\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$

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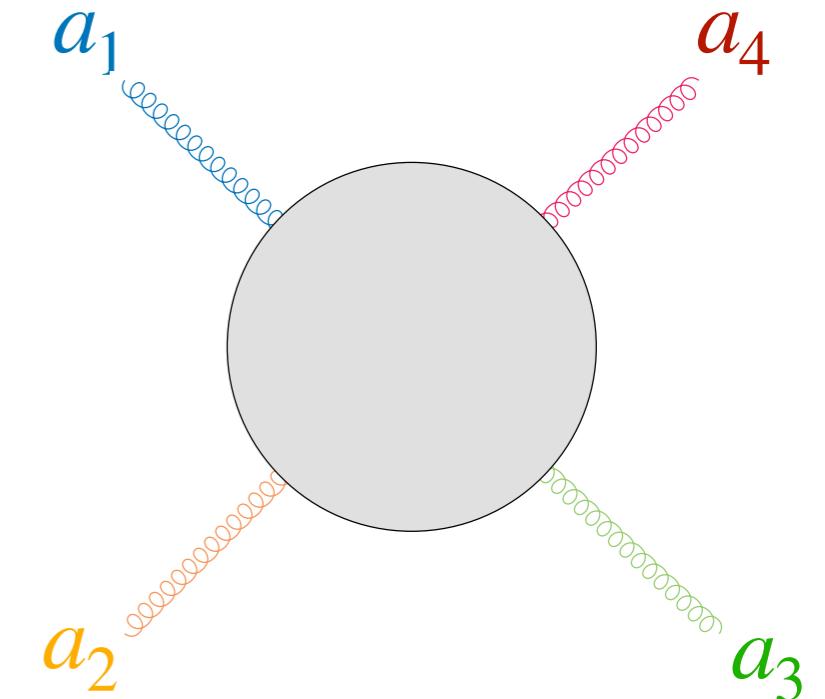
$$\mathcal{C}_4 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}) Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_4})$$

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# Colour Decomposition

$$A = \sum_{i=1}^6 A_i \mathcal{C}_i$$



$$\mathcal{C}_1 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}) + Tr(\mathbf{T}^{a_4}\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$

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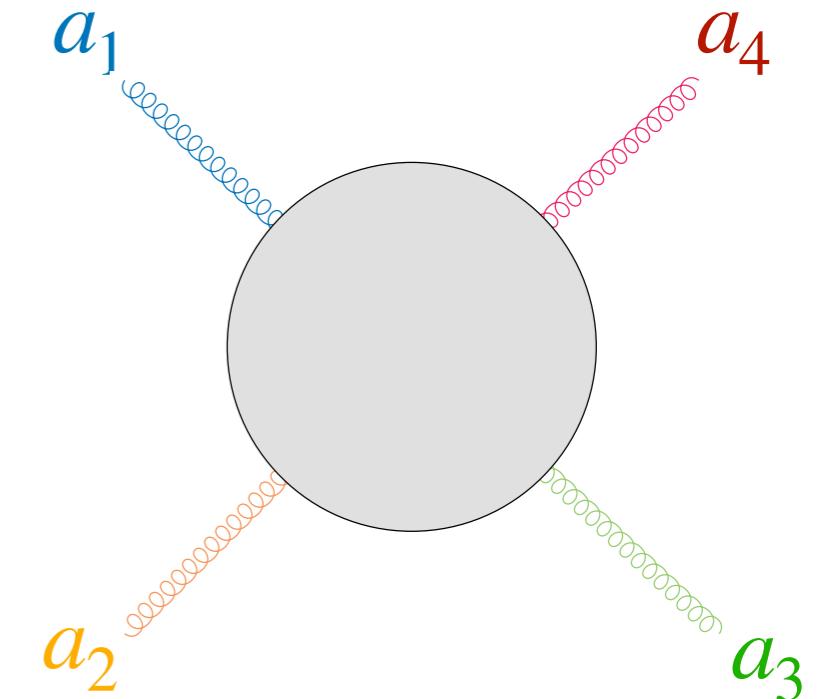
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# Colour Decomposition

$$A = \sum_{i=1}^6 A_i \mathcal{C}_i$$



$$\mathcal{C}_1 = Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}) + Tr(\mathbf{T}^{a_4}\mathbf{T}^{a_3}\mathbf{T}^{a_2}\mathbf{T}^{a_1})$$

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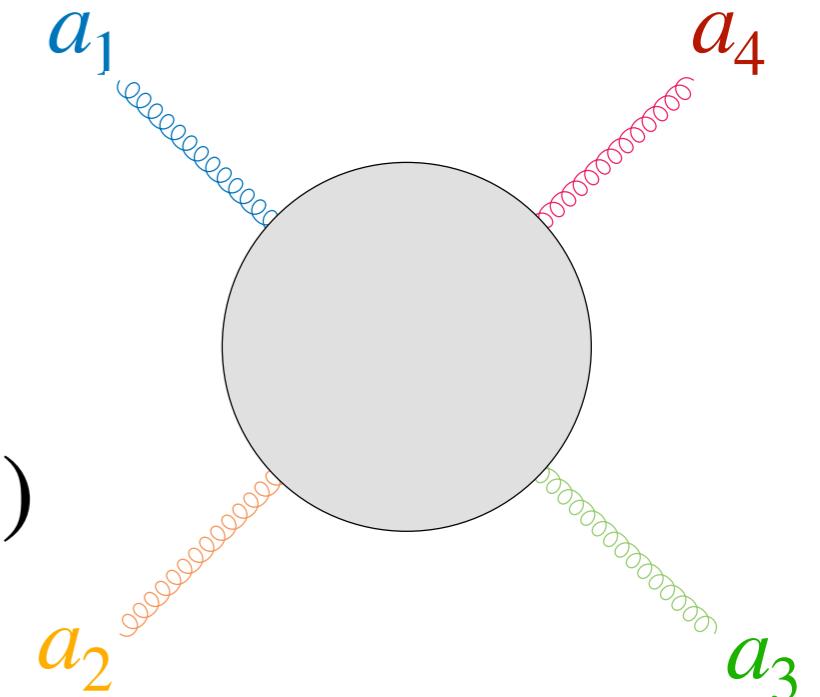
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# Colour Decomposition

$$A = \sum_{i=1}^6 A_i \mathcal{C}_i = A_1 \mathcal{C}_1 + A_4 \mathcal{C}_4 + (\text{perms})$$



$$\mathcal{C}_1 = Tr(\mathbf{T}^{a_1} \mathbf{T}^{a_2} \mathbf{T}^{a_3} \mathbf{T}^{a_4}) + Tr(\mathbf{T}^{a_4} \mathbf{T}^{a_3} \mathbf{T}^{a_2} \mathbf{T}^{a_1})$$

$$\mathcal{C}_2 = Tr(\mathbf{T}^{a_1} \mathbf{T}^{a_2} \mathbf{T}^{a_4} \mathbf{T}^{a_3}) + Tr(\mathbf{T}^{a_3} \mathbf{T}^{a_4} \mathbf{T}^{a_2} \mathbf{T}^{a_1})$$

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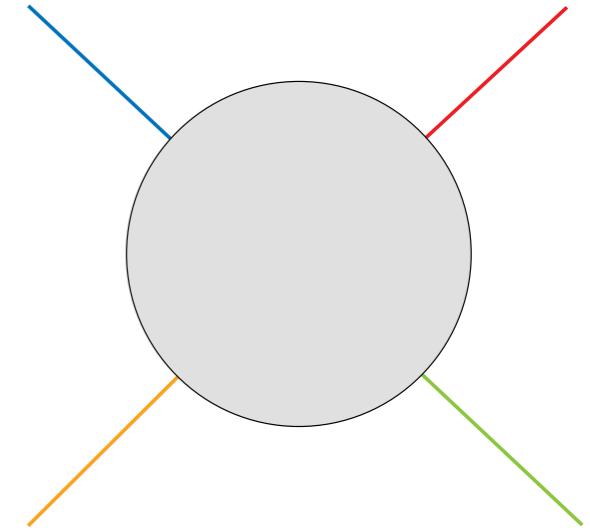
$$\mathcal{C}_5 = Tr(\mathbf{T}^{a_1} \mathbf{T}^{a_3}) Tr(\mathbf{T}^{a_2} \mathbf{T}^{a_4})$$

$$\mathcal{C}_6 = Tr(\mathbf{T}^{a_1} \mathbf{T}^{a_4}) Tr(\mathbf{T}^{a_2} \mathbf{T}^{a_3})$$

# Tensor Projection

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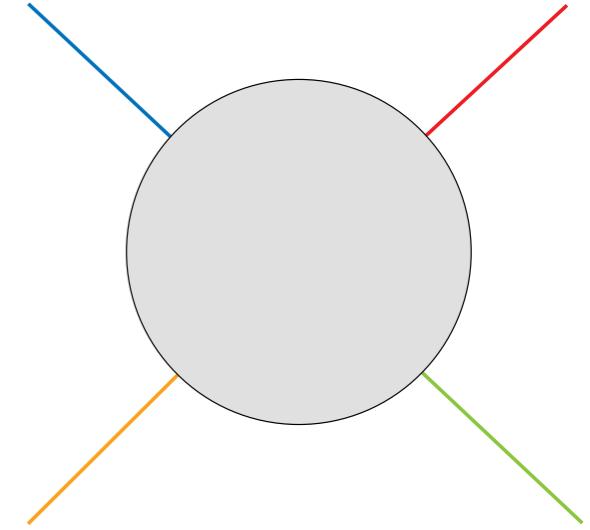
$$A = \sum_i^N F_i T_i$$



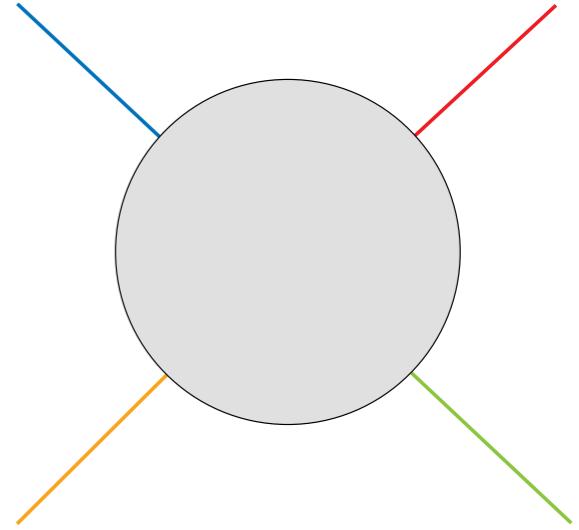
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$$A = \sum_i^N F_i T_i$$

Form Factors: Only contain **scalar integrals!**



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$$A = \sum_i^N F_i T_i$$

Form Factors:

Only contain **scalar integrals!**

Computed in **Dim. Reg.**

# Tensor Projection

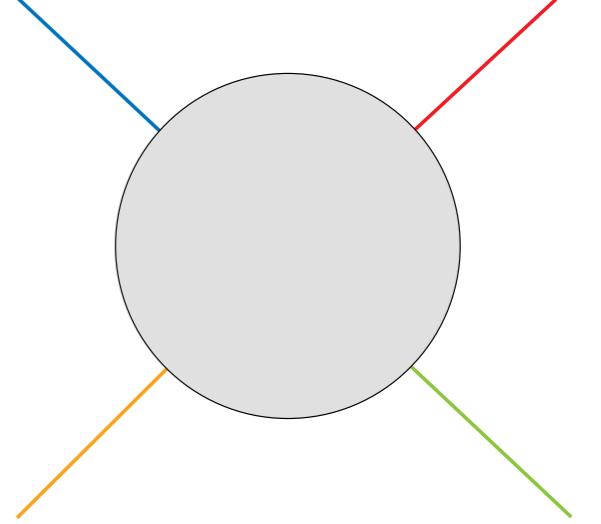
$$A = \sum_i F_i T_i$$

Form Factors:

Only contain **scalar integrals!**

Computed in **Dim. Reg.**

Ward Identities/gauge fixing



A diagram showing a central gray circle representing a tensor loop. Four lines extend from the circle to the right: a blue line at the top, a red line at the top-right, an orange line at the bottom-right, and a green line at the bottom-left.

$N$

Ward Identities/gauge fixing

# Tensor Projection

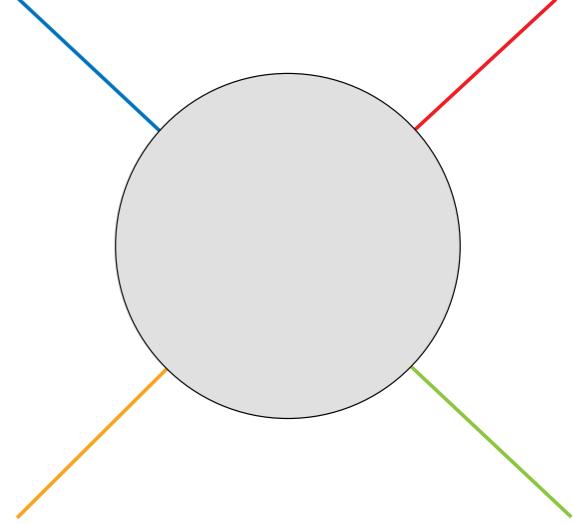
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Computed in **Dim. Reg.**

Ward Identities/gauge fixing



A diagram showing a central gray circle representing a tensor loop. Four lines extend from the circle to the outside: a blue line at the top-left, a red line at the top-right, an orange line at the bottom-left, and a green line at the bottom-right.

