Towards improving the logarithmic accuracy of PS: Higher-order kernels

arXiv:2109.07496v3

Dissecting the collinear structure of quark splitting at NNLL

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A bird's eye view

- (Semi)-analytic resummation has achieved an impressive accuracy (NNLL and N^{3}LL) over previous decades.

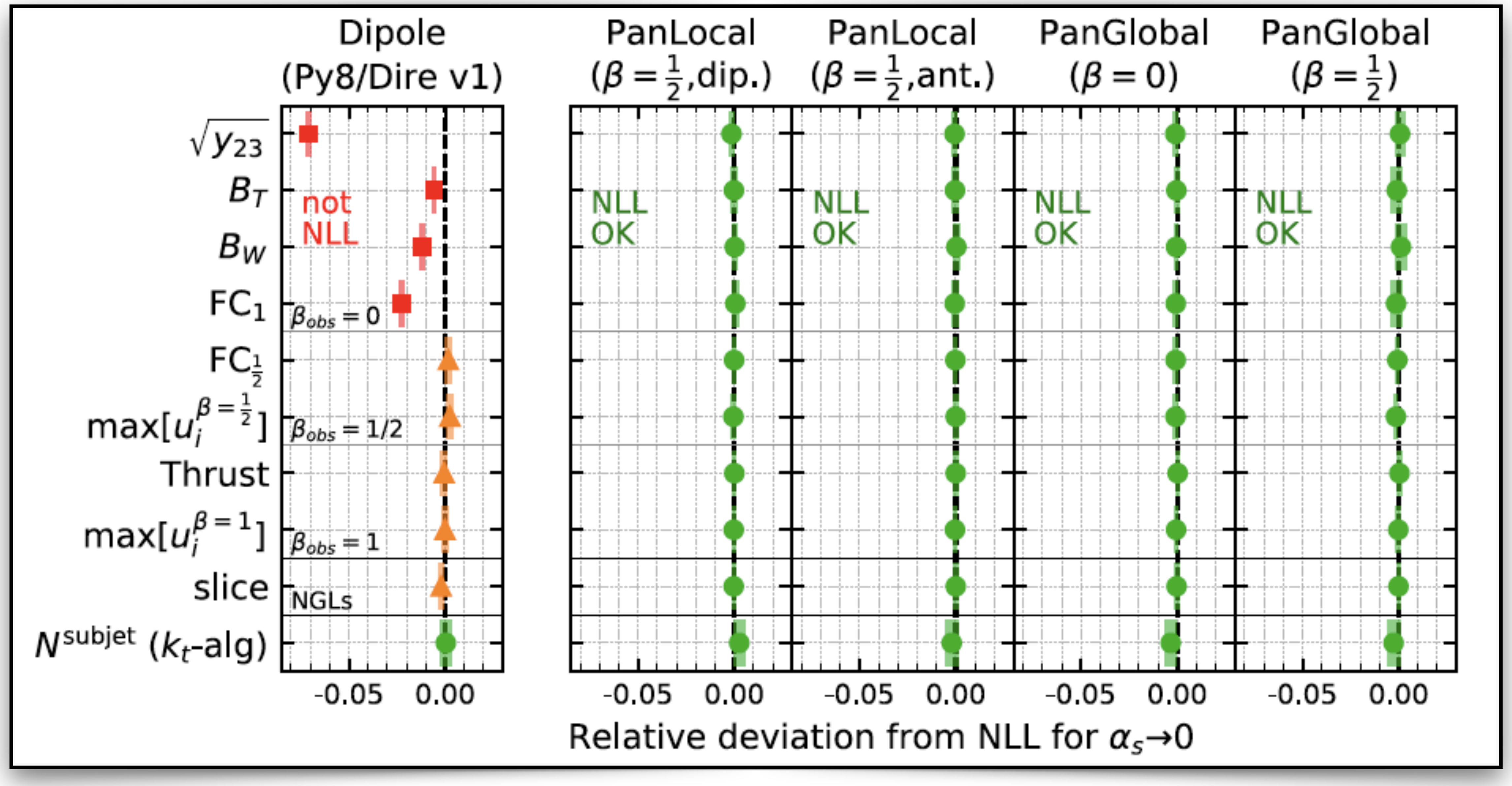
$1-T$	0803.0342,1006.3080,1105.4560
ρ_H	1005.1644
B_T, B_W	1210.0580
C-parameter	1411.6633
EEC	hep-ph/0407241,1708.04093,1801.02627
Angularities	1806.10622,1807.11487
D-parameter	1912.09341

- Parton showers (PS) have not kept up with such progress.
- PS are essential due to their versatility: It is much more efficient to simulate QCD dynamics than to resum a specific observable.

Motivation: Recent progress in NLL accurate PS

- The PanScales family of PS has been able to achieve NLL accuracy for any recursive IRC safe observable:

Dasgupta et. al. (2002.11114), color and spin (2011.10054,2103.16526,2111.01161), G. Salam ``The power and limits of parton showers "https://gsalam.web.cern.ch/gsalam/talks/repo/202109-SLAC-seminar\\-SLAC-panscales-seminar.pdf"



- The crux of this development is simply choosing recoil maps that preserve the correct physical limits required for NLL.

Outline

- What do we need to achieve NNLL in PS?
  - The hard-collinear facet: B_2
- Tool kit: triple-collinear splitting functions
- The physical coupling beyond the soft limit
- Extracting a differential B_2
  - A definition of an effective splitting probability
- Outlook

Look back at NLL

- Over 30 years ago Catani, Marchesini & Webber introduced the notion of a soft physical coupling:

$$\mathrm{d}\mathcal{P}_{\mathrm{sc}} = C_i \frac{\alpha_s^{\mathrm{phys}}}{\pi} \frac{\mathrm{d}k_t^2}{k_t^2} \frac{\mathrm{d}z}{1-z}, \qquad \alpha_s^{\mathrm{phys}} = \alpha_s(k_t^2)\left(1 + K_{\mathrm{CMW}} \frac{\alpha_s(k_t^2)}{2\pi}\right)$$

- The CMW coupling represents the intensity of soft gluon radiation.

$$K_{\mathrm{CMW}} = \left(\frac{67}{18} - \frac{\pi^2}{6}\right) C_A - \frac{10}{9} T_F$$

- For showers that intertwine real and virtual corrections through unitarity, specifying the scheme and scale of the coupling is the sole NLO ingredient to achieve NLL accuracy.

