

N³LO splitting functions and the OPE

High Precision for Hard Processes
Giulio Falcioni

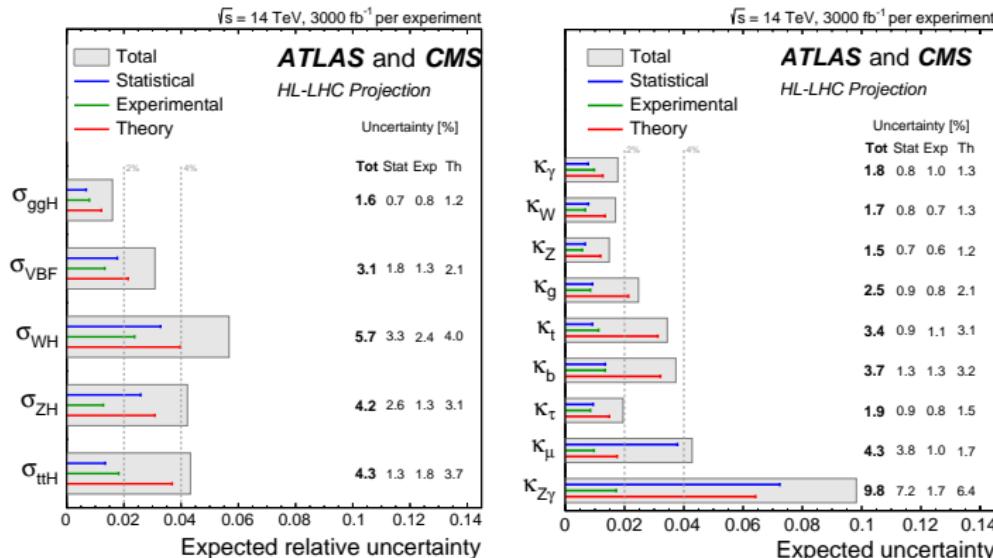
Based on **JHEP 05 (2022) 177**; arxiv:[2203.11181](https://arxiv.org/abs/2203.11181)
in collaboration with Franz Herzog



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1% uncertainties at the LHC

Projections for HL-LHC: experimental uncertainties $\simeq 1\%$
 Error budget dominated by theory.



Vol. 7/2019 of CERN Yellow Reports: Monograph.

LHC phenomenology to N³LO in QCD

N³LO corrections are *crucial* to reach 1% accuracy.

	Q [GeV]	$\delta\sigma^{\text{N}^3\text{LO}}$	$\delta(\text{scale})$	$\delta(\text{PDF-TH})$
$gg \rightarrow \text{Higgs}$	m_H	3.5%	+0.21% -2.37%	$\pm 1.2\%$
$b\bar{b} \rightarrow \text{Higgs}$	m_H	-2.3%	+3.0% -4.8%	$\pm 2.5\%$
NCDY	30	-4.8%	+1.53% -2.54%	$\pm 2.8\%$
	100	-2.1%	+0.66% -0.79%	$\pm 2.5\%$
CCDY(W^+)	30	-4.7%	+2.5% -1.7%	$\pm 3.2\%$
	150	-2.0%	+0.5% -0.5%	$\pm 2.1\%$
CCDY(W^-)	30	-5.0%	+2.6% -1.6%	$\pm 3.2\%$
	150	-2.1%	+0.6% -0.5%	$\pm 2.13\%$

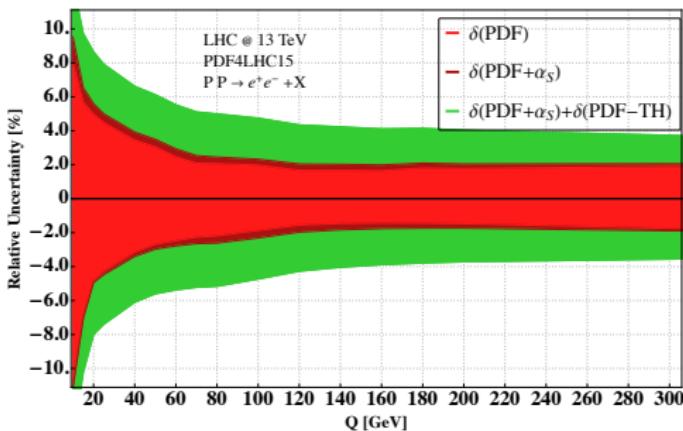
Numbers from J. Baglio, C. Duhr, B. Mistlberger, R. Szafron 2209.06138

$$\delta\sigma^{\text{N}^3\text{LO}} = \frac{\sigma^{\text{N}^3\text{LO}} - \sigma^{\text{NNLO}}}{\sigma^{\text{NNLO}}}$$

Missing N³LO PDFs

Additional errors on N³LO predictions from missing N³LO PDFs

$$\delta(\text{PDF} - \text{TH}) = \frac{1}{2} \left| \frac{\sigma^{\text{NNLO, NNLO-PDFs}}(Q^2) - \sigma^{\text{NNLO, NLO-PDFs}}(Q^2)}{\sigma^{\text{NNLO, NNLO-PDFs}}(Q^2)} \right| \simeq O(2\% - 3\%)$$



Neutral Current DY at N3LO ([Duhr, Mistlberger JHEP 03 \(2022\) 116](#)).