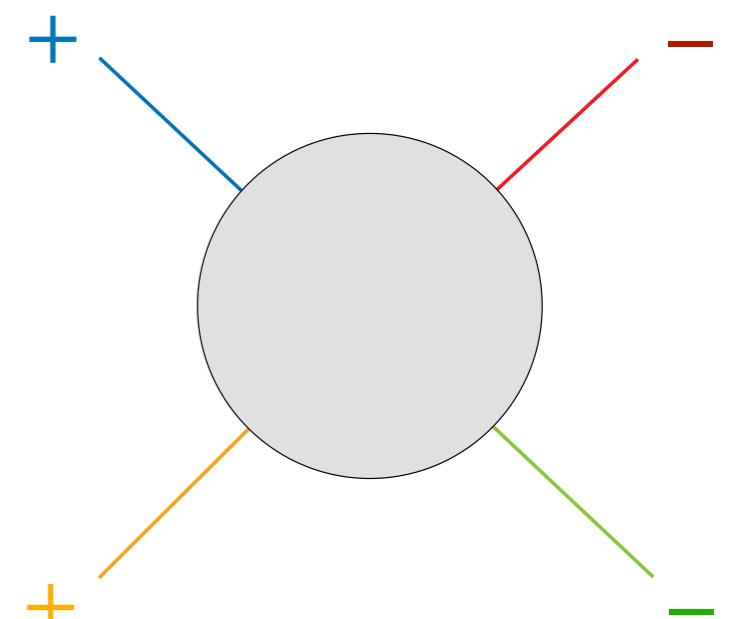
**tHV** scheme

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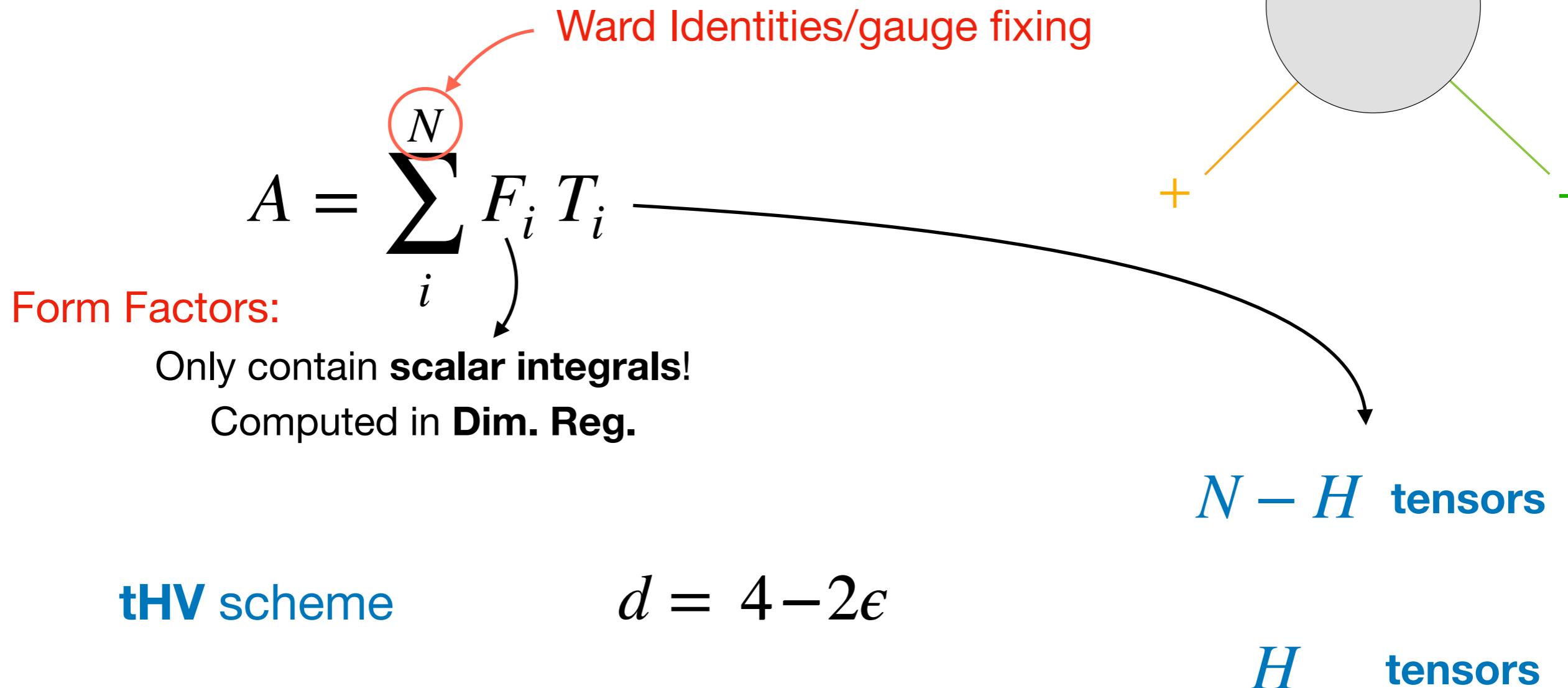
Form Factors: Only contain **scalar integrals!**  
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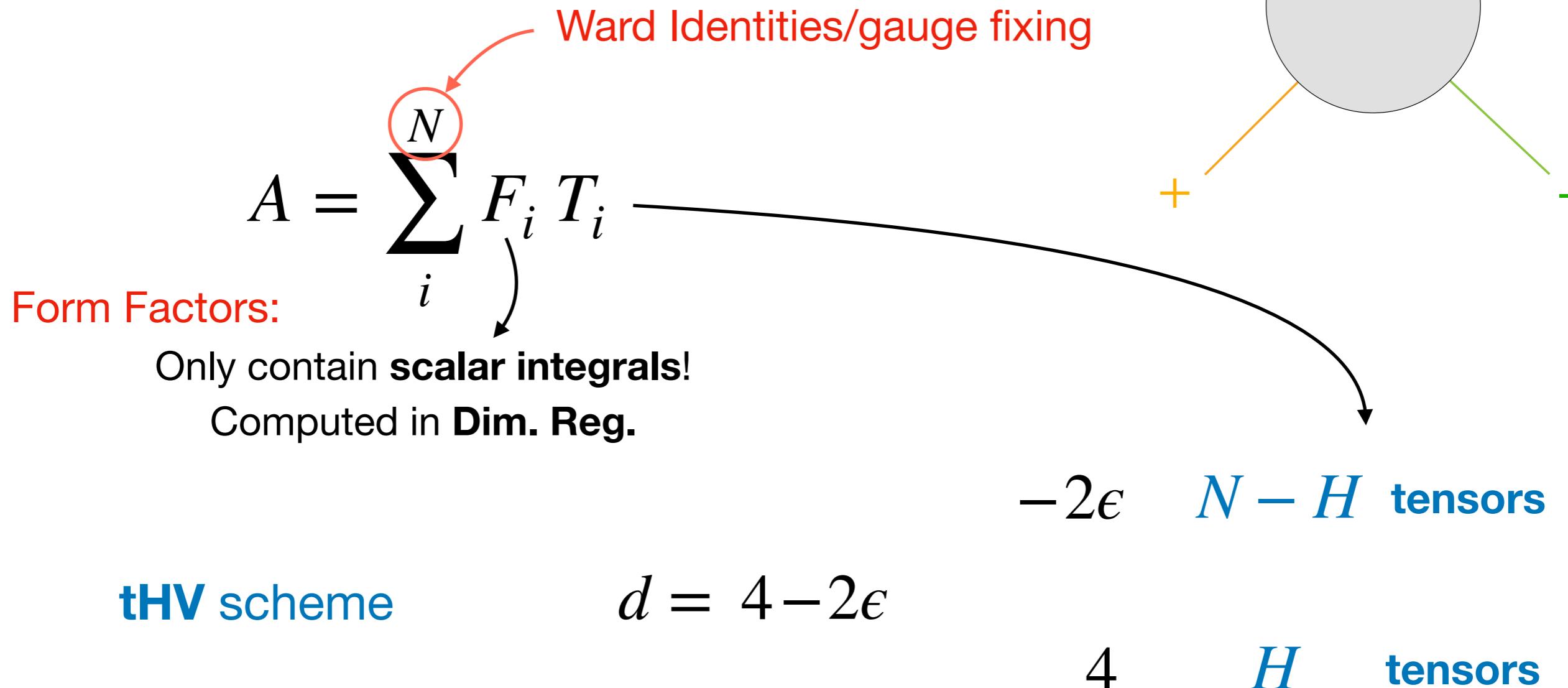
tHV scheme

$$d = 4 - 2\epsilon$$

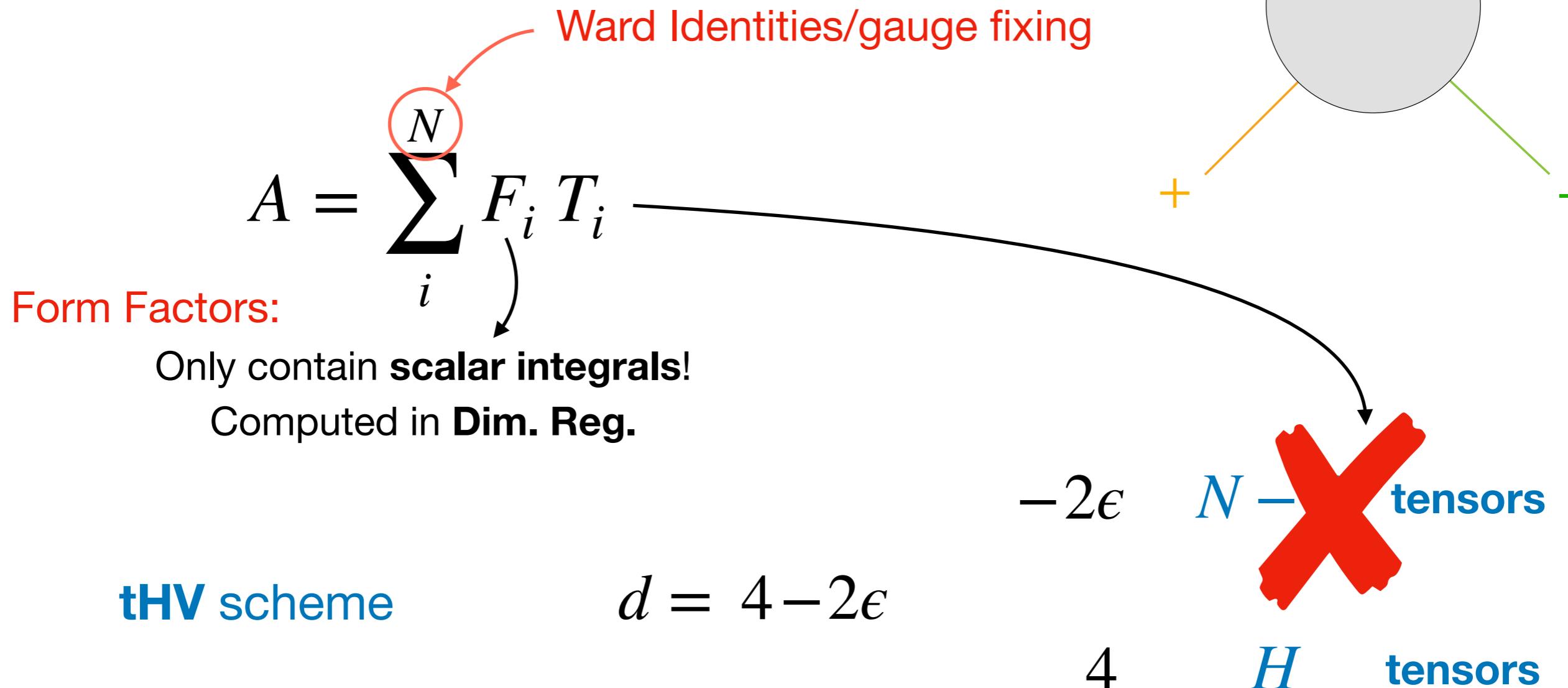
# Tensor Projection



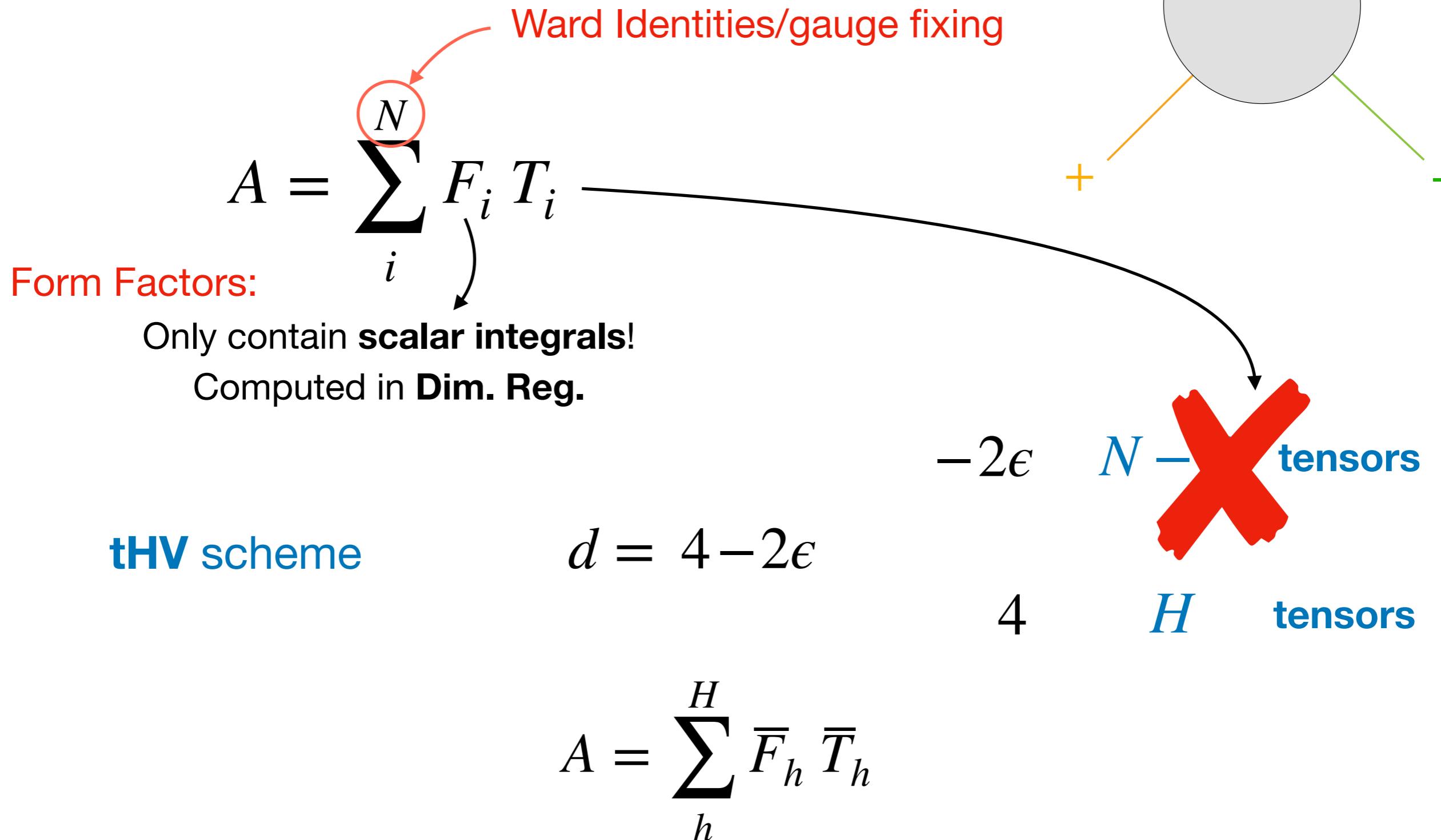
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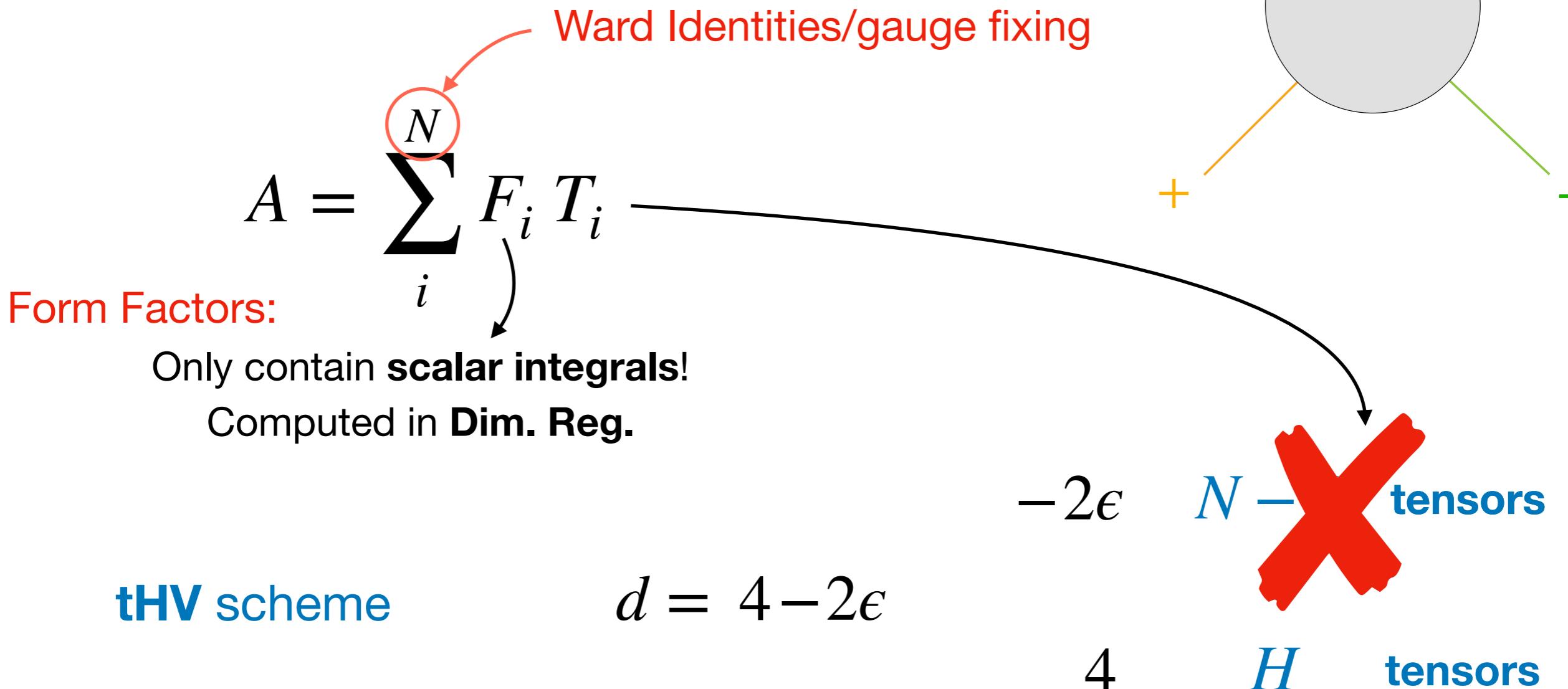
# Tensor Projection