Questions for NNLL PS

- What is the scale of the coupling beyond the soft limit?

$$\boxed{k_t^2 \to k_t^2 * f(z), \quad f(z) = ?}$$

- The inclusive limit of the double-soft function defines the CMW coupling. Can we furnish a commensurate understanding of the triple-collinear splitting functions?

- What is the underlying physics of the coefficient B_2? Can we define a suitable differential version thereof?

- Can we extend the notion of the web beyond the soft limit?

Introduction into B_2

- So what exactly is $B_2^{q/g}$?
- Let us take an example from the transverse momentum distribution in hadronic collisions:

$$\frac{\mathrm{d}\sigma_{ab\to F}}{\mathrm{d}p_t^2} = \frac{1}{2}\int b\,\mathrm{d}b J_0(bp_t)\, W_{ab}^F(s, Q, b)$$

de Florian & Grazzini hep-ph/0108273 (see also the references therein)

- The interesting piece is the function W_{ab}^F, which includes the quark/gluon form factor:

$$S_{q/g}(Q,b) = \exp\left(-\int_{b_0^2/b^2}^{Q^2}\frac{\mathrm{d}q^2}{q^2}\left[A_{q/g}(\alpha_s)\ln\frac{Q^2}{q^2} + B_{q/g}(\alpha_s)\right]\right)$$

Introduction into B_2

- Each function has a perturbative expansion. The A functions has a soft origin, while the B function has a hard-collinear origin.

$$A_{q/g} = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n A_{(n)}^{q/g}, \quad B_{q/g} = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n B_{(n)}^{q/g}$$

- Let us focus on the B series. Going back to direct space, one finds a hard-collinear logarithm:

$$\left(\frac{\alpha_s}{2\pi}\right) B_1^{q/g} \quad || \quad \left(\frac{\alpha_s}{2\pi}\right)^2 B_2^{q/g}$$

- This talk is about B_2^q and a suitably defined differential version $\mathscr{B}_2^q(z)$.

Introduction into B_2

- What do we know about the structure of B_2^q?
- In e^+e^-, there exists a complete framework to resum any recursive IRC safe observable to NNLL accuracy - ARES
- For any observable, we have:

Banfi, BKE & Monni 1807.11487, Banfi et. al. 1412.2126

$$\boxed{B_2^q = -\gamma_q^{(2)} + C_F b_0 X_v, \quad b_0 = \frac{11}{6}C_A - \frac{2}{3}T_R n_f}$$

See also de Florian & Grazzini hep-ph/0407241, Davies & Stirling Nucl. Phys. B 244 (1984)

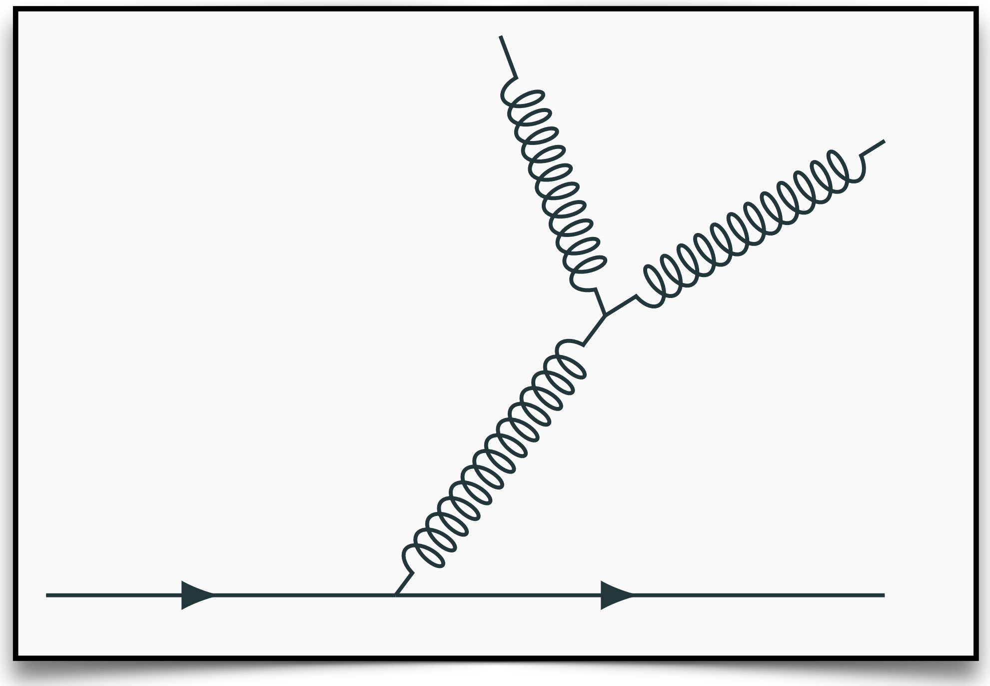
- We have two pieces. An observable dependent constant, X_v, that comes multiplied by b_0. The other pieces, $\gamma_q^{(2)}$, is universal and represents the endpoint contribution, i.e. $\delta(1-x)$, to the NLO non-singlet DGLAP kernel obtained from sum rules.