Scale evolution of the PDFs

N3LO PDFs → 4-loop evolution kernels

$$\mu^2 \frac{d}{d\mu^2} f_i(x, \mu^2) = \int_x^1 \frac{dy}{y} P_{ij}(\alpha_s, y) f_j\left(\frac{x}{y}, \mu^2\right), \quad i = g, q, \bar{q}$$

$$P_{ij}(\alpha_s, x) = \underbrace{a P_{ij}^{(0)}}_{\text{LO}} + \underbrace{a^2 P_{ij}^{(1)}}_{\text{NLO}} + \underbrace{a^3 P_{ij}^{(2)}}_{\text{NNLO}} + \underbrace{a^4 P_{ij}^{(3)}}_{\text{N}^3\text{LO}}, \quad a = \frac{\alpha_s}{4\pi}$$

Formulation in Mellin space

$$\gamma_{ij}^{(N)} = - \int_0^1 x^{N-1} P_{ij}(\alpha_s, x) dx$$

Strategies to compute $N^3\text{LO}$ splitting functions

- Expansion of the DIS Compton amplitude
(Gorishnii, Larin, Tkachov 1983)
 - Calculation of NNLO evolution (Moch, Vermaseren, Vogt 2004)
 - Provided a subset $N = 2 \dots 8$ at $N^3\text{LO}$
(Moch, Ruijl, Ueda, Vermaseren, Vogt 2021)
 - Going beyond may be too expensive with this method.
- **Operator Product Expansion**
 - Complete $N^3\text{LO}$ evolution of Non Singlet (NS) PDFs
(Moch, Ruijl, Ueda, Vermaseren, Vogt 2017)
 - NS and polarised singlet at NNLO
(Blümlein, Marquard, Schneider, Schönwald 2021)²

Warning!

Conceptual problem for gluon and quark singlet PDFs!

Pitfalls in the singlet sector

- Discrepancies in the evolution of the gluon PDF at **2 loops**
 - **Feynman gauge** calculation by different groups (1977-1980) in agreement with each other
 - Another approach in **axial gauge** gives a **different** result (Curci, Furmanski, Petronzio 1980)

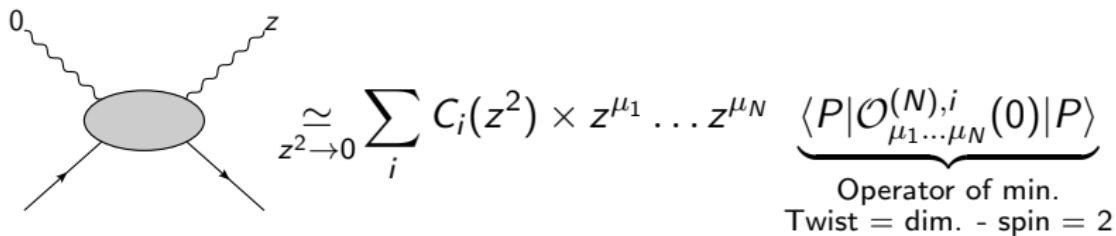
Inconsistency

The evolution of PDFs **cannot** depend on the gauge choice.

- **Axial** gauge result correctly fulfils supersymmetry Ward id.
- Conflict was resolved **only** at 2 loops (Hamberg, van Neerven 1993).

The OPE approach

Lightcone OPE of DIS (Gross, Wilczek 1973; Georgi, Politzer 1973)



- Separate short-distance (C_i) from long-distance physics $\langle P | \dots | P \rangle$.
- Scale evolution of the operators \rightarrow scale evolution of the PDFs

$$\mu^2 \frac{d}{d\mu^2} \mathcal{O}_{\mu_1 \dots \mu_N}^{(N),i} = -\gamma_{ij}^{(N)} \mathcal{O}_{\mu_1 \dots \mu_N}^{(N),j}$$

Moments of the splitting functions from renormalisation of twist 2.

Twist-2 operators

Twist-2 operators of QCD fields

$$\mathcal{O}_{g;\mu_1 \dots \mu_N}^{(N)} = \frac{1}{2} \mathcal{S}_T \left\{ F_{\rho \mu_1}^{a_1} D_{\mu_2}^{a_1 a_2} \dots D_{\mu_{N-1}}^{a_{N-2} a_{N-1}} F_{\mu_N}^{a_{N-1}; \rho} \right\},$$

$$\mathcal{O}_{q;\mu_1 \dots \mu_N}^{(N)} = \mathcal{S}_T \left\{ \bar{\psi}_{i_1} \gamma_{\mu_1} D_{\mu_2}^{i_1 i_2} \dots D_{\mu_N}^{i_{N-1} i_N} \psi_{i_N} \right\},$$

$$\mathcal{O}_{ns;\mu_1 \dots \mu_N}^{(N),\rho} = \mathcal{S}_T \left\{ \bar{\psi}_{i_1} (\lambda^\rho) \gamma_{\mu_1} D_{\mu_2}^{i_1 i_2} \dots D_{\mu_N}^{i_{N-1} i_N} \psi_{i_N} \right\},$$

$\lambda^\rho \rightarrow$ generator of $SU(n_f)$.