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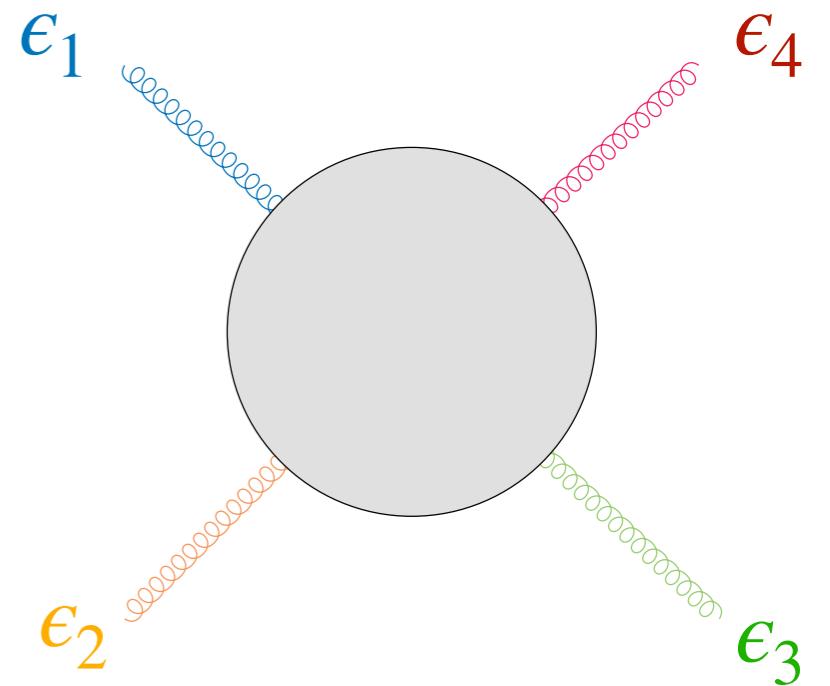


$$A = \sum_h^H \bar{F}_h \bar{T}_h$$

$$A^{++++} = \bar{F}_1 \bar{T}_1$$
$$A^{++--} = \bar{F}_2 \bar{T}_2$$

# Example

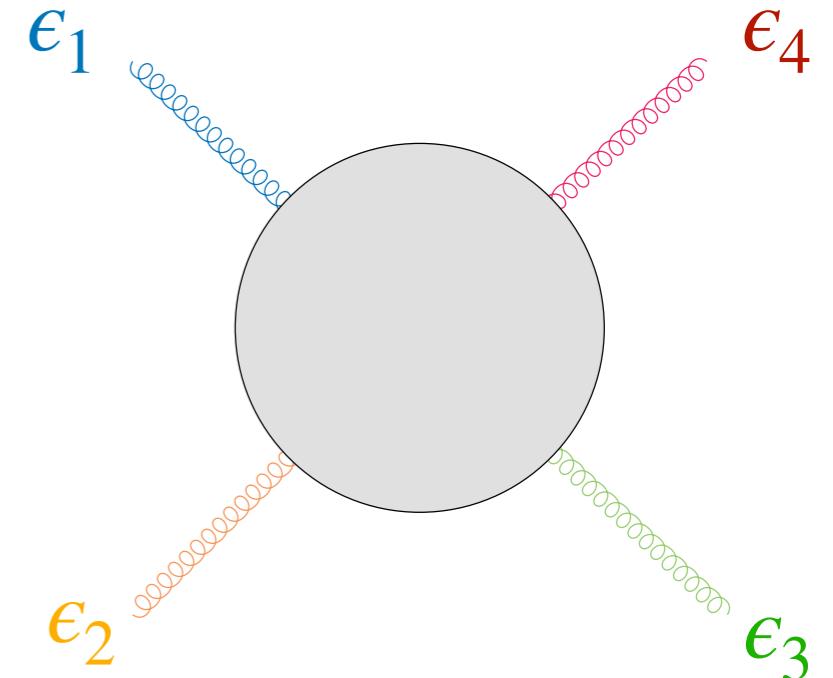
$$A = \sum_i^N F_i T_i$$



# Example

$$A = \sum_i^N F_i T_i$$

$$T = \epsilon_1^{\mu_1} \epsilon_2^{\mu_2} \epsilon_3^{\mu_3} \epsilon_4^{\mu_4} (p_1, p_2, p_3, \eta_{\mu\nu})_{\mu_1 \mu_2 \mu_3 \mu_4}$$

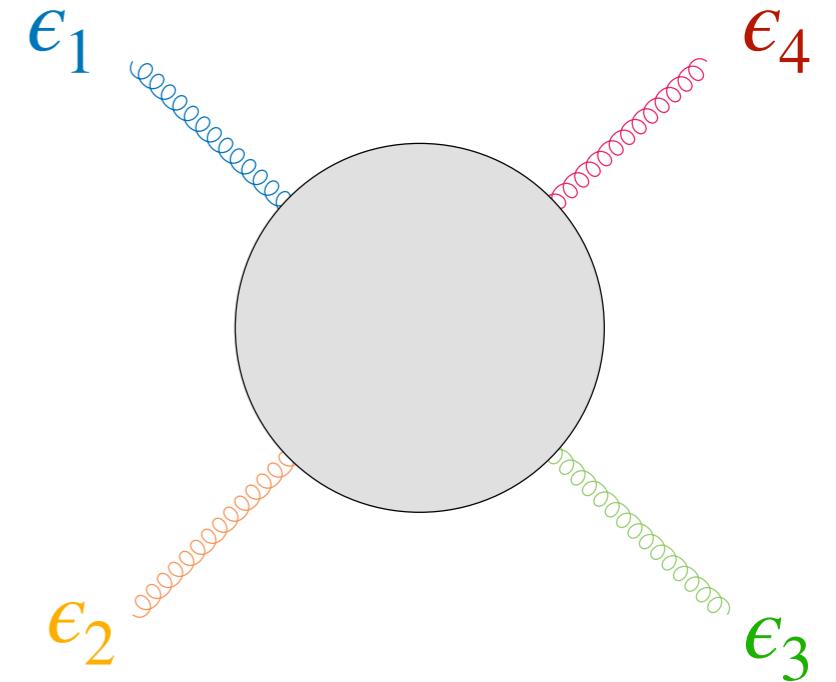


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138 tensors!



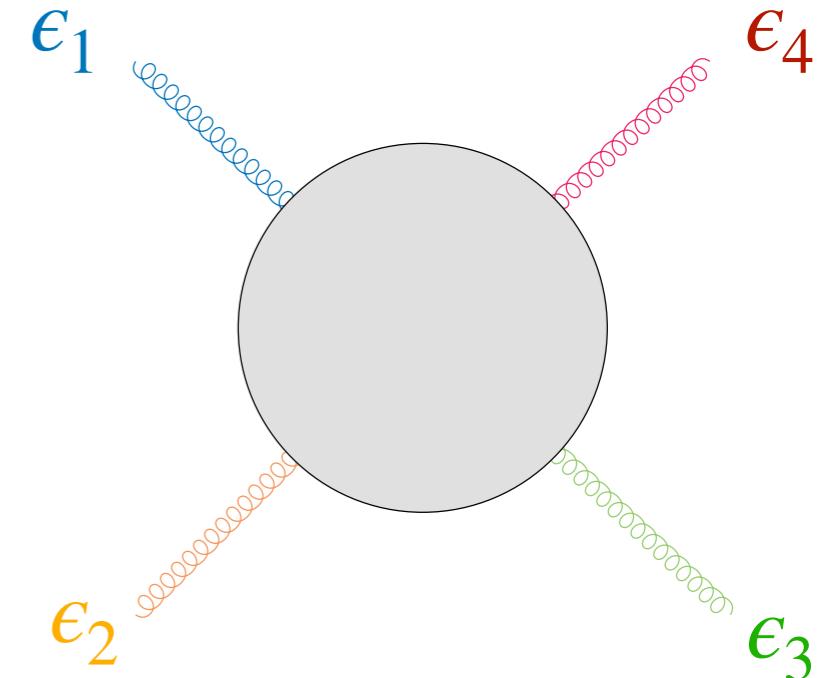
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138 tensors!

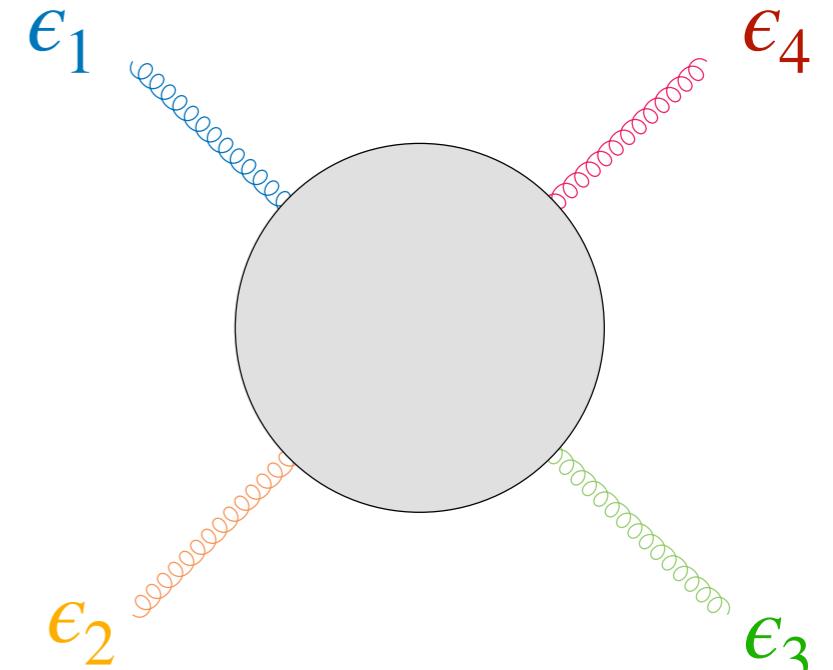
$$\epsilon_i \perp p_i$$



# Example

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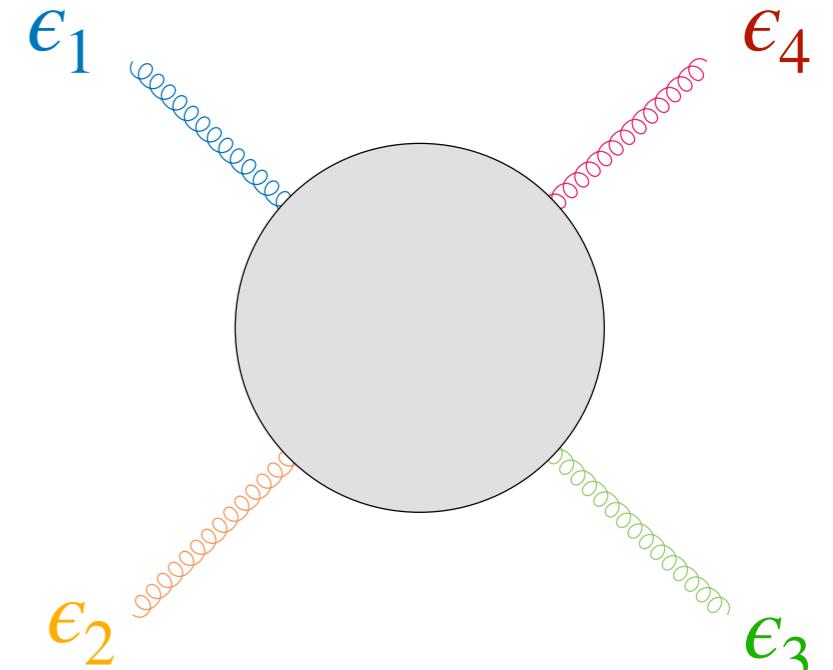
138 tensors!

$$\epsilon_i \perp p_i \quad \text{and} \quad \epsilon_i \perp p_{i+1}$$

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**138 tensors!**

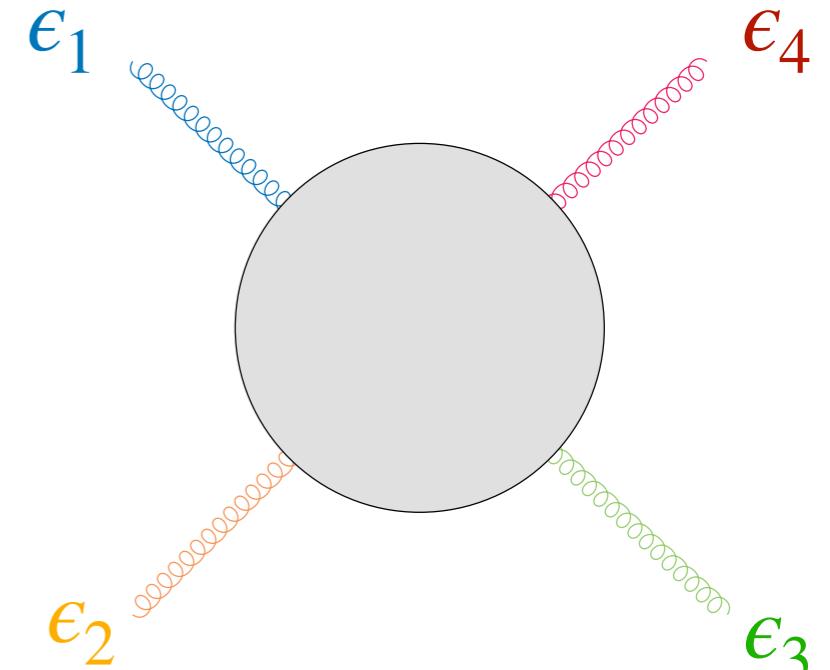
$$\epsilon_i \perp p_i \quad \text{and} \quad \epsilon_i \perp p_{i+1}$$

**10 tensors!**

# Example

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138 tensors!

$$\epsilon_i \perp p_i \quad \text{and} \quad \epsilon_i \perp p_{i+1}$$

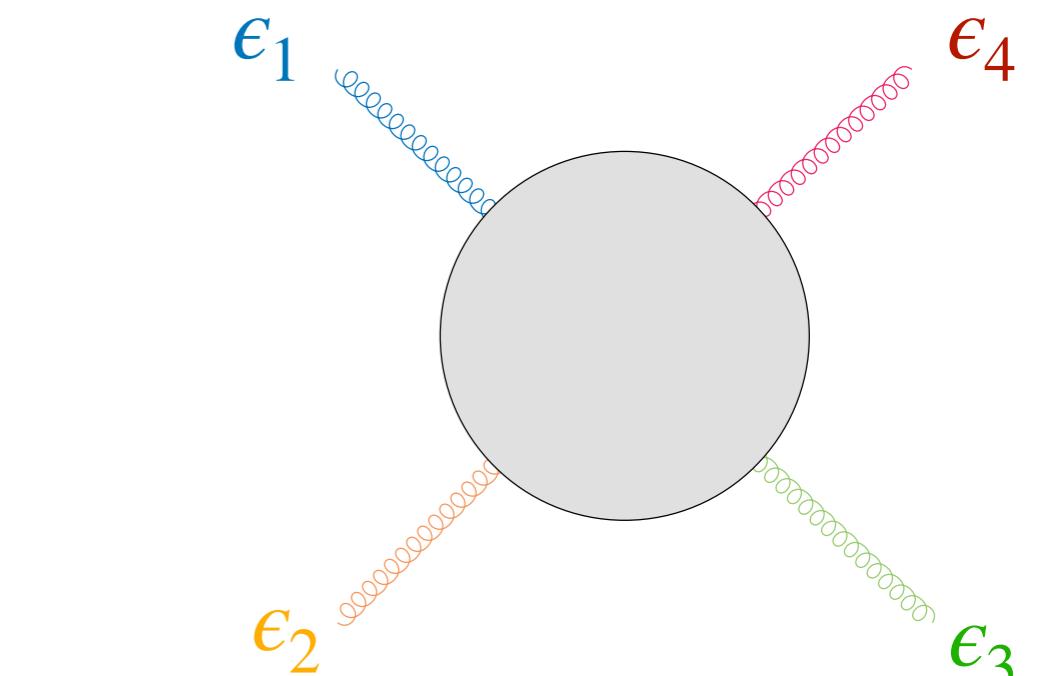
10 tensors!