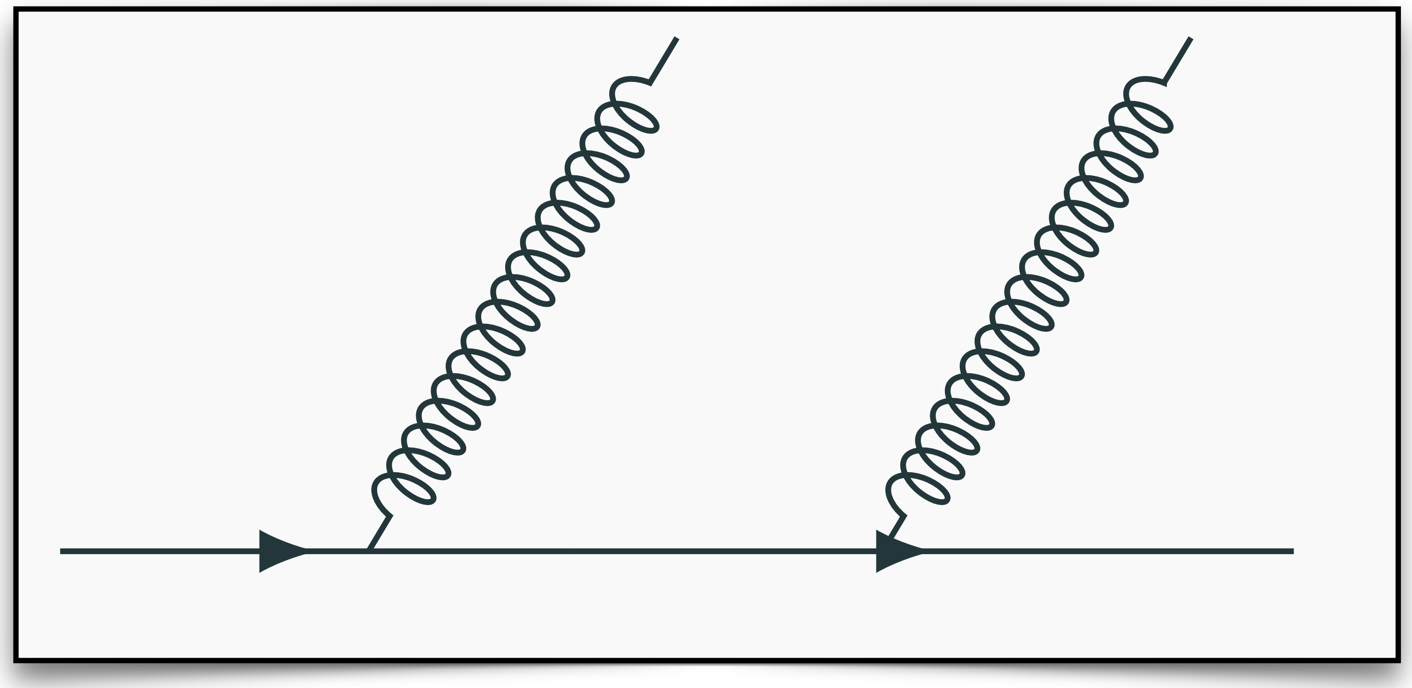
Triple-collinear splitting functions

Catani & Grazzini hep-ph/9810389

- At NLO, a nearly off-shell quark can emit either two gluons or a $q\bar{q}$ pair.

  - $q \to g_1 g_2 q_3$





- These two graphs are squared, summed over spin and colour.

- We are interested in the limit when all pairwise angles are small $\theta_{ij} \to 0$.

Triple-collinear splitting functions

- Therefore, we end up with an abelian channel, C_F^2 , and a non-abelian channel, $C_F C_A$.

$$\boxed{\langle \hat{P}_{g_1 g_2 q_3} \rangle = C_F^2 \langle \hat{P}^{(\mathrm{ab})}_{g_1 g_2 q_3} \rangle + C_F C_A \langle \hat{P}^{(\mathrm{nab})}_{g_1 g_2 q_3} \rangle}$$

- These are functions of the invariant masses, $s_{ij} \simeq z_i z_j \theta_{ij}^2$, where z_i is the light-cone momentum fraction of parton i.

$$\begin{aligned}
\langle \hat{P}^{(\mathrm{ab})}_{g_1 g_2 q_3} \rangle = & \left\{ \frac{s_{123}^2}{2 s_{13} s_{23}} z_3 \left[\frac{1+z_3^2}{z_1 z_2} - \epsilon \frac{z_1^2 + z_2^2}{z_1 z_2} - \epsilon(1+\epsilon) \right] \right. \\
& + \frac{s_{123}}{s_{13}} \left[\frac{z_3(1-z_1) + (1-z_2)^3}{z_1 z_2} + \epsilon^2 (1+z_3) - \epsilon (z_1^2 + z_1 z_2 + z_2^2) \frac{1-z_2}{z_1 z_2} \right] \\
& \left. + (1-\epsilon) \left[\epsilon - (1-\epsilon) \frac{s_{23}}{s_{13}} \right] \right\} + (1 \leftrightarrow 2)
\end{aligned}$$

Triple-collinear splitting functions

- The non-abelian channel is much more involved but features a nice property: angular ordering