$\mathcal{S}_T \rightarrow$ symmetrise over $\mu_1 \dots \mu_N$ and remove trace terms.

- Construct projectors with a **lightlike** vector Δ^μ

$$F_\nu^a = F_{\nu \mu}^a \Delta^\mu, \quad D = D_\mu \Delta^\mu, \quad \partial = \partial_\mu \Delta^\mu, \quad A^a = A_\mu^a \Delta^\mu,$$

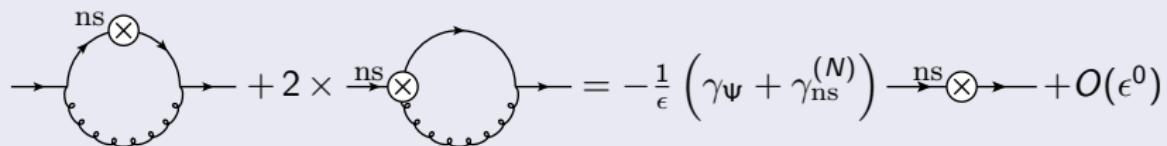
- In this talk \rightarrow focus on **gluonic** operator

$$\mathcal{O}_1^{(N)} = \mathcal{O}_{g;\mu_1 \dots \mu_N}^{(N)} \Delta^{\mu_1} \dots \Delta^{\mu_N} = \frac{1}{2} \text{Tr} \left[F_\rho D^{N-1} F^\rho \right]$$

Renormalisation: Non-singlet vs Singlet operators

Non-singlet operators • Diagonal RGE

$$\mathcal{O}_{\text{ns}}^{(N),R}(\mu^2) = Z_{\text{ns}}^{(N)}(\mu^2) \mathcal{O}_{\text{ns}}^{(N),\text{bare}}$$



Renormalisation: Non-singlet vs Singlet operators

Non-singlet operators • Diagonal RGE

$$\mathcal{O}_{\text{ns}}^{(N),R}(\mu^2) = Z_{\text{ns}}^{(N)}(\mu^2) \mathcal{O}_{\text{ns}}^{(N),\text{bare}}$$

$$+ 2 \times \text{loop diagram} = -\frac{1}{\epsilon} \left(\gamma_\Psi + \gamma_{\text{ns}}^{(N)} \right) + O(\epsilon^0)$$

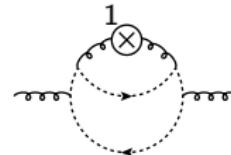
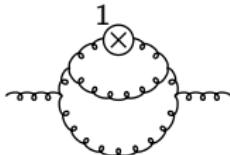
Singlet operators • Mixing

$$\mathcal{O}_1^{(N),R}(\mu^2) = Z_{1i}^{(N)}(\mu^2) \mathcal{O}_i^{(N),\text{bare}}$$

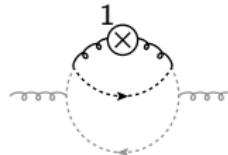
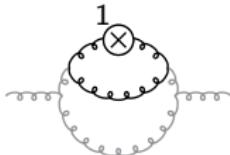
$$+ 2 \times \text{loop diagram} = -\frac{1}{\epsilon} \left(\gamma_A + \gamma_{11}^{(N)} \right) \text{line with cross and } 1$$

$$- \sum_{i>1} \frac{\gamma_{1i}^{(N)}}{\epsilon} \text{ Alien operators}$$

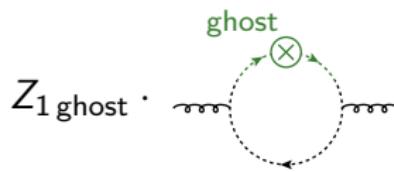
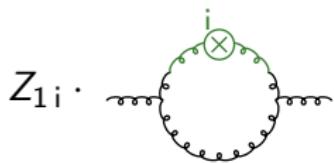
Multiloop renormalisation



Multiloop renormalisation



Alien operators, including **ghost** operators enter in subdivergences



To renormalise the gauge invariant operator we need

- The whole **set of aliens**, including *ghost* operators.
- The mixing counterterms Z_{1i} .