tHV scheme basis →

# Example

$$A = \sum_i^N F_i T_i$$

$$T = \epsilon_1^{\mu_1} \epsilon_2^{\mu_2} \epsilon_3^{\mu_3} \epsilon_4^{\mu_4} (p_1, p_2, p_3, \eta_{\mu\nu})_{\mu_1 \mu_2 \mu_3 \mu_4}$$



**138 tensors!**

$$\epsilon_i \perp p_i \quad \text{and} \quad \epsilon_i \perp p_{i+1}$$

**10 tensors!**

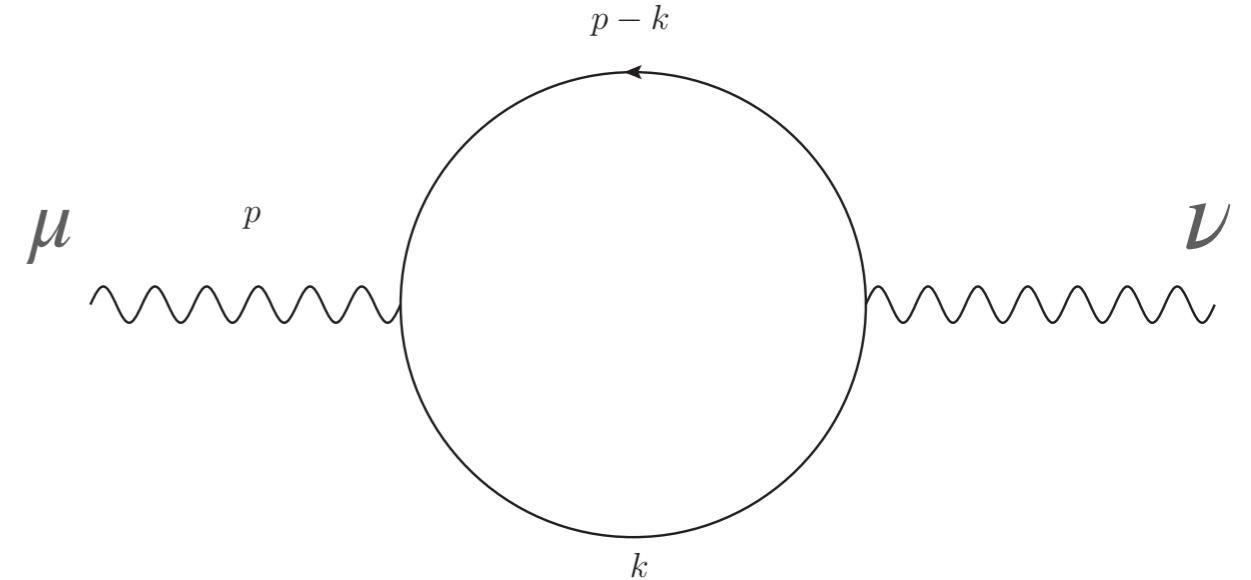
**tHV scheme basis → 8 tensors**

# **Example**

## **QED vacuum polarisation**

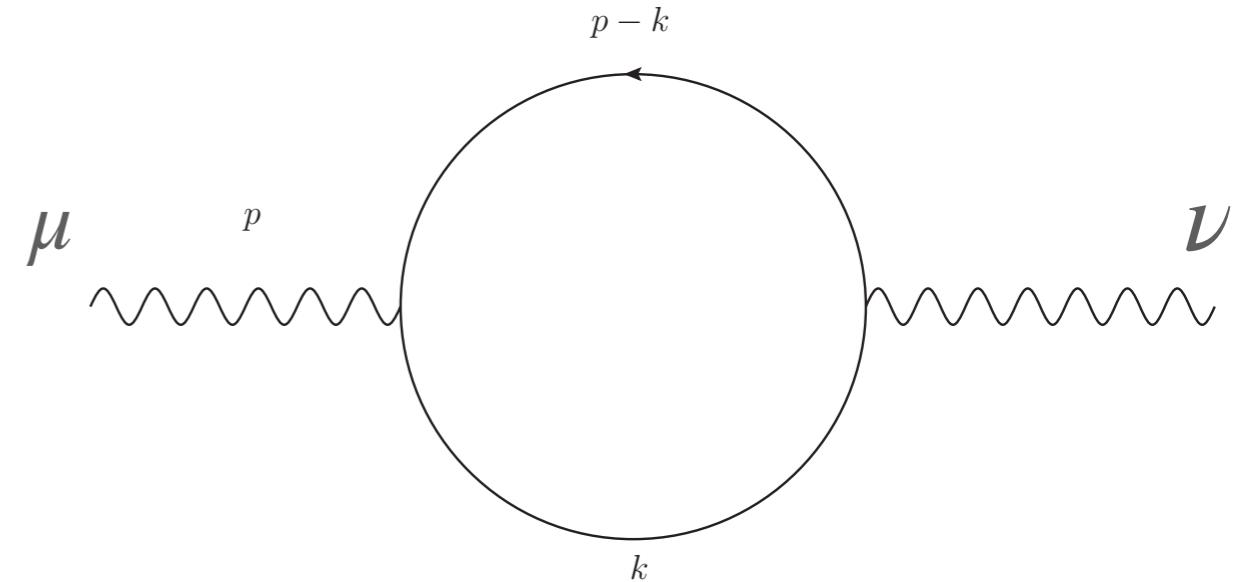
# Example

## QED vacuum polarisation



# Example

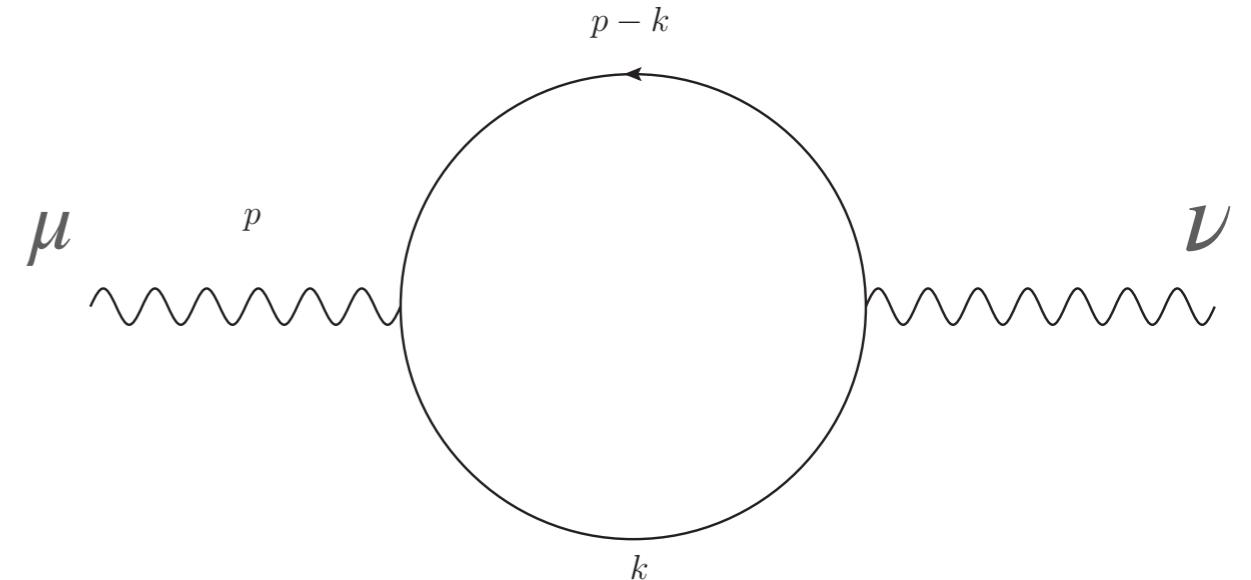
## QED vacuum polarisation



$$A^{\mu\nu} = e^2 \int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr}[\gamma^\mu \not{k} \gamma^\nu (\not{p} - \not{k})]}{(k^2 + i\epsilon)((k - p)^2 + i\epsilon)}$$

# Example

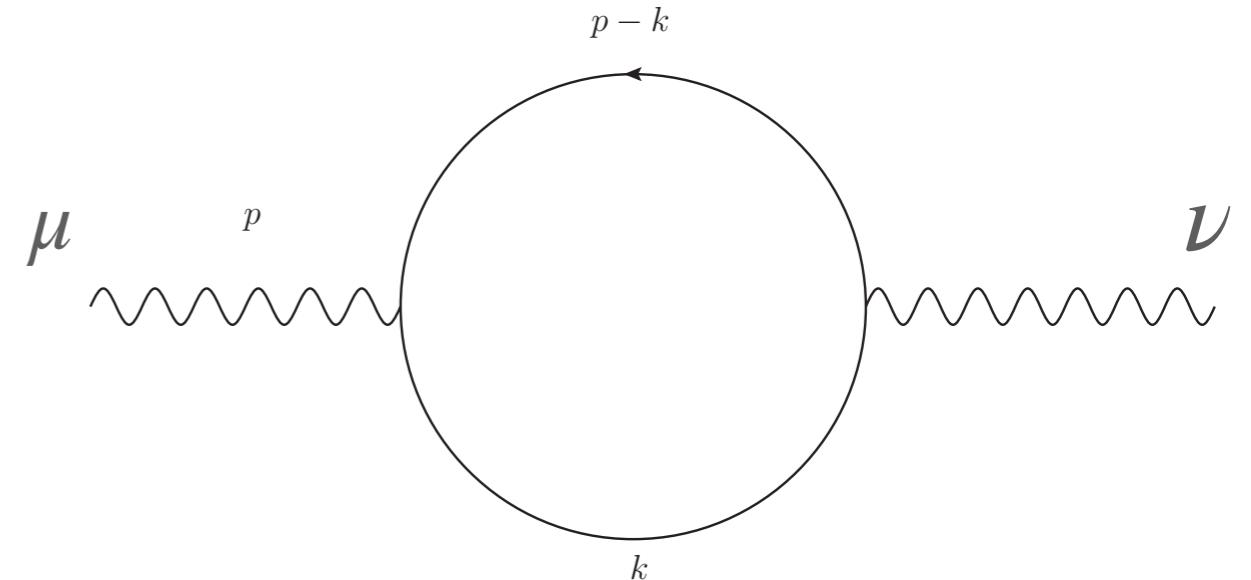
## QED vacuum polarisation



$$A^{\mu\nu} = e^2 \int \frac{d^d k}{(2\pi)^d} \frac{4(k^\mu(p^\nu - k^\nu) + k^\nu(p^\mu - k^\mu) - g^{\mu\nu}(k \cdot p - k^2))}{(k^2 + i\epsilon)((k - p)^2 + i\epsilon)}$$

# Example

## QED vacuum polarisation



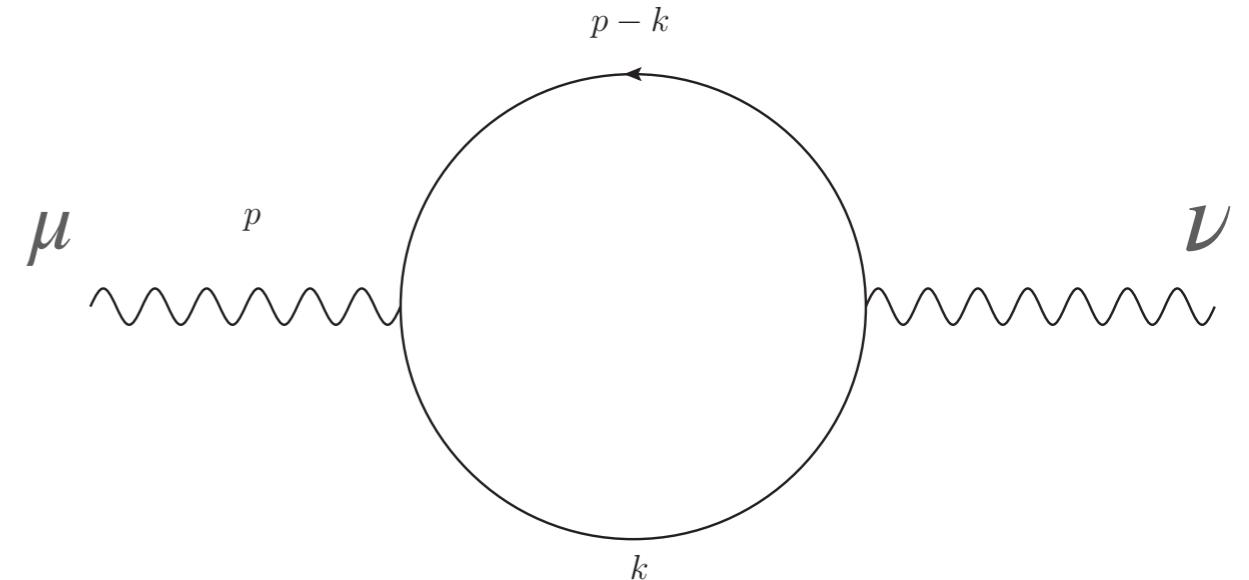
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Tensor basis:

$$T_1^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \quad T_2^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}$$

# Example

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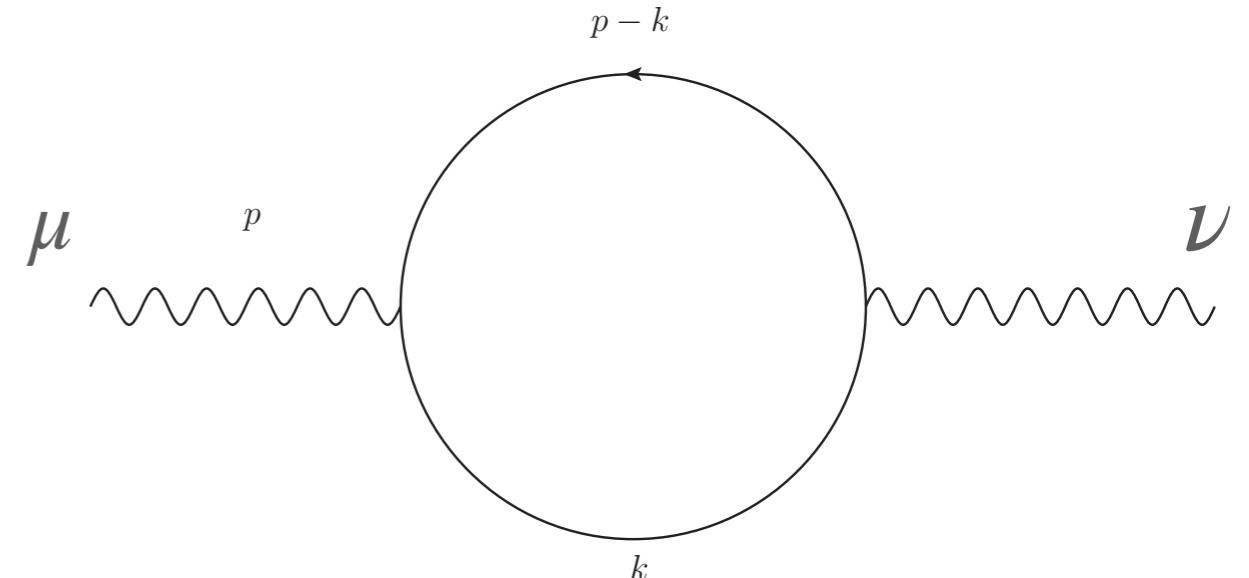
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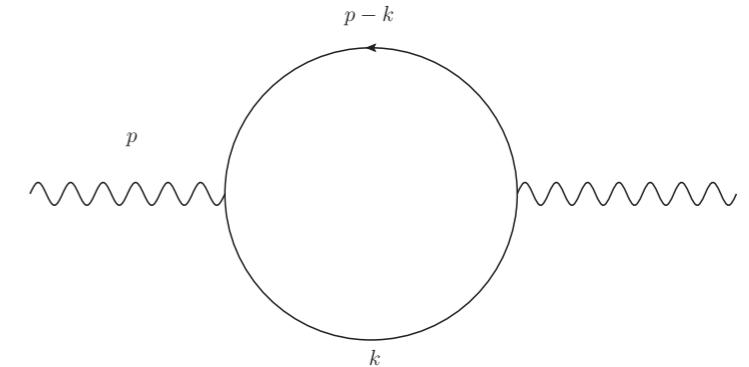
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Lorentz  
indices are  
now outside  
the integrals!