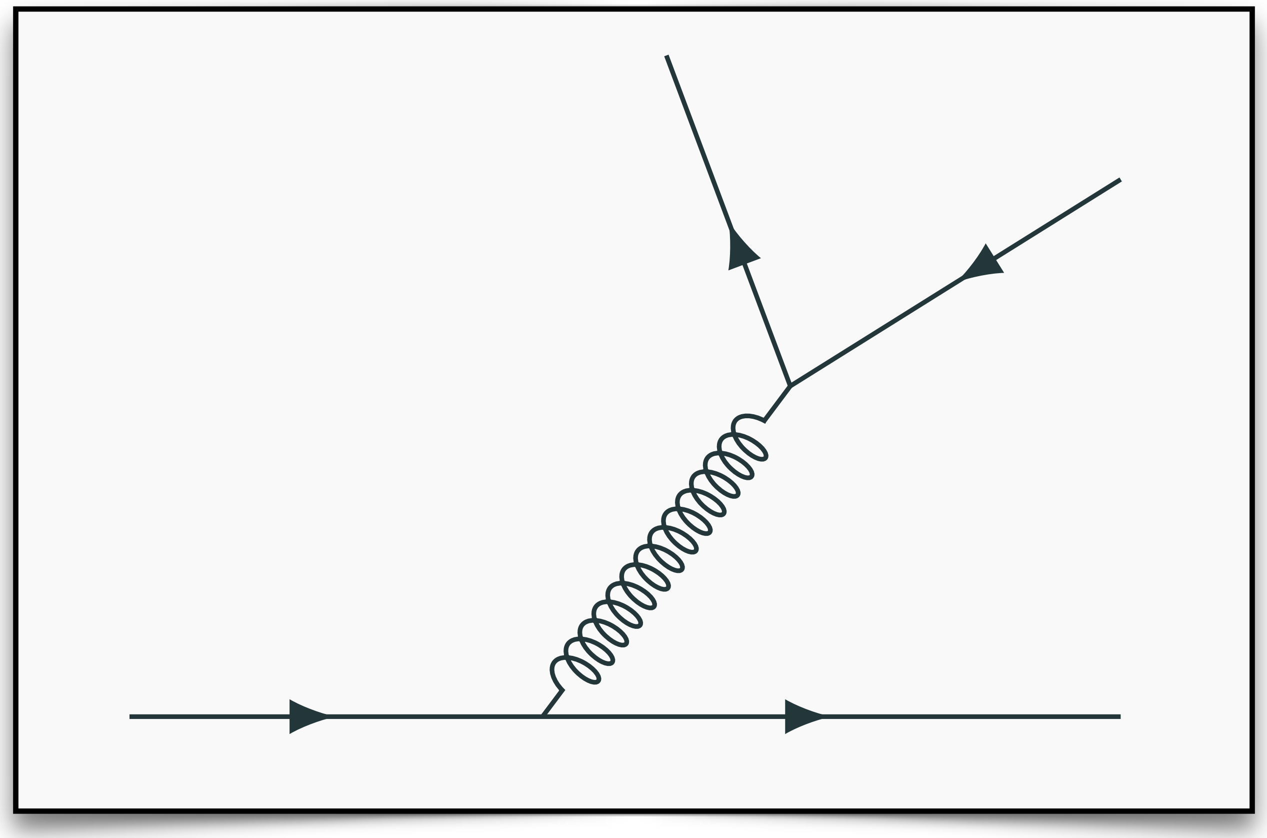
$$
\begin{aligned}
\langle \hat{P}^{(\mathrm{nab})}_{g_1 g_2 q_3} \rangle = & \Bigg\{ (1-\epsilon)\left(\frac{t_{12,3}^2}{4 s_{12}^2} + \frac{1}{4} - \frac{\epsilon}{2} \right) \\
& + \frac{s_{123}^2}{2 s_{12} s_{13}} \left[\frac{(1-z_3)^2(1-\epsilon) + 2z_3}{z_2} + \frac{z_2^2(1-\epsilon) + 2(1-z_2)}{1-z_3} \right] \\
& - \frac{s_{123}^2}{4 s_{13} s_{23}} z_3 \left[\frac{(1-z_3)^2(1-\epsilon) + 2z_3}{z_1 z_2} + \epsilon(1-\epsilon) \right] \\
& + \frac{s_{123}}{2 s_{12}} \left[(1-\epsilon) \frac{z_1(2 - 2z_1 + z_1^2) - z_2(6 - 6z_2 + z_2^2)}{z_2(1-z_3)} + 2\epsilon \frac{z_3(z_1 - 2z_2) - z_2}{z_2(1-z_3)} \right] \\
& + \frac{s_{123}}{2 s_{13}} \Bigg[(1-\epsilon) \frac{(1-z_2)^3 + z_3^2 - z_2}{z_2(1-z_3)} - \epsilon \left(\frac{2(1-z_2)(z_2 - z_3)}{z_2(1-z_3)} - z_1 + z_2 \right) \\
& - \frac{z_3(1-z_1) + (1-z_2)^3}{z_1 z_2} + \epsilon(1-z_2)\left(\frac{z_1^2 + z_2^2}{z_1 z_2} - \epsilon \right) \Bigg] \Bigg\} + (1 \leftrightarrow 2)
\end{aligned}
$$

- The only collinear pole is when the angle between the gluon pair, θ_{12}, goes to zero.

Triple-collinear splitting functions

- The quark can also split to a quark plus a $q\bar{q}$ pair, of identical (different) flavour.

  - ✪ $q \to q_1 \bar{q}_2 q_3$
  - ✪ $q \to q_1' \bar{q}_2' q_3$



- Summing over flavour, we find two different structures:

$$\sum_f \langle \hat{P}_{q_1^f \bar{q}_2^f q_3} \rangle = n_f \langle \hat{P}_{q_1' \bar{q}_2' q_3} \rangle + \langle \hat{P}^{(\mathrm{id})}_{q_1 \bar{q}_2 q_3} \rangle$$

Triple-collinear splitting functions

- The two structures have distinct properties (as well as colour factors)

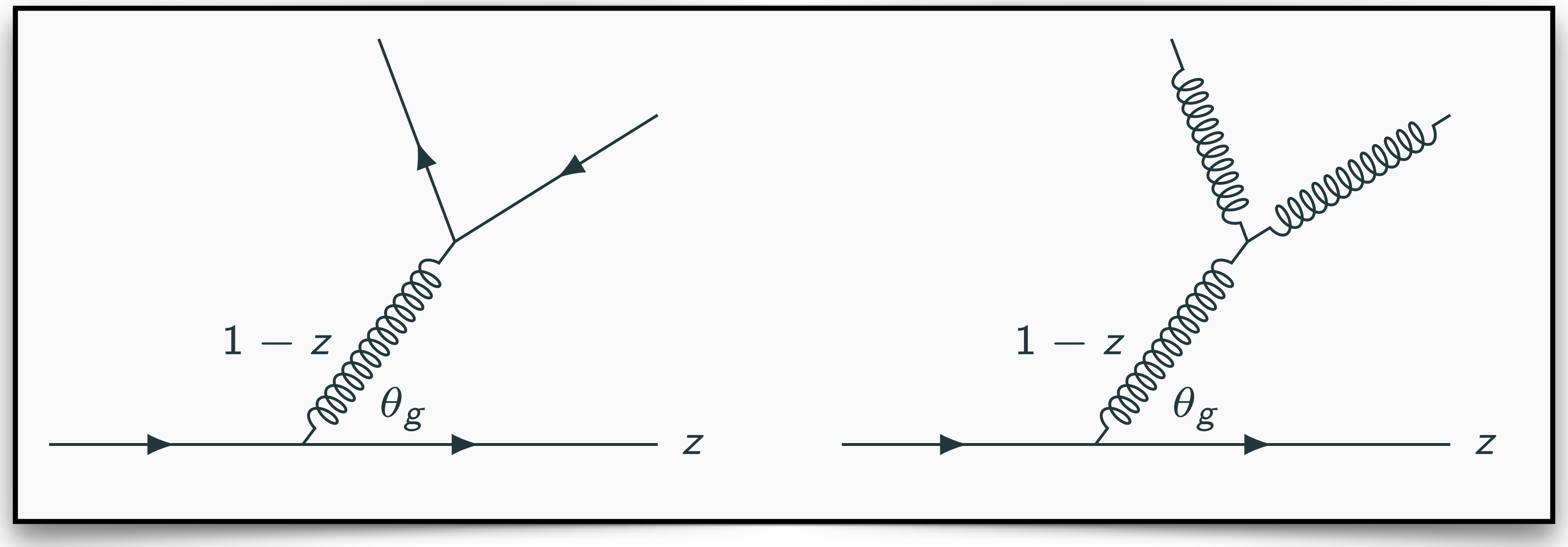
$$\langle \hat{P}_{q_1'\bar{q}_2'q_3} \rangle = \frac{1}{2}\, C_F\, T_R\, \frac{s_{123}}{s_{12}} \left[-\frac{t_{12,3}^2}{s_{12}s_{123}} + \frac{4z_3 + (z_1 - z_2)^2}{z_1 + z_2} + (1 - 2\epsilon)\left(z_1 + z_2 - \frac{s_{12}}{s_{123}} \right) \right]$$