What do we know about aliens

- Basis of all Alien operators entering at **2 loops** (Dixon, Taylor 1974)

$$\overline{O}_A = \bar{F}^{aa} \bar{D}_a^{ab} \partial^{m-2} \bar{A}^b - \bar{g} f^{abc} \bar{F}_a^a \sum_{i=1}^{m-2} \frac{\kappa_i}{\eta} \partial^a [(\partial^{i-1} \bar{A}^b)(\partial^{m-2-i} \bar{A}^c)] + O(\bar{g}^2) \quad (2.9)$$

$$\overline{O}_\omega = -\xi^a \partial^m \bar{\omega}^a - \bar{g} f^{abc} \bar{\xi}_a^a \sum_{i=1}^{m-2} \frac{\eta_i}{\eta} \partial [(\partial^{m-2-i} \bar{A}^b)(\partial^i \bar{\omega}^c)] + O(\bar{g}^2) \quad (2.10)$$

Solve puzzle in the gluon evolution (Hamberg, van Neerven 1993).

- General theorem: characterisation of all aliens (Joglekar, Lee 1975)
 - BRST-exact operators
 - Operators proportional to the Equation of Motion

This talk

- General construction of aliens and **explicit** basis to **4 loops**
- Renormalisation mixing constants $Z_{1i}^{(N)}$ and physical ones

$$\gamma_{gg}^{(N)} = a \frac{\partial}{\partial a} Z_{11}^{(N)} \Big|_{\frac{1}{\epsilon}}$$

Setup $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{G.F.}+\text{G}}$

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu}^a F^{a;\mu\nu}$$

$$\mathcal{L}_{\text{GF+G}} = s \left[\bar{c}^a \left(\partial^\mu A_\mu^a - \frac{\xi_L}{2} b^a \right) \right] = \bar{s} \left[c^a \left(\frac{\xi_L}{2} b^a - \partial^\mu A_\mu^a \right) \right]$$

- Gauge invariance of \mathcal{L}_0 under

$$\delta_\omega A_\mu^a = D_\mu^{ab} \omega^b = \partial_\mu \omega^a + g f^{abc} A_\mu^b \omega^c$$

- BRST(s)/anti-BRST(\bar{s}) invariance, s.t. $s^2 = \bar{s}^2 = \{s, \bar{s}\} = 0$

$$s(A_\mu^a) = D_\mu^{ab} c^b$$

$$\bar{s}(A_\mu^a) = D_\mu^{ab} \bar{c}^b$$

$$s(c^a) = -\frac{g}{2} f^{abc} c^b c^c$$

$$\bar{s}(\bar{c}^a) = -\frac{g}{2} f^{abc} \bar{c}^b \bar{c}^c$$

$$s(\bar{c}^a) = -b^a$$

$$\bar{s}(c^a) = -\bar{b}^a$$

$$s(b^a) = 0$$

$$\bar{s}(\bar{b}^a) = 0$$

Equation of Motion Operators

- Variations of $S_0 = \int \mathcal{L}_0 d^d x$ under $A_\mu^a \rightarrow A_\mu^a + \mathcal{G}_\mu^a$

$$\mathcal{O}_{\text{EOM}}^{(N)} = \frac{\delta S_0}{\delta A_\mu^a} \mathcal{G}_\mu^a(A_{\mu_1}^{a_1}, \partial_{\mu_1} A_{\mu_2}^{a_1}, \dots) = \underset{\text{Twist}}{\underset{\text{Lead}}{(D^\nu F_\nu)^a}} \mathcal{G}^a(A^{a_1}, \partial A^{a_1} \dots)$$

Equation of Motion Operators

- Variations of $S_0 = \int \mathcal{L}_0 d^d x$ under $A_\mu^a \rightarrow A_\mu^a + \mathcal{G}_\mu^a$

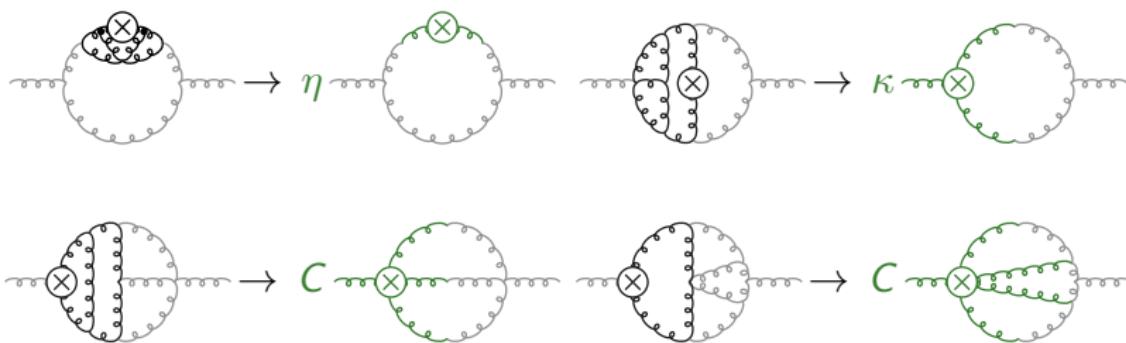
$$\mathcal{O}_{\text{EOM}}^{(N)} = \frac{\delta S_0}{\delta A_\mu^a} \mathcal{G}_\mu^a(A_{\mu_1}^{a_1}, \partial_{\mu_1} A_{\mu_2}^{a_1}, \dots) = \underset{\substack{\text{Lead} \\ \text{Twist}}}{(D^\nu F_\nu)^a} \mathcal{G}^a(A^{a_1}, \partial A^{a_1} \dots)$$