# Example

## QED vacuum polarisation



Tensor basis:

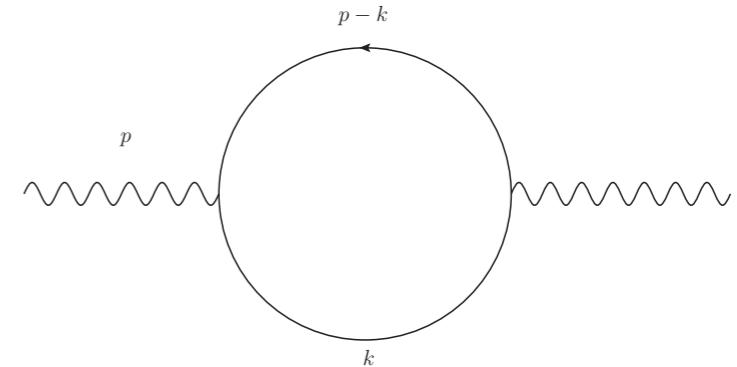
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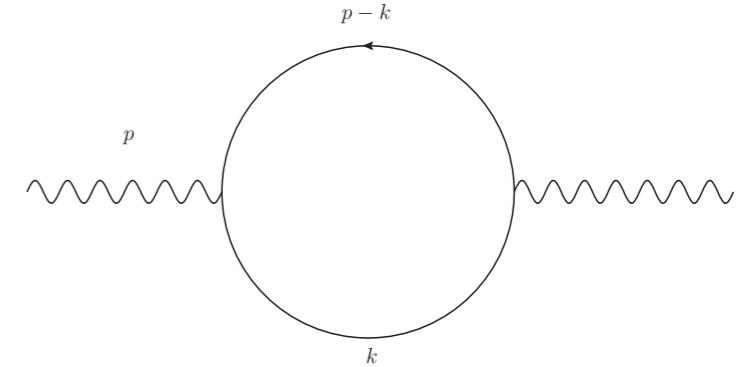
Projectors:

$$P_{1\mu\nu} = \frac{1}{d-1} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right)$$

$$P_{2\mu\nu} = \frac{p^\mu p^\nu}{p^2}$$

# Example

## QED vacuum polarisation



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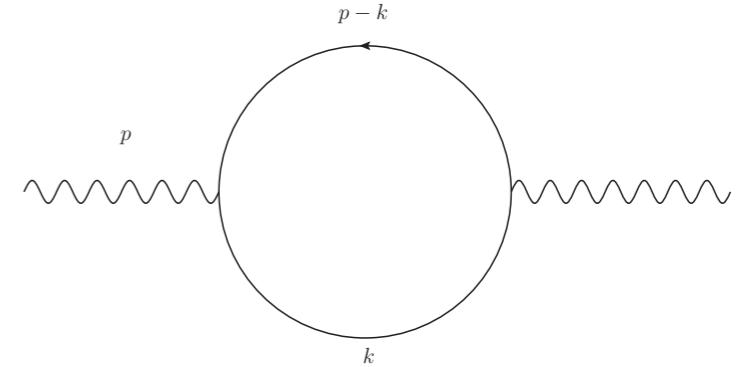


$$P_{1\mu\nu} A^{\mu\nu} = F_1$$

$$P_{2\mu\nu} A^{\mu\nu} = F_2$$

# Example

## QED vacuum polarisation



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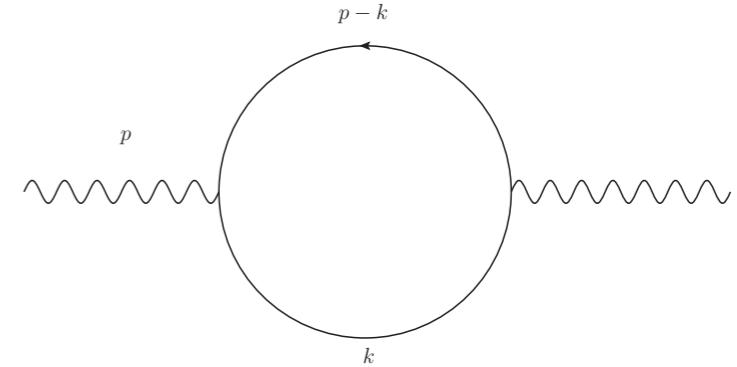
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# Example

## QED vacuum polarisation



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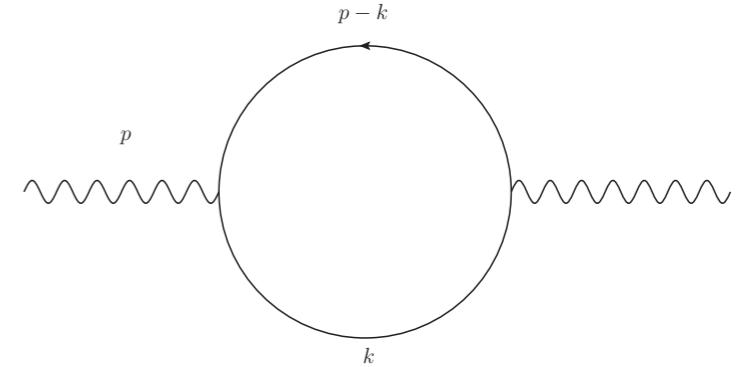
$$P_{1\mu\nu} A^{\mu\nu} = F_1$$

Ward Identity:  $p_\mu A^{\mu\nu} = 0$

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# Example

## QED vacuum polarisation



Tensor basis:

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$$P_{1\mu\nu} A^{\mu\nu} = F_1$$

Ward Identity:  $p_\mu A^{\mu\nu} = 0$

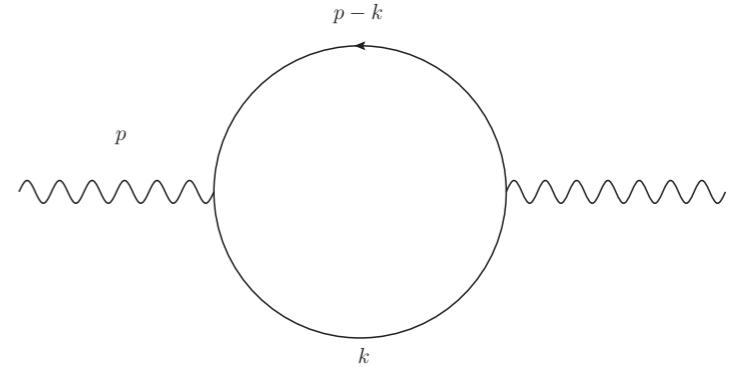
$$P_{2\mu\nu} A^{\mu\nu} = F_2$$



$$F_2 = 0$$

# Example

## QED vacuum polarisation



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$$F_2 = 0$$

# General Case

$$A^X = \sum_{i=1}^N F_i T_i^X$$

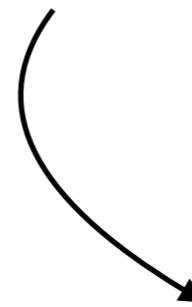
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Computed in Dim. Reg.

# General Case

$$A^X = \sum_{i=1}^N F_i T_i^X$$

• Ward Identities/gauge fixing

Only contain **scalar integrals!**

Computed in Dim. Reg.

The diagram features a mathematical equation  $A^X = \sum_{i=1}^N F_i T_i^X$ . A red circle highlights the number  $N$  in the upper limit of the summation. Two black curved arrows originate from this red circle: one points to the text 'Ward Identities/gauge fixing' located above the equation, and the other points to the text 'Only contain scalar integrals!' located below the equation.

# General Case

$$A^X = \sum_{i=1}^N F_i T_i^X$$

• Ward Identities/gauge fixing  
• 't Hooft-Veltman scheme

Only contain **scalar integrals!**

Computed in Dim. Reg.

# General Case

$$A^X = \sum_{i=1}^N F_i T_i^X$$

$N$

- Ward Identities/gauge fixing
- 't Hooft-Veltman scheme

Only contain **scalar integrals!**

Computed in Dim. Reg.

$$P_i \cdot A = F_i$$

# General Case

$$A^X = \sum_{i=1}^N F_i T_i^X$$

• Ward Identities/gauge fixing  
• 't Hooft-Veltman scheme

Only contain **scalar integrals!**

Computed in Dim. Reg.

$$P_i \cdot A = F_i$$
$$\bar{u}(p) \cdot u(p) = \gamma_\mu p^\mu$$
$$\epsilon^*(p) \cdot \epsilon(p) = -\eta^{\mu\nu} + \dots$$

# General Case

$$P_i \cdot T_i = \delta_{ij}$$

$$P_i \cdot A = F_i$$

$$\bar{u}(p) \cdot u(p) = \gamma_\mu p^\mu$$

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$$A^X = \sum_{i=1}^N F_i T_i^X$$

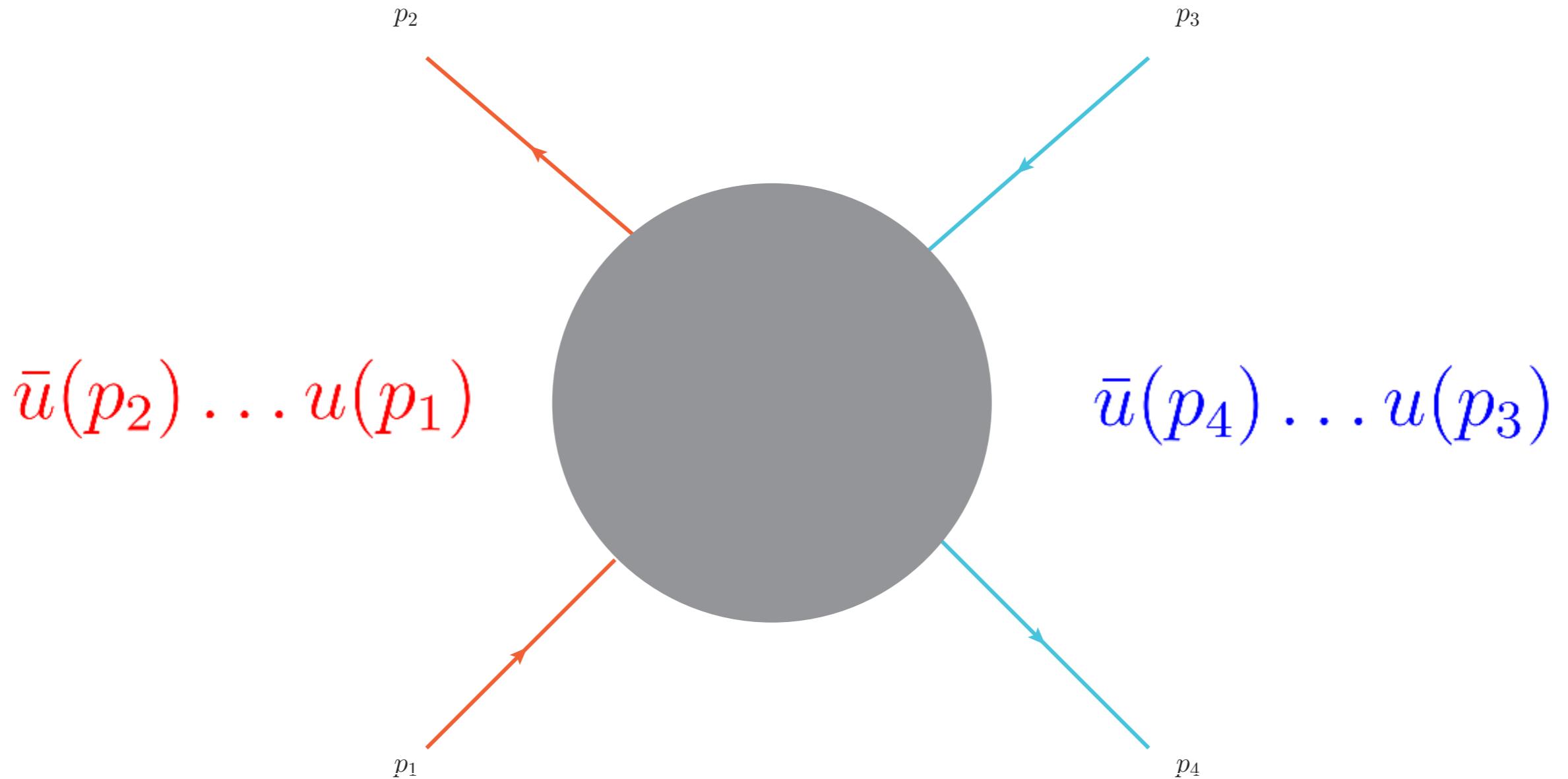
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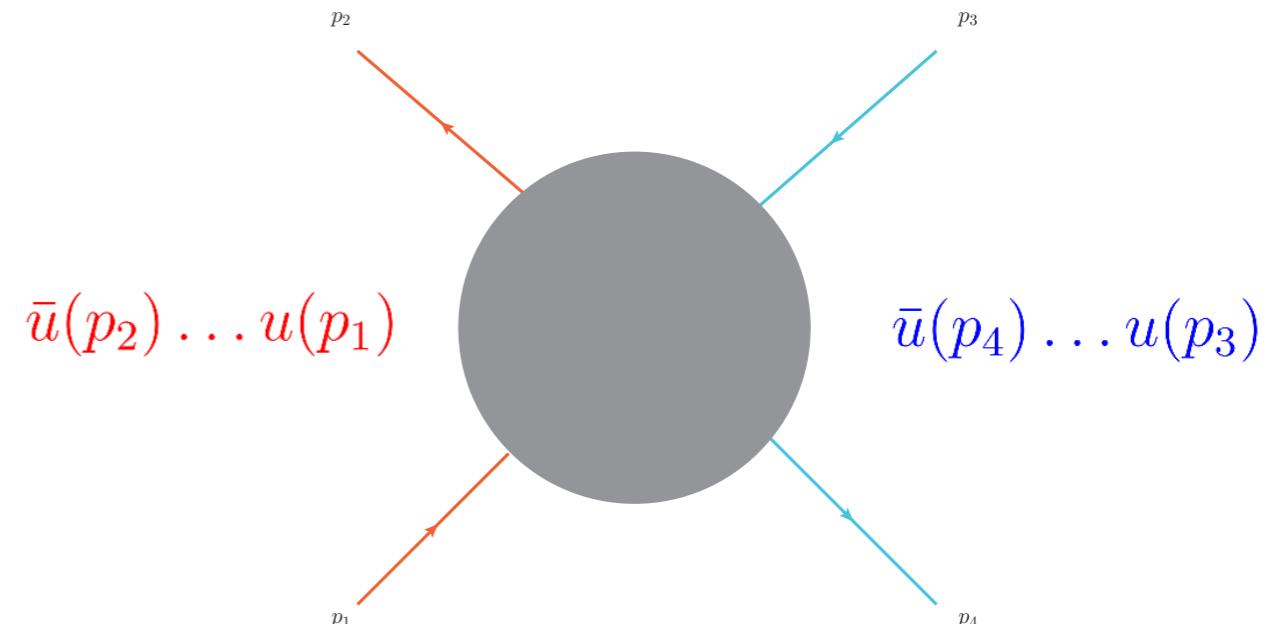
Computed in Dim. Reg.