$$\begin{aligned}
\langle \hat{P}^{(\mathrm{id})}_{q_1\bar{q}_2q_3} \rangle = C_F \left(C_F - \frac{1}{2} C_A \right) \Bigg\{ & (1-\epsilon)\left(\frac{2s_{23}}{s_{12}} - \epsilon \right) + \frac{s_{123}}{s_{12}} \left[\frac{1+z_1^2}{1-z_2} - \frac{2z_2}{1-z_3} \right. \\
& \left. - \epsilon \left(\frac{(1-z_3)^2}{1-z_2} + 1 + z_1 - \frac{2z_2}{1-z_3} \right) - \epsilon^2 (1 - z_3) \right] \\
& - \frac{s_{123}^2}{s_{12}s_{13}} \frac{z_1}{2} \left[\frac{1+z_1^2}{(1-z_2)(1-z_3)} - \epsilon \left(1 + 2\frac{1-z_2}{1-z_3} \right) - \epsilon^2 \right] \Bigg\} + (2 \leftrightarrow 3)
\end{aligned}$$

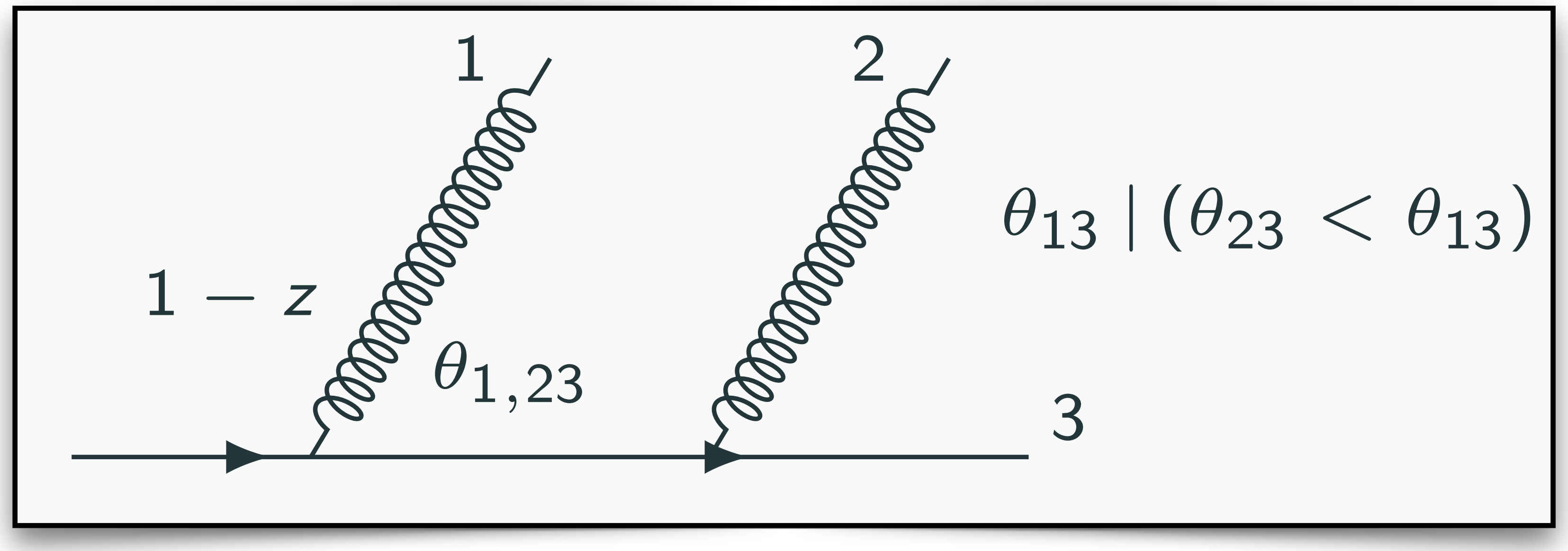
- The “non-identical” functions has a pole as θ_{12} goes to zero, while the “identical” function is fully finite.

Road map

- What variables do we fix?
- The goal is to retain differential information over “a chosen angle” and “a suitable energy”.
- These choices must be IRC safe.
  - ✪ Gluon decay