- Concrete ansatz with general couplings

$$\begin{aligned} \mathcal{O}_{\text{EOM}}^{(N)} &= (D^\mu F_\mu)^a \left[\eta \partial^{N-2} A^a + g f^{aa_1 a_2} \sum_{i_1+i_2=N-3} \kappa_{i_1 i_2} (\partial^{i_1} A^{a_1})(\partial^{i_2} A^{a_2}) \right. \\ &\quad + g^2 \sum_{i_1+i_2+i_3=N-4} \left(\underbrace{\kappa_{i_1 i_2 i_3}^{(1)} f^{aa_1 z} f^{a_2 a_3 z} + \kappa_{i_1 i_2 i_3}^{(2)} d_4^{aa_1 a_2 a_3} + \kappa_{i_1 i_2 i_3}^{(3)} d_{4ff}^{aa_1 a_2 a_3}}_{C_{i_1 i_2 i_3}^{aa_1 a_2 a_3}} \right) (\partial^{i_1} A^{a_1}) .. (\partial^{i_3} A^{a_3}) \\ &\quad \left. + g^3 \sum_{i_1+..+i_4=N-5} \left(\underbrace{\kappa_{i_1 \dots i_4}^{(1)} (f f f)^{aa_1 a_2 a_3 a_4} + \kappa_{i_1 \dots i_4}^{(2)} d_{4f}^{aa_1 a_2 a_3 a_4}}_{C_{i_1 \dots i_4}^{aa_1 \dots a_4}} \right) (\partial^{i_1} A^{a_1}) .. (\partial^{i_4} A^{a_4}) + O(g^4) \right] \end{aligned}$$

Diagrammatic interpretation

- $\mathcal{O}_{\text{EOM}}^{(N)}$ cancel subdivergences in **gluonic** diagrams at 4 loops



Coefficients (e.g. $C_{i_1 \dots i_n}^{a_1 \dots a_n}$) have colour structure of n -point graphs.

Generalised Gauge Invariance

Define \mathcal{L}_{EGI} to include all purely *gluonic* terms

$$S_{\text{EGI}} = \int \mathcal{L}_{\text{EGI}} d^d x = \int \underbrace{\mathcal{L}_0 + \mathcal{C}_1 \mathcal{O}_1^{(N)}}_{\text{Gauge invariant}} + \underbrace{\mathcal{O}_{\text{EOM}}^{(N)}}_{\sim \delta S_0} d^d x$$

$\mathcal{C}_1 \rightarrow$ coupling of $\mathcal{O}_1^{(N)}$, analogous to η , κ_{ij} , $\kappa_{ijk}^{(1)}$, ...

$\mathcal{O}_{\text{EOM}}^{(N)}$ gauge variant \rightarrow Generalised gauge transf. cancels $\delta_\omega \mathcal{O}_{\text{EOM}}^{(N)}$

$$\delta A_\mu^a = \delta_\omega A_\mu^a + \delta_\omega^\Delta A_\mu^a \quad \text{such that} \quad \delta S_{\text{EGI}} = 0$$

$$\delta_\omega^\Delta A_\mu^a = -\delta_\omega \mathcal{G}_\mu^a + g f^{abc} \mathcal{G}_\mu^b \omega^c$$

Generalised BRST transformations

Generalised gauge transf. \rightarrow Generalised BRST transf.

$$\delta A_\mu^a = \delta_\omega A_\mu^a + \delta_\omega^\Delta A_\mu^a \xrightarrow{\omega^a \rightarrow c^a} s'(A_\mu^a) = s(A_\mu^a) + s_\Delta(A_\mu^a)$$

From the expression of $\delta_\omega^\Delta A_\mu^a$ we get

$$s_\Delta A_\mu^a = -s(\mathcal{G}_\mu^a) + g f^{abc} \mathcal{G}_\mu^b c^c \quad (1)$$

Rest of BRST transf. **unchanged** i.e. $s_\Delta(c^a) = s_\Delta(\bar{c}^a) = s_\Delta(b^a) = 0$.
Automatically nilpotent by eq.(1)

$$s'^2(A_\mu^a) = \{s, s_\Delta\} A_\mu^a = 0$$

Ghost operators

Define complete Lagrangian invariant under generalised BRST

$$\widetilde{\mathcal{L}} = \underbrace{\mathcal{L}_0 + \mathcal{C}_1 \mathcal{O}_1^{(N)} + \mathcal{O}_{\text{EOM}}^{(N)}}_{\mathcal{L}_{\text{EGI}}} + s' \underbrace{\left[\bar{c}^a \left(\partial^\mu A_\mu^a - \frac{\xi_L}{2} b^a \right) \right]}_{\text{Original ancestor for gauge fix.}}$$

Ghost operators given by s_Δ , once \mathcal{G}_μ^a is known

$$\mathcal{O}_G^{(N)} = -\bar{c}^a \partial^\mu (s_\Delta A_\mu^a) = \bar{c}^a \partial^\mu \left(s(\mathcal{G}_\mu^a) - g f^{abc} \mathcal{G}_\mu^b c^c \right)$$

EOM and ghost operators are related to each other.