# Quarks Example

**qQ → qQ**



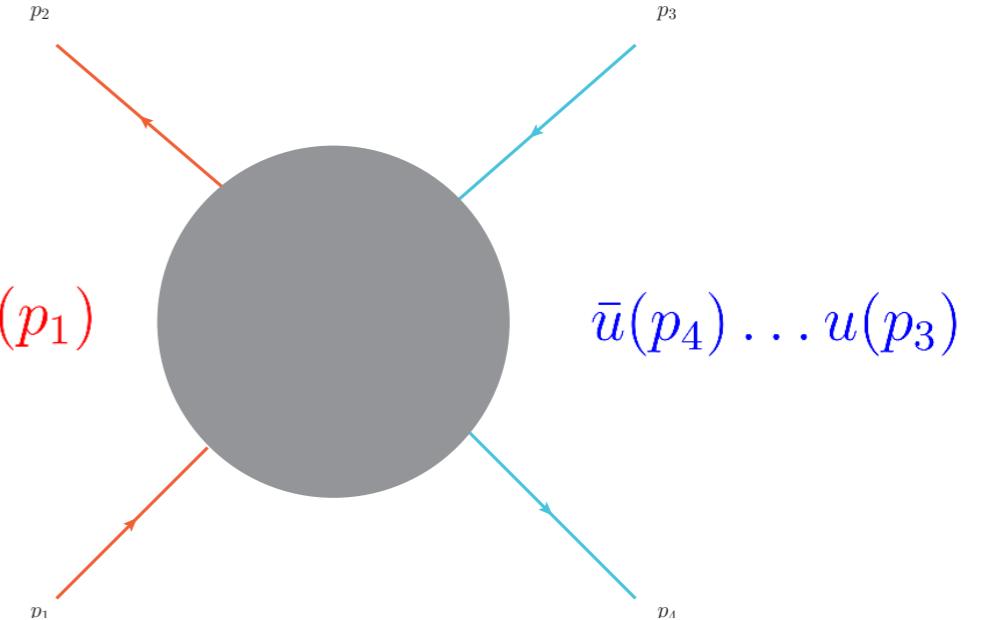
**qQ → qQ**



**qQ → qQ**

$$A^X = \sum_{i=1}^N F_i T_i^X$$

$\bar{u}(p_2) \dots u(p_1)$

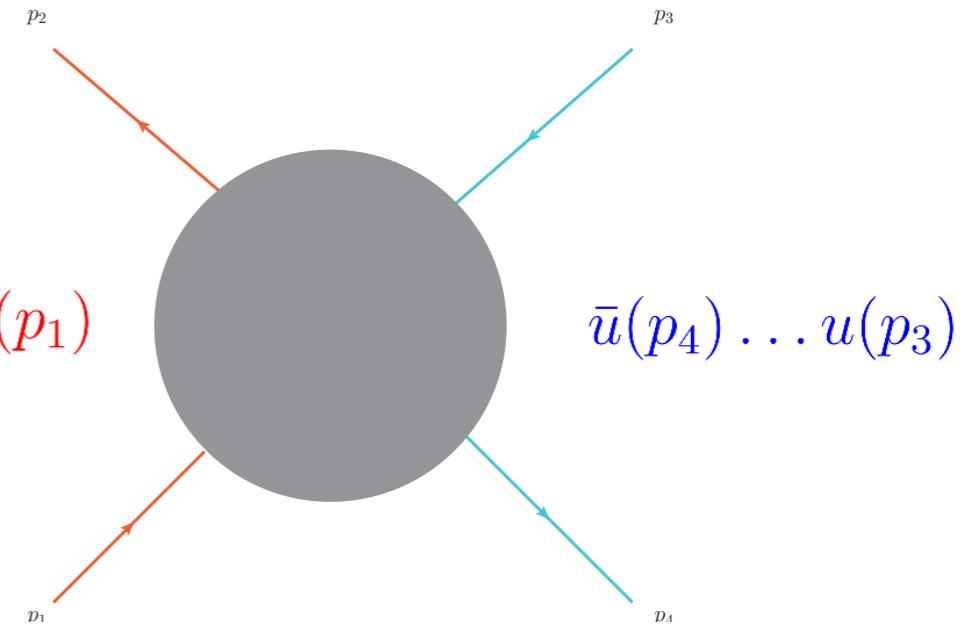


# $\mathbf{qQ} \rightarrow \mathbf{qQ}$

$$A^X = \sum_{i=1}^N F_i T_i^X$$

$$\bar{u}(p_2) \dots u(p_1)$$

$$\bar{u}(p_4) \dots u(p_3)$$



$$T_1 = \bar{u}(p_2) \gamma_{\mu_1} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} u(p_3)$$

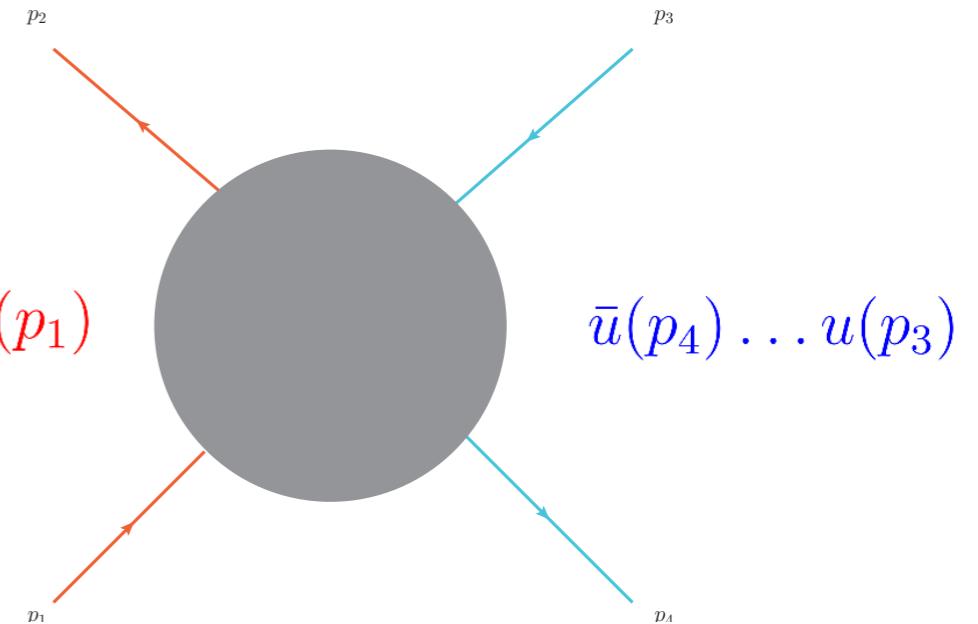
$$T_2 = \bar{u}(p_2) \not{p}_3 u(p_1) \times \bar{u}(p_4) \not{p}_1 u(p_3)$$

# $\mathbf{qQ} \rightarrow \mathbf{qQ}$

$$A^X = \sum_{i=1}^N F_i T_i^X$$

$$\bar{u}(p_2) \dots u(p_1)$$

$$\bar{u}(p_4) \dots u(p_3)$$



$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$

$$T_2 = \bar{u}(p_2)\not{p}_3u(p_1) \times \bar{u}(p_4)\not{p}_1u(p_3)$$

$$T_3 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}u(p_3)$$

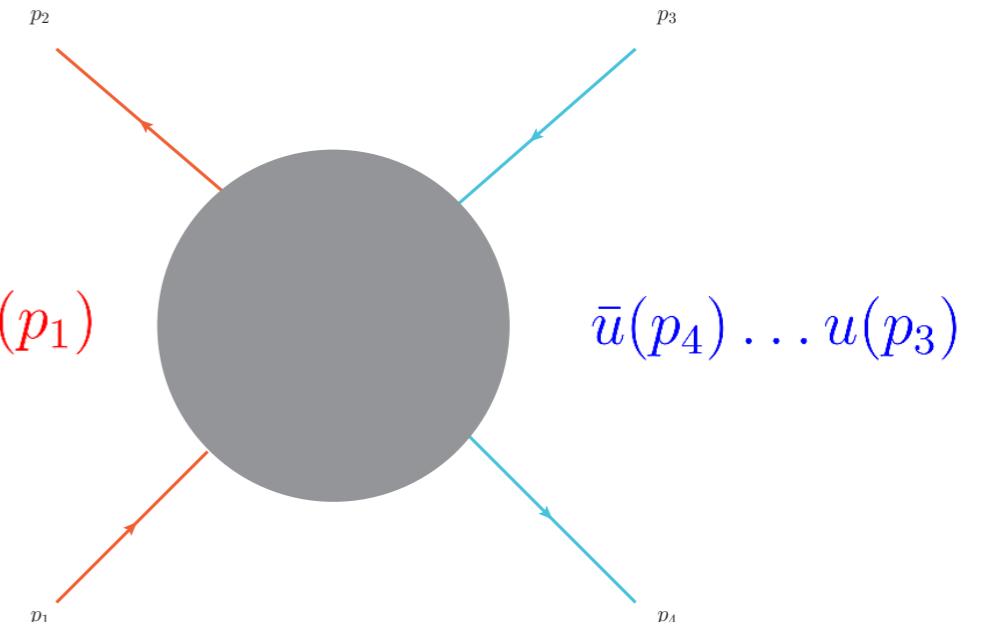
$$T_4 = \bar{u}(p_2)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\not{p}_1\gamma^{\mu_3}u(p_3)$$

# $\mathbf{qQ} \rightarrow \mathbf{qQ}$

$$A^X = \sum_{i=1}^N F_i T_i^X$$

$$\bar{u}(p_2) \dots u(p_1)$$

$$\bar{u}(p_4) \dots u(p_3)$$



$$T_1 = \bar{u}(p_2) \gamma_{\mu_1} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} u(p_3)$$

$$T_2 = \bar{u}(p_2) \not{p}_3 u(p_1) \times \bar{u}(p_4) \not{p}_1 u(p_3)$$

$$T_3 = \bar{u}(p_2) \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} u(p_3)$$

$$T_4 = \bar{u}(p_2) \gamma_{\mu_1} \not{p}_3 \gamma_{\mu_3} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} \not{p}_1 \gamma^{\mu_3} u(p_3)$$

$$T_5 = \bar{u}(p_2) \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} \gamma_{\mu_5} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5} u(p_3)$$

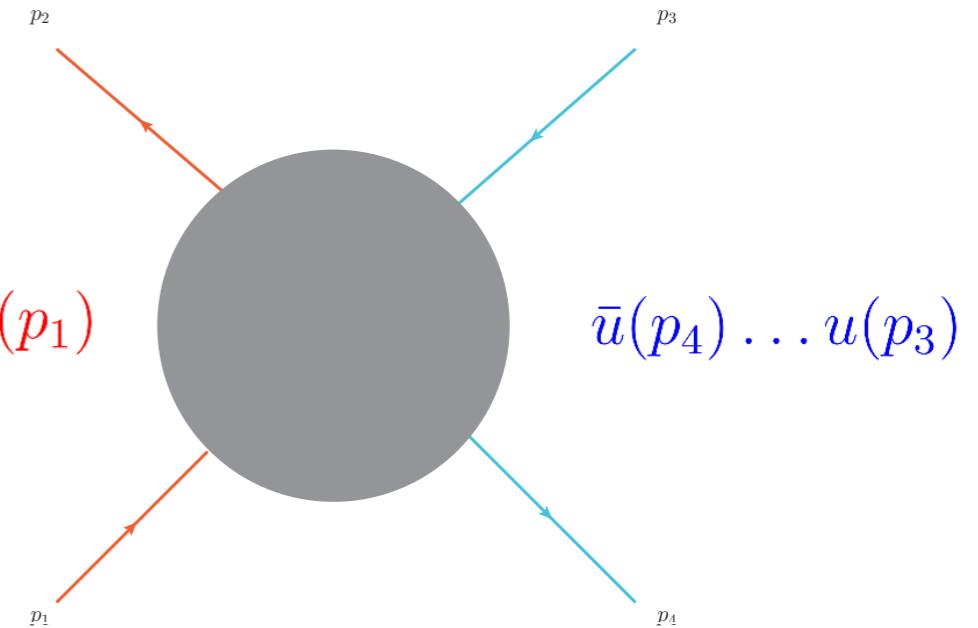
$$T_6 = \bar{u}(p_2) \gamma_{\mu_1} \gamma_{\mu_2} \not{p}_3 \gamma_{\mu_4} \gamma_{\mu_5} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} \gamma^{\mu_2} \not{p}_1 \gamma^{\mu_4} \gamma^{\mu_5} u(p_3)$$

# $\mathbf{qQ} \rightarrow \mathbf{qQ}$

$$A^X = \sum_{i=1}^N F_i T_i^X$$

$$\bar{u}(p_2) \dots u(p_1)$$

$$\bar{u}(p_4) \dots u(p_3)$$



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$$T_2 = \bar{u}(p_2)\not{p}_3u(p_1) \times \bar{u}(p_4)\not{p}_1u(p_3)$$

$$T_3 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}u(p_3)$$

$$T_4 = \bar{u}(p_2)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\not{p}_1\gamma^{\mu_3}u(p_3)$$

$$T_5 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}u(p_3)$$

$$T_6 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\not{p}_1\gamma^{\mu_4}\gamma^{\mu_5}u(p_3)$$

$$T_7 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_{\mu_6}\gamma_{\mu_7}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}\gamma^{\mu_6}\gamma^{\mu_7}u(p_3)$$

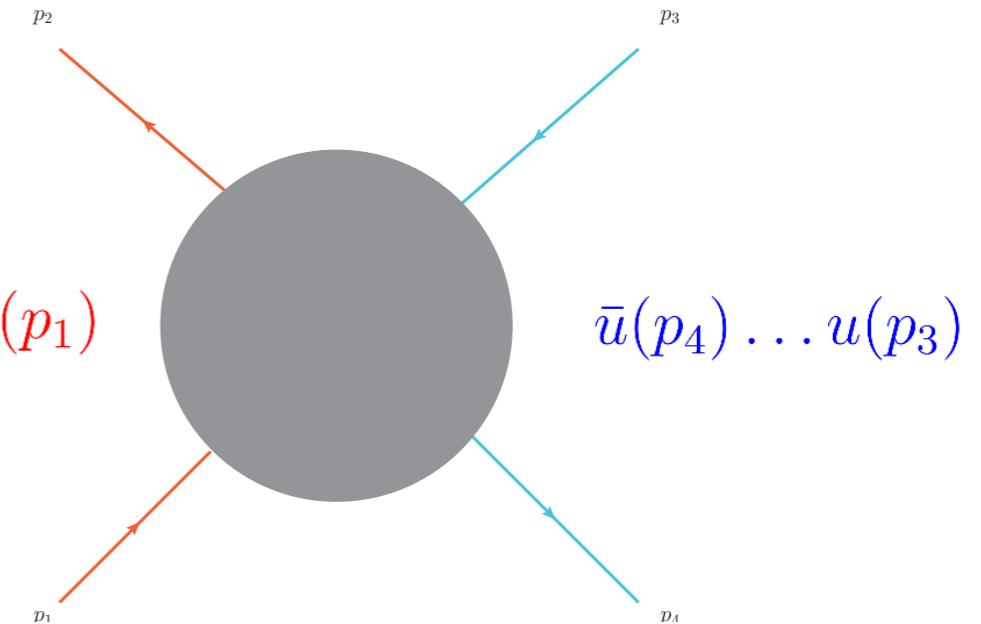
$$T_8 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\not{p}_3\gamma_{\mu_5}\gamma_{\mu_6}\gamma_{\mu_7}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\not{p}_1\gamma^{\mu_5}\gamma^{\mu_6}\gamma^{\mu_7}u(p_3)$$