  - ✪ Gluon emission

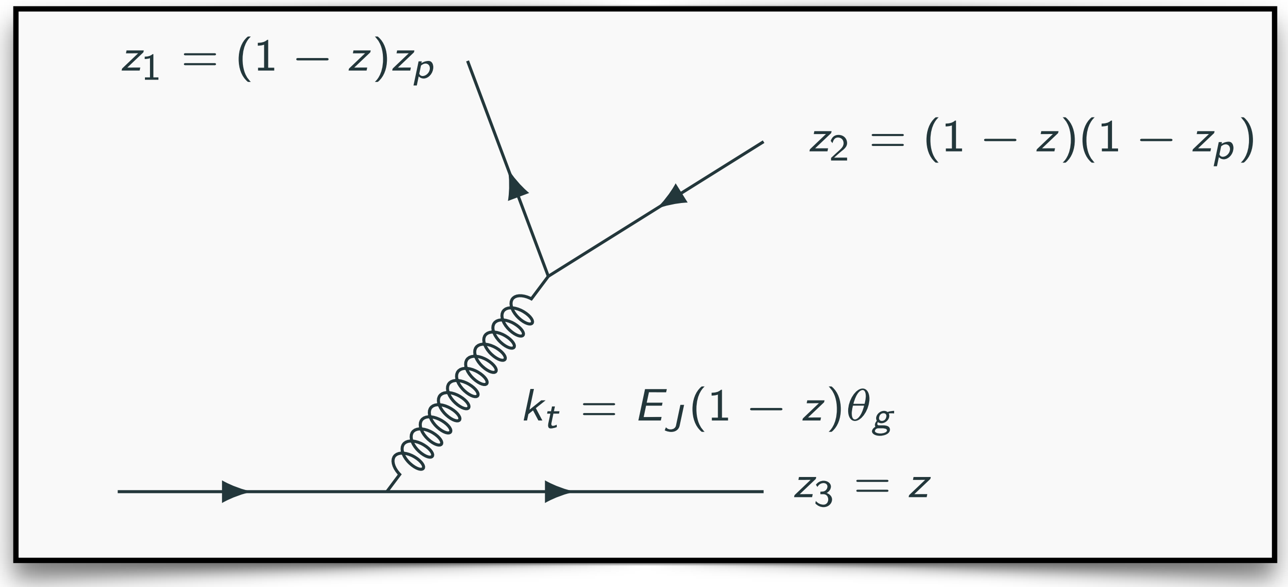


Gluon decay: web variables

- It is essential to perform an analytic computation to uncover the scale of the coupling. This could be achieved by parameterising the triple-collinear phase space in terms of the "web variables".

$$\mathrm{d}\Phi^{\mathrm{web}}_{1\to 3} = \frac{(4\pi)^{2\epsilon}}{256\pi^4}\,\frac{2z^{1-2\epsilon}dz}{1-z}\,\frac{1}{\Gamma(1-\epsilon)}\,\frac{d^{2-2\epsilon}k_\perp}{\Omega_{2-2\epsilon}}\,\frac{ds_{12}}{(s_{12})^\epsilon}\,\frac{dz_p}{(z_p(1-z_p))^\epsilon}\,\frac{1}{\Gamma(1-\epsilon)}\,\frac{d\Omega_{2-2\epsilon}}{\Omega_{2-2\epsilon}}$$

- The meaning of the different variables is as follows:



- The invariants, s_{13} & s_{23}, are written in terms of these variables.

The θ_g distribution: $C_F T_R n_f$

- Using the web variables, we get:

$$\boxed{\left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\theta_g^2\,dz}\right)^{C_F T_R n_f} = C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 z^{-3\epsilon}\left((1-z)^2\theta_g^2\right)^{-2\epsilon} \left(-\frac{2}{3\epsilon}p_{qq}(z,\epsilon) - \frac{10}{9}p_{qq}(z) - \frac{2}{3}(1-z)\right)}$$

- Due to the angular ordering property of the splitting function:

$$\boxed{\text{max.}\{s_{12}\} \to \infty}$$

- The last ingredient is the one-loop virtual corrections to the $1 \to 2$ splitting function

$$\boxed{\left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}_{\text{virt.}}}{d\theta_g^2\,dz}\right)^{C_F T_R n_f} = C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 z^{-2\epsilon}(1-z)^{-2\epsilon}\left(\theta_g^2\right)^{-\epsilon}\left(\frac{2}{3\epsilon}p_{qq}(z,\epsilon)\right)}$$

The θ_g distribution: $C_F T_R n_f$

- The double-differential distribution then reads:

$$\boxed{\left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\theta_g^2\, dz}\right)^{C_F T_R n_f} = C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{1+z^2}{1-z}\left(\frac{2}{3}\ln\left(z(1-z)^2\theta_g^2\right) - \frac{10}{9}\right) - \frac{2}{3}(1-z)\right)}$$

- To glean the physics, we compute the jet mass distribution $\rho = s_{123}/E^2$:

$$\boxed{\left(\frac{\rho}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\rho\, dz}\right)^{C_F T_R n_f} = C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{1+z^2}{1-z}\left(\frac{2}{3}\ln\left((1-z)\rho\right) - \frac{10}{9}\right) - \frac{2}{3}(1-z)\right)}$$

- We can move between different distributions using the LO relation → scale of the coupling

$$\boxed{\rho = z(1-z)\theta_g^2}$$

Extracting $\mathscr{B}_2^q(z)$: $C_F T_R n_f$

- To extract the NNLL structure, we subtract the LL and NLL (soft-enhanced) pieces:

$$C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 \left[\frac{2}{1-z}\left(\frac{2}{3}\ln\left((1-z)^2\theta_g^2\right) - \frac{10}{9}\right) - \frac{2}{3}(1+z)\ln\theta_g^2\right]$$

- This defines a purely collinear object (no singularity as $z \to 1$):

$$\mathcal{B}_2^{q,n_f}(z;\theta_g^2) = \left(\frac{1+z^2}{1-z}\,\frac{2}{3}\ln z - (1+z)\left(\frac{2}{3}\ln(1-z)^2 - \frac{10}{9}\right) - \frac{2}{3}(1-z)\right)$$

- The integral over $z \in (0,1)$ yields the resummation coefficient:

$$B_2^{q,\theta_g^2,n_f} = C_F T_R n_f \left(\frac{\alpha_s}{2\pi}\right)^2 \int_0^1 dz\, \mathcal{B}_2^{q,n_f}(z;\theta_g^2) = -\gamma_q^{(2,n_f)} + C_F b_0^{(n_f)} X_{\theta_g^2}$$

✪ $X_\rho = \dfrac{\pi^2}{3} - \dfrac{7}{2}$

✪ $X_{\theta_g^2} = \dfrac{2\pi^2}{3} - \dfrac{13}{2}$

The θ_g distribution: $C_F(C_F - C_A/2)$

- This colour structure is finite.

$$\left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\theta_g^2\, dz}\right)^{(\text{id.})} = C_F\left(C_F - \frac{C_A}{2}\right)\left(\frac{\alpha_s}{2\pi}\right)^2 \left[\left(4z - \frac{7}{2}\right) + \frac{5z^2-2}{2(1-z)}\ln z + \frac{1+z^2}{1-z}\left(\frac{\pi^2}{6} - \ln z \ln(1-z) - \mathrm{Li}_2(z)\right)\right]$$

- This channel does not contribute to the LL or NLL structures.