Diagrammatic interpretation

Spin $N = 2$

$$\mathcal{G}^a = \eta A^a$$

$$\mathcal{O}_{\text{FOOM}}^{(2)} = (D^\nu F_\nu)^a \mathcal{G}^a = \eta (D^\nu F_\nu)^a A^a, \quad \mathcal{O}_G^{(2)} = \eta \bar{c}^a \partial^2 c^a$$

There is a **single** alien operator

$$\mathcal{O}_2^{(2)} = \eta \left[(D^\nu F_\nu)^a A^a + \bar{c}^a \partial^2 c^a \right].$$

Same mixing counterterm into gluon and ghost operators



Generalised anti-BRST invariance

$$\tilde{\mathcal{L}} = \mathcal{L}_0 + \mathcal{C}_1 \mathcal{O}_1^{(N)} + \mathcal{O}_{\text{EOM}}^{(N)} + \bar{s}' \left[c^a \left(\frac{\xi_L}{2} b^a - \partial^\mu A_\mu^a \right) \right]$$

where

$$\bar{s}' A_\mu^a = \bar{s} A_\mu^a + \bar{s}_\Delta A_\mu^a, \quad \bar{s}_\Delta A_\mu^a = -\bar{s} (\mathcal{G}_\mu^a) + g f^{abc} \mathcal{G}_\mu^b \bar{c}^c$$

New expression for **ghost operators** \rightarrow new non-trivial relations

$$\mathcal{O}_G^{(N)} = \underbrace{-\bar{c}^a \partial^\mu (s_\Delta A_\mu^a)}_{\text{from BRST}} = \underbrace{c^a \partial^\mu (\bar{s}_\Delta A_\mu^a)}_{\text{from anti-BRST}}$$

For instance, at $N = 4$, this implies $2\kappa_{01} = \eta$.

Summary

- Complete set of gluonic gauge invariant + Aliens

$$\tilde{\mathcal{L}} = \mathcal{L}_0 - \frac{(\partial^\mu A_\mu^a)^2}{2\xi_L} - \bar{c}^a \partial^\mu D_\mu^{ab} c^b + \underbrace{\mathcal{C}_1 \mathcal{O}_1^{(N)}}_{\text{Gauge inv.}} + \underbrace{\sum_{i>1} \mathcal{C}_i \mathcal{O}_i^{(N)}}_{\mathcal{O}_{\text{EOM}}^{(N)} + \mathcal{O}_G^{(N)}}$$

- $\mathcal{O}_{\text{EOM}}^{(N)} = (D^\mu F_\mu)^a \mathcal{G}^a$, where \mathcal{G}^a has known ansatz.
- $\mathcal{O}_G^{(N)} = -(\partial \bar{c}^a) (s(\mathcal{G}^a) - g f^{abc} \mathcal{G}^b c^c)$, **automatically**.
- Consistency with anti-BRST $\rightarrow \{\mathcal{C}_{i>2}\}$ restricted to minimal set.
 E.g. for $N=4$,

$$\{\mathcal{C}_i\} = \{\eta, \kappa_{000}^{(2)}\} \text{ or } \{\mathcal{C}_i\} = \{\kappa_{01}, \kappa_{000}^{(2)}\}.$$

Renormalising operators with $N = 2$

$$\mathcal{O}_1^{(2)} = \frac{1}{2} F_\mu^a F^{a;\mu}, \quad \mathcal{O}_2^{(2)} = (D^\nu F_\nu)^a A^a + \bar{c}^a \partial^2 c^a$$

- Renormalisation constants: mixing

$$\delta Z_{12}^{(2)} = -a \frac{C_A}{2\epsilon} + a^2 C_A^2 \left[\frac{19}{24\epsilon^2} + \frac{5}{48} \frac{\xi}{\epsilon} - \frac{35}{48\epsilon} \right] + a^3 C_A^3 \left[-\frac{779}{432\epsilon^3} + \frac{1}{\epsilon^2} \left(\frac{2807}{864} - \frac{35\xi}{216} + \frac{5\xi^2}{288} \right) + \frac{1}{\epsilon} \left(-\frac{16759}{7776} - \frac{11\zeta_3}{72} + \frac{377\xi}{1728} + \frac{5\zeta_3\xi}{72} - \frac{65\xi^2}{1728} \right) \right] + O(a^4)$$

- Renormalisation constants: gauge invariant operator

$$Z_{11}^{(2)} = 1 + O(a^5)$$

Agreement with general theorem ([Freedman, Muzinich, Weinberg 1974](#)) and explicit calculations ([Moch, Ruijl, Ueda, Vermaseren, Vogt 2021](#)).