# $\mathbf{qQ} \rightarrow \mathbf{qQ}$

$$A^X = \sum_{i=1}^N F_i T_i^X$$

$\bar{u}(p_2) \dots u(p_1)$

$\bar{u}(p_4) \dots u(p_3)$



$$T_1 = \bar{u}(p_2) \gamma_{\mu_1} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} u(p_3)$$

$$T_2 = \bar{u}(p_2) \not{p}_3 u(p_1) \times \bar{u}(p_4) \not{p}_1 u(p_3)$$

$$T_3 = \bar{u}(p_2) \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} u(p_3)$$

$$T_4 = \bar{u}(p_2) \gamma_{\mu_1} \not{p}_3 \gamma_{\mu_3} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} \not{p}_1 \gamma^{\mu_3} u(p_3)$$

$$T_5 = \bar{u}(p_2) \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} \gamma_{\mu_5} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5} u(p_3)$$

$$T_6 = \bar{u}(p_2) \gamma_{\mu_1} \gamma_{\mu_2} \not{p}_3 \gamma_{\mu_4} \gamma_{\mu_5} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} \gamma^{\mu_2} \not{p}_1 \gamma^{\mu_4} \gamma^{\mu_5} u(p_3)$$

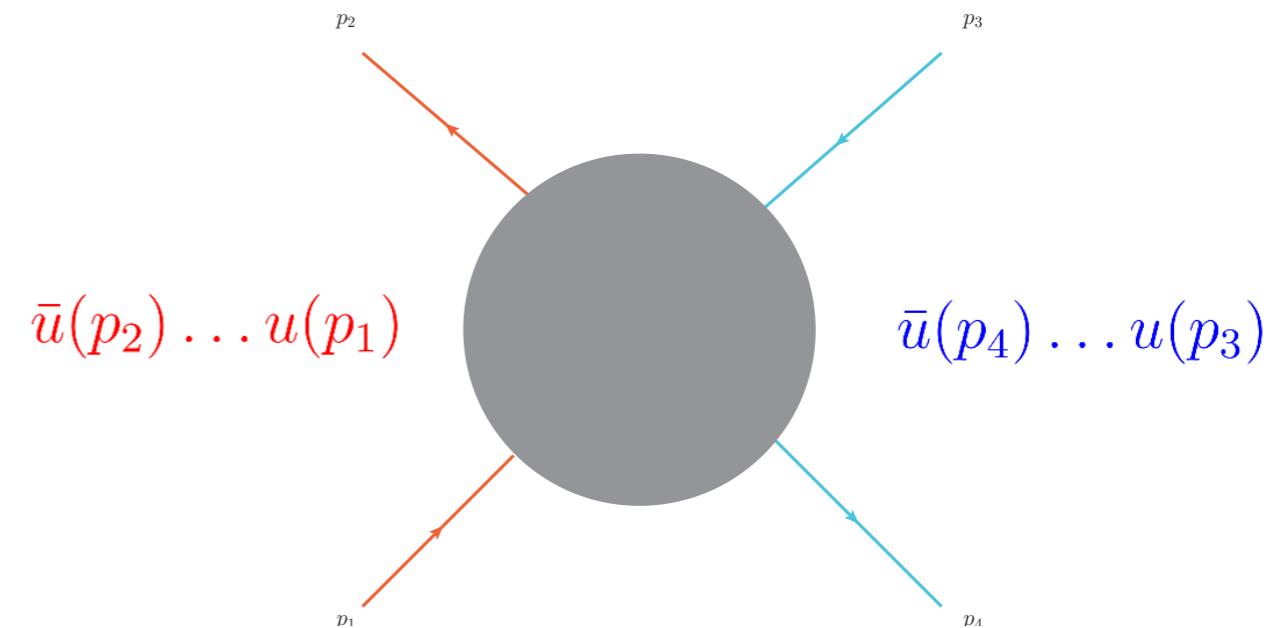
$$T_7 = \bar{u}(p_2) \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \gamma_{\mu_4} \gamma_{\mu_5} \gamma_{\mu_6} \gamma_{\mu_7} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \gamma^{\mu_4} \gamma^{\mu_5} \gamma^{\mu_6} \gamma^{\mu_7} u(p_3)$$

$$T_8 = \bar{u}(p_2) \gamma_{\mu_1} \gamma_{\mu_2} \gamma_{\mu_3} \not{p}_3 \gamma_{\mu_5} \gamma_{\mu_6} \gamma_{\mu_7} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} \gamma^{\mu_2} \gamma^{\mu_3} \not{p}_1 \gamma^{\mu_5} \gamma^{\mu_6} \gamma^{\mu_7} u(p_3)$$

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# $q\bar{Q} \rightarrow q\bar{Q}$

$$A^X = \sum_{i=1}^N F_i T_i^X$$



$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$

$$T_2 = \bar{u}(p_2)\not{p}_3u(p_1) \times \bar{u}(p_4)\not{p}_1u(p_3)$$

$$T_3 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}u(p_3)$$

$$T_4 = \bar{u}(p_2)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\not{p}_1\gamma^{\mu_3}u(p_3)$$

$$T_5 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}u(p_3)$$

$$T_6 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\not{p}_1\gamma^{\mu_4}\gamma^{\mu_5}u(p_3)$$

$$T_7 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}\gamma_{\mu_6}\gamma_{\mu_7}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}\gamma^{\mu_6}\gamma^{\mu_7}u(p_3)$$

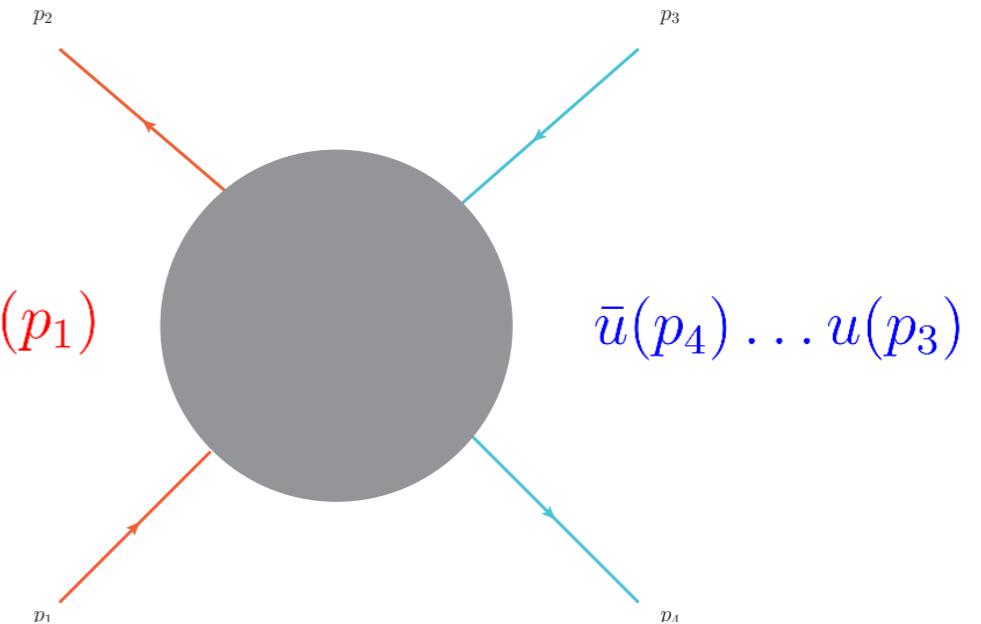
$$T_8 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\not{p}_3\gamma_{\mu_5}\gamma_{\mu_6}\gamma_{\mu_7}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\not{p}_1\gamma^{\mu_5}\gamma^{\mu_6}\gamma^{\mu_7}u(p_3)$$

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# $\mathbf{qQ} \rightarrow \mathbf{qQ}$

$$A^X = \sum_{i=1}^N F_i T_i^X$$

$$\bar{u}(p_2) \dots u(p_1)$$



$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$

$$T_2 = \bar{u}(p_2)\not{p}_3 u(p_1) \times \bar{u}(p_4)\not{p}_1 u(p_3)$$

$$T_3 = T_3^4 + (d-4)T_3^{-2\epsilon}$$

$$T_4 = T_4^4 + (d-4)T_4^{-2\epsilon}$$

$$T_5 = T_5^4 + (d-4)T_5^{-2\epsilon}$$

$$T_6 = T_6^4 + (d-4)T_6^{-2\epsilon}$$

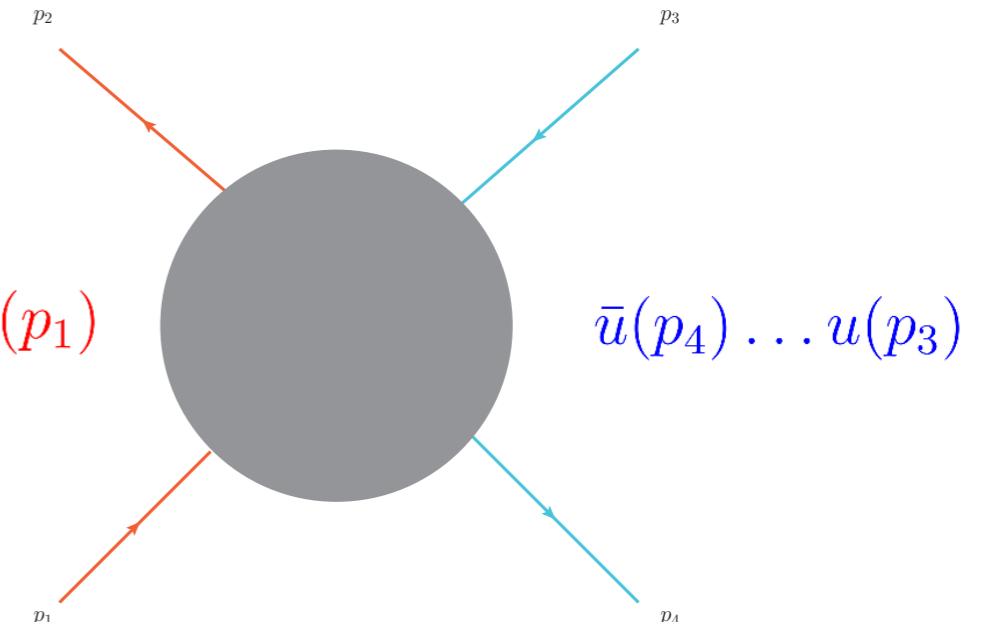
$$T_7 = T_7^4 + (d-4)T_7^{-2\epsilon}$$

$$T_8 = T_8^4 + (d-4)T_8^{-2\epsilon}$$

# $q\bar{Q} \rightarrow q\bar{Q}$

$$A^X = \sum_{i=1}^N F_i T_i^X$$

$$\bar{u}(p_2) \dots u(p_1)$$



$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$

$$T_2 = \bar{u}(p_2)\not{p}_3u(p_1) \times \bar{u}(p_4)\not{p}_1u(p_3)$$

$$T_3 = \cancel{T_3^4} + (d-4)T_3^{-2\epsilon}$$

$$T_4 = \cancel{T_4^4} + (d-4)T_4^{-2\epsilon}$$

$$T_5 = \cancel{T_5^4} + (d-4)T_5^{-2\epsilon}$$

$$T_6 = \cancel{T_6^4} + (d-4)T_6^{-2\epsilon}$$

$$T_7 = \cancel{T_7^4} + (d-4)T_7^{-2\epsilon}$$

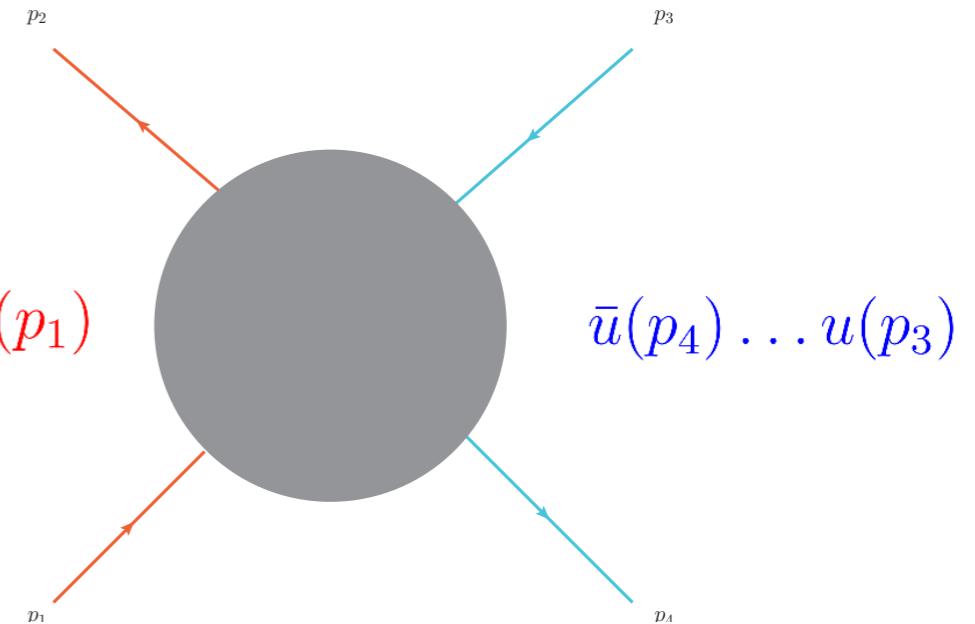
$$T_8 = \cancel{T_8^4} + (d-4)T_8^{-2\epsilon}$$

# $\mathbf{qQ} \rightarrow \mathbf{qQ}$

$$A^X = \sum_{i=1}^N F_i T_i^X$$

$$\bar{u}(p_2) \dots u(p_1)$$

$$\bar{u}(p_4) \dots u(p_3)$$



$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$

$$T_2 = \bar{u}(p_2)\not{p}_3u(p_1) \times \bar{u}(p_4)\not{p}_1u(p_3)$$

$$\overline{T}_3 = (d-4)T_3^{-2\epsilon}$$

$$\overline{T}_4 = (d-4)T_4^{-2\epsilon}$$

$$\overline{T}_5 = (d-4)T_5^{-2\epsilon}$$

$$\overline{T}_6 = (d-4)T_6^{-2\epsilon}$$

$$\overline{T}_7 = (d-4)T_7^{-2\epsilon}$$

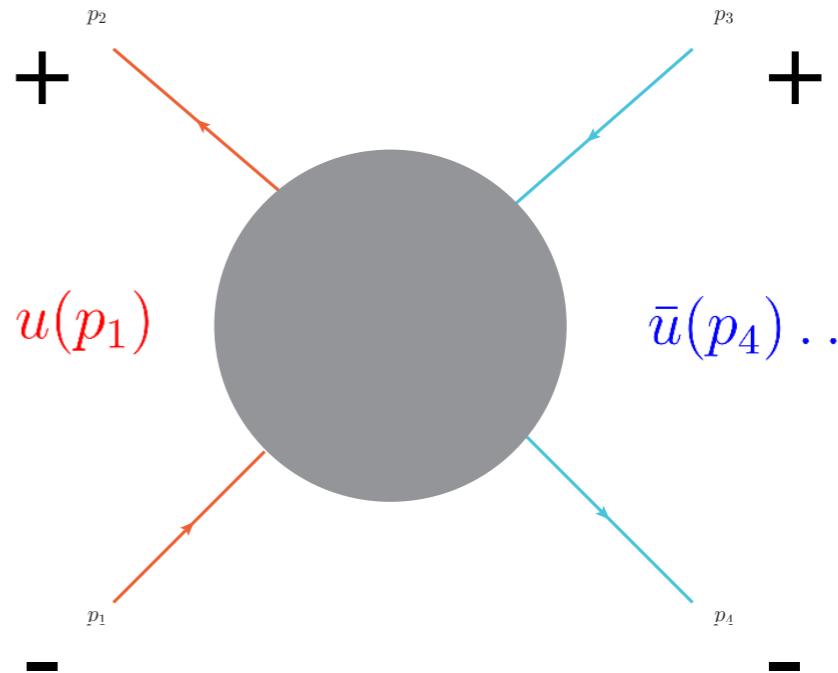
$$\overline{T}_8 = (d-4)T_8^{-2\epsilon}$$