$$\mathcal{B}_2^{q,(\text{id.})}(z) = \left(4z - \frac{7}{2}\right) + \frac{5z^2-2}{2(1-z)}\ln z + \frac{1+z^2}{1-z}\left(\frac{\pi^2}{6} - \ln z \ln(1-z) - \mathrm{Li}_2(z)\right)$$

- We have a contribution to the B_2^q in the C_F^2 & $C_F C_A$ colour factors.

$$\int_0^1 dz\, \mathcal{B}_2^{q,(\text{id.})}(z) = \frac{13}{4} - \frac{\pi^2}{2} + 2\zeta_3$$

The θ_g distribution: $C_F C_A$

- The web variables allows for an analytic computation:

$$\left(\frac{\rho}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\rho\, dz}\right)^{\text{nab.}} = C_F C_A \left(\frac{\alpha_s}{2\pi}\right)^2 \Bigg[\left(\frac{1+z^2}{1-z}\right)\Bigg(-\frac{11}{6}\ln\left(\rho\,(1-z)\right) + \frac{67}{18} - \frac{\pi^2}{6} + \ln^2 z + \mathrm{Li}_2\left(\frac{z-1}{z}\right) + 2\,\mathrm{Li}_2(1-z)\Bigg) + \frac{3}{2}\frac{z^2\ln z}{1-z} + \frac{1}{6}(8-5z)\Bigg]$$

- We can easily move to the θ_g distribution using the LO relation, remove the LL and NLL and construct:

$$\mathcal{B}_2^{q,(\text{nab.})}(z;\theta_g^2) = -\frac{1+z^2}{1-z}\frac{11}{6}\ln z + (1+z)\left(\frac{11}{6}\ln(1-z)^2 - \frac{67}{18} + \frac{\pi^2}{6}\right) + \frac{11}{6}(1-z) + \frac{2z-1}{2} + \frac{1+z^2}{1-z}\left(\ln^2 z + \mathrm{Li}_2\left(\frac{z-1}{z}\right) + 2\mathrm{Li}_2(1-z)\right)$$

The θ_g distribution: $C_F C_A$

- To construct the $C_F C_A$ portion of B_2^q, we must include the interference term:

$$\boxed{\begin{aligned} B_2^{q,\theta_g^2,C_F C_A} &= C_F C_A \left(\frac{\alpha_s}{2\pi}\right)^2 \int_0^1 dz \left(\mathcal{B}_2^{q,(\mathrm{nab.})}(z;\theta_g^2) - \frac{1}{2}\mathcal{B}_2^{q,(\mathrm{id.})}(z;\theta_g^2)\right) \\ &= -\gamma_q^{(2,C_A)} + C_F b_0^{(C_A)} X_{\theta_g^2} \end{aligned}}$$

- The same story holds for the jet mass distribution $X_{\theta_g^2} \to X_\rho$

Take home 1: We can define a suitably differential version of the resummation coefficient B_2^q.

Take home 2: We can move from the "kinematic" distribution to any other observable using the LO relation.

Physical coupling beyond soft limit

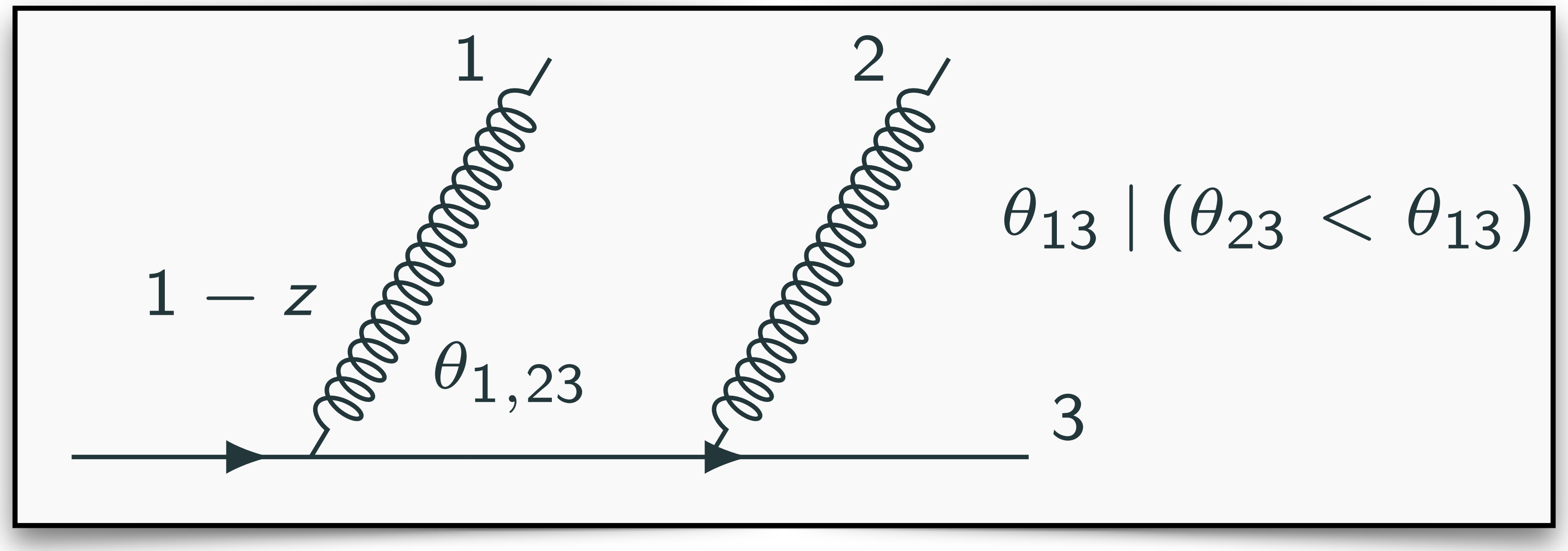
- Let us combine the “gluon decay” results with the LO distribution:

$$\left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma}{d\theta_g^2\,dz}\right)^{\text{tot.}} = \frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(1)}}{d\theta_g^2\,dz} + \left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\theta_g^2\,dz}\right)^{C_F T_R n_f} + \left(\frac{\theta_g^2}{\sigma_0}\frac{d^2\sigma^{(2)}}{d\theta_g^2\,dz}\right)^{\text{nab.}}$$
$$= C_F\, p_{qq}(z)\left[\frac{\alpha_s\left(E^2\right)}{2\pi} + \left(\frac{\alpha_s}{2\pi}\right)^2\left(-b_0 \ln\left((1-z)^2\theta_g^2\right) + K_{\text{CMW}}\right) - \left(\frac{\alpha_s}{2\pi}\right)^2 b_0 \ln z\right]$$
$$+ C_F\, b_0\left(\frac{\alpha_s}{2\pi}\right)^2 (1-z) + \left(\frac{\alpha_s}{2\pi}\right)^2 R^{\text{nab.}}(z)$$

- Red: the CMW coupling (multiplying the full p_{qq}).
- Blue: scale of coupling is actually $z\, k_t^2$.
- Orange: could be absorbed in a new “scheme” for the coupling.
- Black: a remainder function, with $C_F C_A$ colour factor, that encodes the residual dynamics.