Renormalising operators with $N = 4$

- Operator basis and mixing

$$\mathcal{O}_1^{(4)} = \frac{1}{2} \text{Tr}[F_\nu D^2 F^\nu],$$

$$\begin{aligned} \mathcal{O}_2^{(4)} &= (D.F)^a \left[\partial^2 A^a + g f^{abc} A^b \partial A^c \right] - \partial \bar{c}^a \partial^3 c^a \\ &\quad - g f^{abc} \partial \bar{c}^a \left[2A^b \partial^2 c^c + \partial A^b \partial c^c \right] \\ &\quad - g^2 f^{abe} f^{cde} \partial \bar{c}^a A^b A^c \partial c^d, \end{aligned}$$

$$\mathcal{O}_3^{(4)} = d^{abcd} \left[(D.F)^a A^b A^c A^d - 3 \partial \bar{c}^a A^b A^c \partial c^d \right]$$

$$\begin{aligned} \delta Z_{12}^{(4)} &= -\frac{a C_A}{12\epsilon} - a^2 C_A^2 \left[\frac{97}{1440\epsilon^2} - \frac{\xi}{320\epsilon} + \frac{8641}{86400\epsilon} \right] \\ &\quad + a^3 C_A^3 \left[\frac{9437}{86400\epsilon^3} - \frac{1}{\epsilon^2} \left(\frac{1520341}{15552000} - \frac{853\xi}{86400} \right) \right. \\ &\quad \left. - \frac{1}{\epsilon} \left(\frac{166178237}{466560000} + \frac{\zeta_3}{2400} - \frac{37199\xi}{648000} - \frac{37\zeta_3\xi}{9600} \right) \right] \end{aligned}$$

$$\delta Z_{13}^{(4)} = \frac{a C_A}{24\epsilon}$$

- Gauge invariant operator renormalisation agrees with
 (Moch, Ruijl, Ueda, Vermaseren, Vogt 2021)

$$\begin{aligned} \delta Z_{11}^{(4)} &= a \frac{21 C_A}{5\epsilon} + a^2 C_A^2 \left(\frac{28}{25\epsilon^2} + \frac{7121}{1000\epsilon} \right) - a^3 C_A^3 \left(\frac{1316}{1125\epsilon^3} + \frac{151441}{45000\epsilon^2} - \frac{103309639}{4050000\epsilon} \right) \\ &\quad + a^4 \left\{ C_A^4 \left[\frac{11186}{5625\epsilon^4} + \frac{1512989}{450000\epsilon^3} - \frac{5437269017}{162000000\epsilon^2} + \frac{1}{\epsilon} \left(\frac{1502628149}{13500000} + \frac{1146397\zeta_3}{45000} \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{126\zeta_5}{5} \right) \right] + \frac{d_{AA}}{N_A} \left(\frac{21623}{600\epsilon} + \frac{3899\zeta_3}{15\epsilon} - \frac{1512\zeta_5}{5\epsilon} \right) \right\} \end{aligned}$$

Renormalisation constants for $N = 6$

- Mixing with alien operators

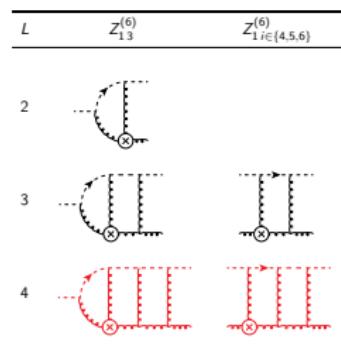
$$\delta Z_{12}^{(6)} = -a \frac{C_A}{30\epsilon} - a^2 C_A^2 \left[\frac{653}{10080\epsilon^2} + \frac{19\xi}{20160\epsilon} + \frac{185093}{4233600\epsilon} \right] + O(a^3)$$

$$\delta Z_{13}^{(6)} = -a \frac{C_A}{48\epsilon} - a^2 C_A^2 \left[\frac{2021}{40320\epsilon^2} + \frac{235813}{8467200\epsilon} + O(\xi) \right] + O(a^3),$$

$$\delta Z_{14}^{(6)} = -a \frac{C_A}{32\epsilon} + O(a^2),$$

$$\delta Z_{15}^{(6)} = a \frac{C_A}{24\epsilon} + O(a^2),$$

$$\delta Z_{16}^{(6)} = O(a^2).$$



- Gauge invariant operator renormalisation agrees with
 (Larin, Nogueira, van Ritbergen, Vermaseren 1996)

$$\begin{aligned} \delta Z_{11}^{(6)} = & a \frac{83 C_A}{14\epsilon} + a^2 C_A^2 \left(\frac{7885}{1176\epsilon^2} + \frac{1506899}{148176\epsilon} \right) + a^3 C_A^3 \left(-\frac{465215}{148176\epsilon^3} \right. \\ & \left. + \frac{243375989}{18670176\epsilon^2} + \frac{96390174479}{2613824640\epsilon} \right) + O(a^4), \end{aligned}$$

Conclusion

- Obj: OPE approach to get *moments* of $N^3\text{LO}$ splitting functions.
- Conceptual obstacle: **alien** operators mixing in renormalisation.