# $qQ \rightarrow qQ$

$$A^X = \sum_{i=1}^N F_i T_i^X$$

$$\bar{u}(p_2) \dots u(p_1)$$

$$\bar{u}(p_4) \dots u(p_3)$$



$$T_1 = \bar{u}(p_2) \gamma_{\mu_1} u(p_1) \times \bar{u}(p_4) \gamma^{\mu_1} u(p_3)$$

$$T_2 = \bar{u}(p_2) \not{p}_3 u(p_1) \times \bar{u}(p_4) \not{p}_1 u(p_3)$$

$$\overline{T}_3 = (d-4) T_3^{-2\epsilon}$$

$$\overline{T}_4 = (d-4) T_4^{-2\epsilon}$$

$$\overline{T}_5 = (d-4) T_5^{-2\epsilon}$$

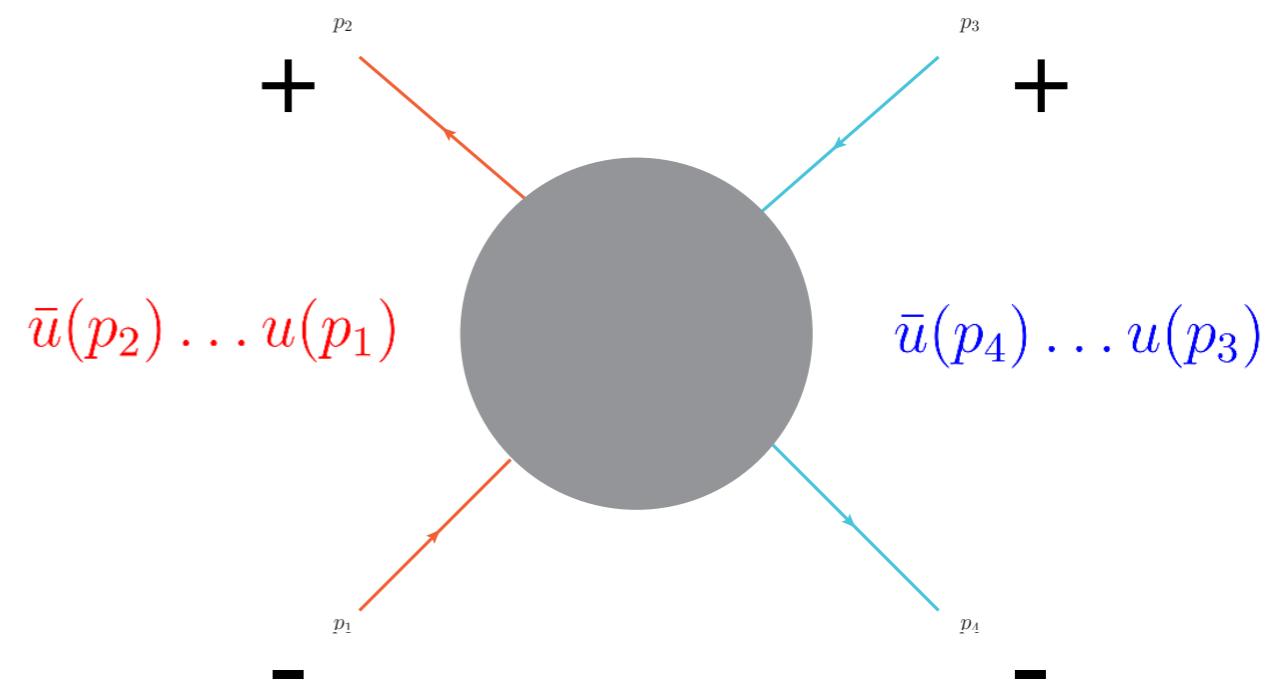
$$\overline{T}_6 = (d-4) T_6^{-2\epsilon}$$

$$\overline{T}_7 = (d-4) T_7^{-2\epsilon}$$

$$\overline{T}_8 = (d-4) T_8^{-2\epsilon}$$

**qQ → qQ**

$$A^X = \sum_{i=1}^N F_i T_i^X$$



$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$

$$T_2 = \bar{u}(p_2)\not{p}_3 u(p_1) \times \bar{u}(p_4)\not{p}_1 u(p_3)$$

$$\overline{T}_3 = (d-4)T_3^{-2\epsilon}$$

$$\overline{T}_4 = (d-4)T_4^{-2\epsilon}$$

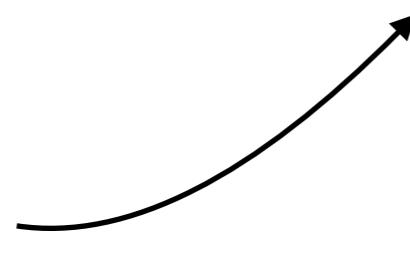
$$\overline{T}_5 = (d-4)T_5^{-2\epsilon}$$

$$\overline{T}_6 = (d-4)T_6^{-2\epsilon}$$

$$\overline{T}_7 = (d-4)T_7^{-2\epsilon}$$

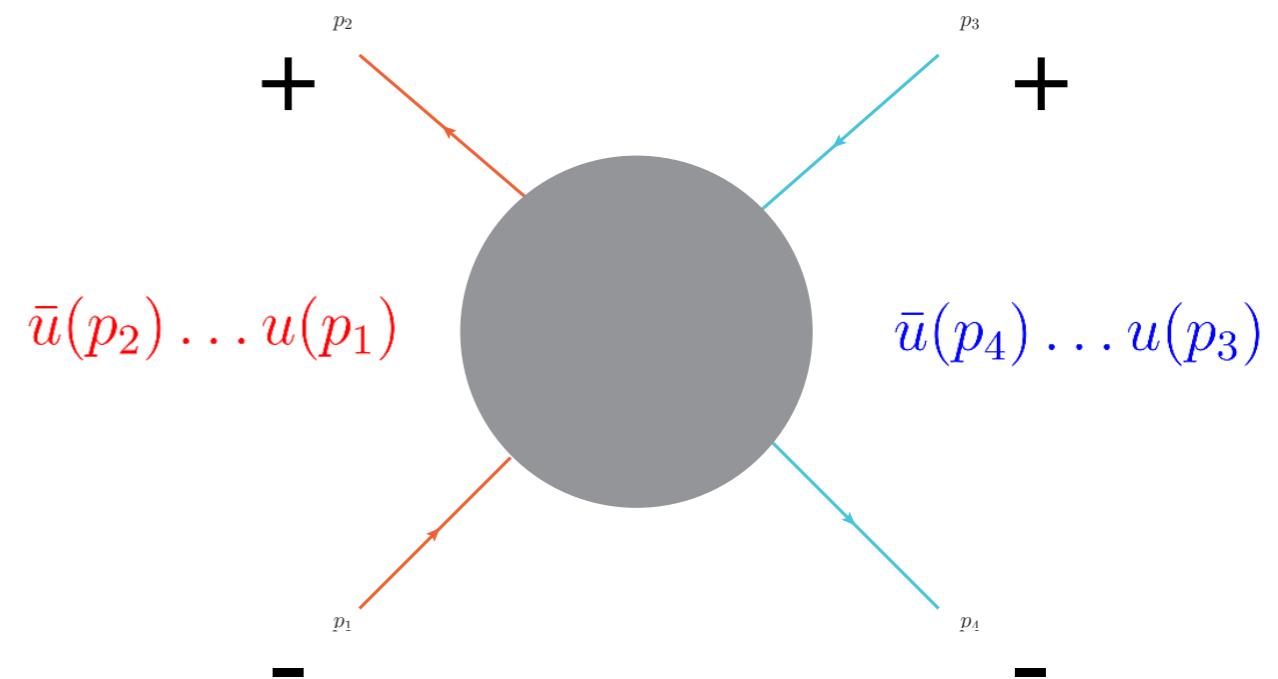
$$\overline{T}_8 = (d-4)T_8^{-2\epsilon}$$

Orthogonal  
&  
zero in d=4 !!



**qQ → qQ**

$$A^X = \sum_{i=1}^N F_i T_i^X$$



$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1) \times \bar{u}(p_4)\gamma^{\mu_1}u(p_3)$$

$$T_2 = \bar{u}(p_2)\not{p}_3u(p_1) \times \bar{u}(p_4)\not{p}_1u(p_3)$$

$$\overline{T}_3 = (d-4)T_3^{-2\epsilon}$$

$$\overline{T}_4 = (d-4)T_4^{-2\epsilon}$$

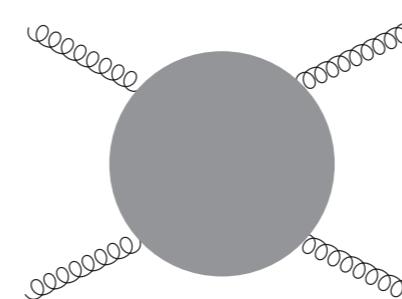
$$\overline{T}_5 = (d-4)T_5^{-2\epsilon}$$

$$\overline{T}_6 = (d-4)T_6^{-2\epsilon}$$

$$\overline{T}_7 = (d-4)T_7^{-2\epsilon}$$

$$\overline{T}_8 = (d-4)T_8^{-2\epsilon}$$

Orthogonal  
&  
zero in d=4 !!



From 138 to 8 tensors!

# Integration by Parts

# Integration by Parts

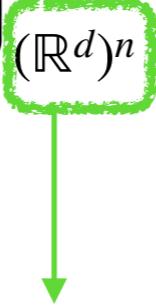
# Integration by Parts

$$I(d, \{p\}) = \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l \quad f(k_1, \dots, k_l, p_1, \dots, p_n)$$

# Integration by Parts

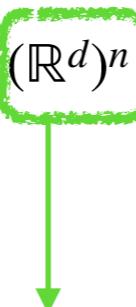
$$I(d, \{p\}) = \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l \quad f(k_1, \dots, k_l, p_1, \dots, p_n)$$

# Integration by Parts

$$I(d, \{p\}) = \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l f(k_1, \dots, k_l, p_1, \dots, p_n)$$


Invariance:

# Integration by Parts

$$I(d, \{p\}) = \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l \quad f(k_1, \dots, k_l, p_1, \dots, p_n)$$


**Invariance:**  $k_i \rightarrow A_{ij} k_j + B_{ih} p_h$

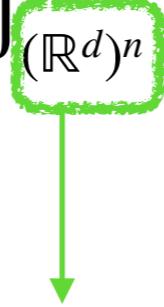
# Integration by Parts

$$I(d, \{p\}) = \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l \quad f(k_1, \dots, k_l, p_1, \dots, p_n)$$


**Invariance:**  $k_i \rightarrow A_{ij} k_j + B_{ih} p_h$

**Generators:**

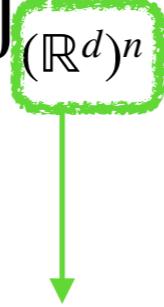
# Integration by Parts

$$I(d, \{p\}) = \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l \quad f(k_1, \dots, k_l, p_1, \dots, p_n)$$


**Invariance:**  $k_i \rightarrow A_{ij} k_j + B_{ih} p_h$

**Generators:**  $k_i \rightarrow k_i + \alpha q_j$

# Integration by Parts

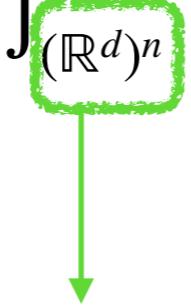
$$I(d, \{p\}) = \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l \ f(k_1, \dots, k_l, p_1, \dots, p_n)$$


**Invariance:**  $k_i \rightarrow A_{ij} k_j + B_{ih} p_h$

**Generators:**  $k_i \rightarrow k_i + \alpha q_j$

$$(q_1, \dots, q_{l+n}) = (k_1, \dots, k_l, p_1, \dots, p_n)$$

# Integration by Parts

$$I(d, \{p\}) = \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l \ f(k_1, \dots, k_l, p_1, \dots, p_n)$$


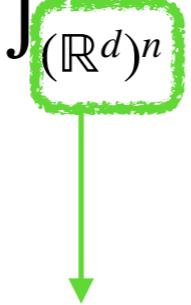
**Invariance:**  $k_i \rightarrow A_{ij} k_j + B_{ih} p_h$

**Generators:**  $k_i \rightarrow k_i + \alpha q_j$

$$(q_1, \dots, q_{l+n}) = (k_1, \dots, k_l, p_1, \dots, p_n)$$

$$I(d, \{p\}) \rightarrow \int_{(\mathbb{R}^d)^n} d^d k_1 \dots (1 + \alpha d \delta_{q_j k_i}) d^d k_i \dots d^d k_l \left( f + \alpha q_j \cdot \frac{\partial}{\partial k_i} f \right)$$

# Integration by Parts

$$I(d, \{p\}) = \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l \ f(k_1, \dots, k_l, p_1, \dots, p_n)$$


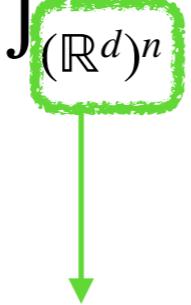
**Invariance:**  $k_i \rightarrow A_{ij} k_j + B_{ih} p_h$

**Generators:**  $k_i \rightarrow k_i + \alpha q_j$

$$(q_1, \dots, q_{l+n}) = (k_1, \dots, k_l, p_1, \dots, p_n)$$

$$I(d, \{p\}) \rightarrow I(d, \{p\}) + \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l \left( \alpha d \delta_{q_j k_i} f + \alpha q_j \cdot \frac{\partial}{\partial k_i} f \right)$$

# Integration by Parts

$$I(d, \{p\}) = \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l \ f(k_1, \dots, k_l, p_1, \dots, p_n)$$


**Invariance:**  $k_i \rightarrow A_{ij} k_j + B_{ih} p_h$

**Generators:**  $k_i \rightarrow k_i + \alpha q_j$

$$(q_1, \dots, q_{l+n}) = (k_1, \dots, k_l, p_1, \dots, p_n)$$

$$I(d, \{p\}) \rightarrow I(d, \{p\}) + \alpha \int_{(\mathbb{R}^d)^n} d^d k_1 \dots d^d k_l \ \frac{\partial}{\partial k_i} \cdot (\mathbf{q}_j f)$$

# Differential Equations

Invariants:  $x_i \rightarrow I(d; x_1, \dots, x_{4n-10})$

$$I(d; x_1, \dots, x_{4n-10}) = \sum c_i M_i(d; x_1, \dots, x_{4n-10})$$

$$d\vec{M} = \mathbf{A}(d; x_i) \cdot \vec{M}$$

Canonical basis:  $d\vec{M}_c = (d - 4) \mathbf{A}_{\log}(x_i) \cdot \vec{M}_c$

$$\begin{pmatrix} \frac{1}{x} - \frac{1}{1+x} & 0 \\ \frac{1}{x} & \frac{1}{1+x} \end{pmatrix} dx = \begin{pmatrix} d\log(x) - d\log(1+x) & 0 \\ d\log(x) & d\log(1+x) \end{pmatrix}$$

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