The abelian channel

- The physics of “gluon emission” is quite distinct from “gluon decay”



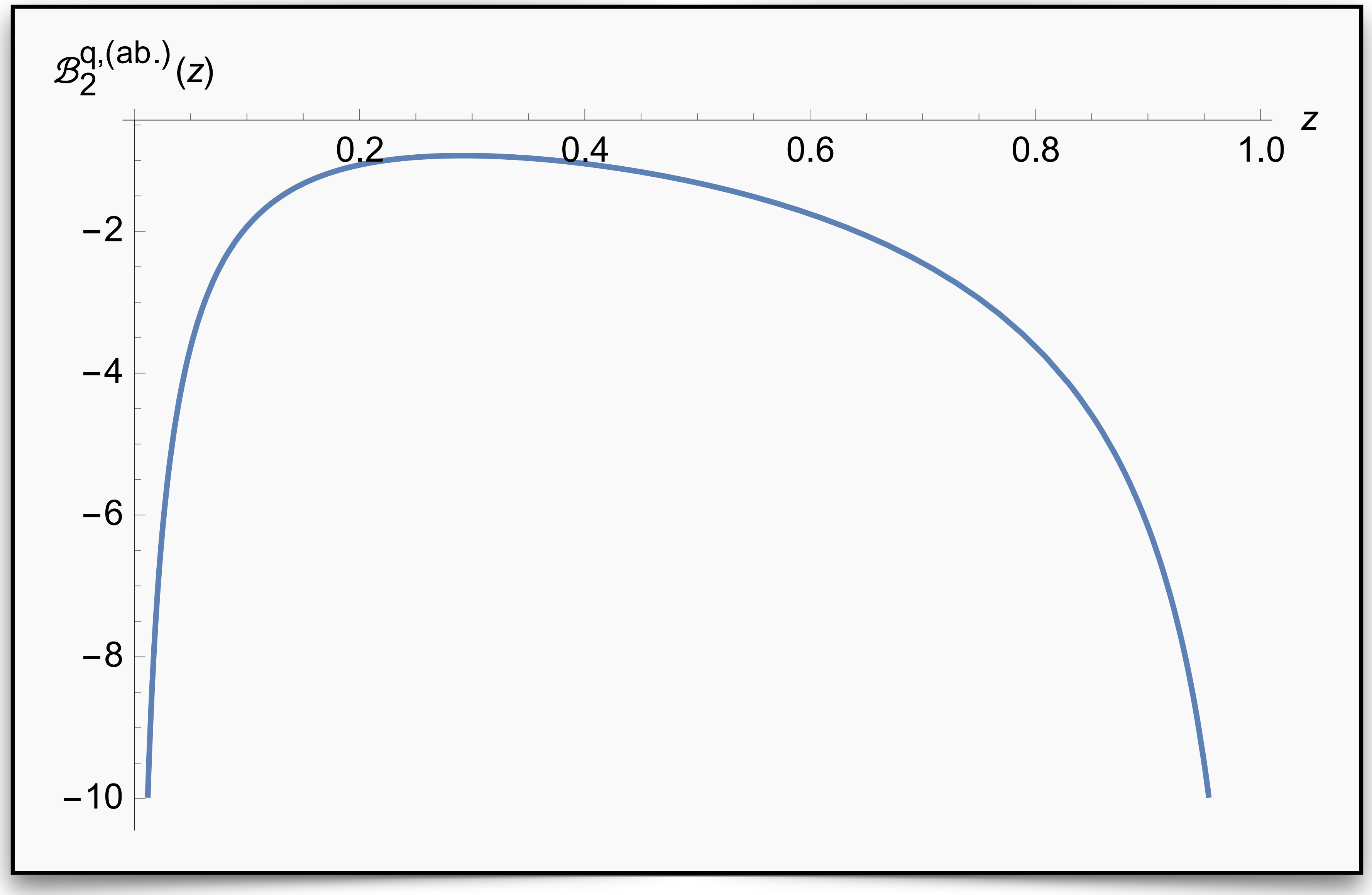
- All LL+NLL dynamics is encoded in the “strongly-ordered” in angle limit: $\theta_{23} \ll \theta_{13}$

- Collinear factorisation → iterated LO splittings

$$\mathscr{B}_2^{q,(ab.)}(z;\theta^2) = \left(\frac{\theta^2}{\sigma_0}\frac{d^2\sigma}{dz\,d\theta^2}\right)^{d-r} - \left(\frac{\theta^2}{\sigma_0}\frac{d^2\sigma}{dz\,d\theta^2}\right)^{s-o} + \left(\frac{\theta^2}{\sigma_0}\frac{d^2\sigma}{dz\,d\theta^2}\right)^{r-v}$$

The abelian channel

- The phase space cut renders the computation non analytic: elliptic integrals.
- We can still cast the result as a 1-fold integral:



- We can also use the PSLQ algorithm to fit the integral: $\boxed{\int_0^1 \mathrm{d}z\, \mathcal{B}_2^{q,(\mathrm{ab.})}(z;\theta^2) = \pi^2 - 8\zeta(3) - \frac{29}{8}}$

Outlook

- One practical side of this work is the ability to provide resummed predictions for groomed observables *á la* ARES.

  "Direct QCD resummation for groomed jet observables at NNLL+NLO" arXiv: 22xx.xxxxx

  M. Dasgupta, BKE, J. Helliwell

- The collinear dynamics of gluon jets shows many interesting features.

  To appear soon, M. Dasgupta, BKE, P. F. Monni

- The inclusion of the results in PS algorithm is the ultimate goal.