This work

- General method to construct all the aliens.
- Explicit bases of aliens up to 4 loops.
- Results of moments $N = 2, 4$ to 4 loops and $N = 6$ to 3 loops.

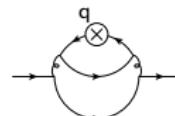
In progress

- Extension to fermionic operators.
- Calculation of higher moments to 4 loops.

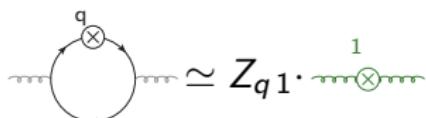
Thank you for the attention!

Quark operators mixing

Consider quark pure singlet e.g.

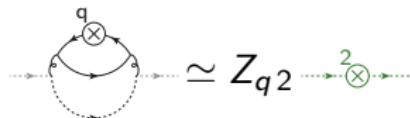
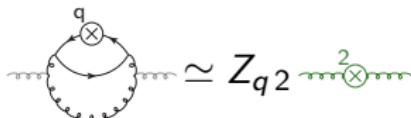


Mixing of quark singlet operators into the aliens starts at 3 loops



Only the physical operator $\mathcal{O}_1^{(N)}$ enters **subdiagrams** at **1 loop**

By BRST invariance, aliens involve both gluon and ghost.



Ghost diagrams start at 2 loops \rightarrow subdivergence at 3 loops.

Comparison with Joglekar and Lee

$\mathcal{O}_{\text{EOM}}^{(N)}$ generated by *classical* EOM. $\mathcal{O}_G^{(N)} = s_\Delta (\bar{c} \partial^\mu A_\mu^a)$ generalised-BRST-exact. But

$$\mathcal{O}_{\text{EOM}}^{(N)} = \frac{\delta S_0}{\delta A_\mu^a} \mathcal{G}_\mu^a, \quad \mathcal{O}_G^{(N)} = -s (\bar{c}^a \partial^\mu \mathcal{G}_\mu^a) + \underbrace{\left[\frac{1}{\xi_L} \partial^\mu (\partial^\nu A_\nu^a) + g f^{abc} (\partial^\mu \bar{c}^b) c^c \right]}_{\text{Gauge fix. and ghost terms in EOM of } A_\mu^a} \mathcal{G}_\mu^b$$

$$\boxed{\mathcal{O}_{\text{EOM}}^{(N)} + \mathcal{O}_G^{(N)} = s ((\partial^\mu \bar{c}^a) \mathcal{G}_\mu^a) + \frac{\delta S}{\delta A_\mu^a} \mathcal{G}_\mu^a}$$

Now we compare with Joglekar and Lee operators of class I and II

$$\mathcal{O}_{\text{I}} = s (F(A, c, \bar{c})) + \frac{\delta S}{\delta A_\mu^a} \frac{\partial F(A, c, \bar{c})}{\partial (\partial_\mu \bar{c}^a)} \Rightarrow F = (\partial^\mu \bar{c}^a) \mathcal{G}_\mu^a$$

$$\mathcal{O}_{\text{II}} = \frac{\delta S}{\delta c^a} X^a (A, c, \bar{c}) \Rightarrow X^a = 0 \quad \frac{\delta S}{\delta c^a} \text{ is twist-3!}$$

Space of alien operators

Spin N	2	4	6	8	10	12	14	16
w aBRST	1	2	5	12	25	50	87	140
w/o aBRST	1	3	11	30	66	126	215	339

Generalised Gauge transformation (I)

Expression of EOM operators

$$\mathcal{G}^a = \sum_{k=1}^{\infty} g^{k-1} \sum_{\substack{i_1 + \dots + i_k \\ = N-k-1}} C_{i_1 \dots i_k}^{a; a_1 \dots a_k} \left(\partial^{i_1} A^{a_1} \right) \dots \left(\partial^{i_k} A^{a_k} \right)$$

$C_{i_1 \dots i_k}^{a; a_1 \dots a_k}$ given up to 4 loops. Then $\delta_\omega^\Delta A_\mu^a = -\Delta_\mu (\delta_\omega \mathcal{G}^a - g f^{abc} \mathcal{G}^b \omega^c)$ reads

$$\begin{aligned} \delta_\omega^\Delta A_\mu^a &= -\Delta_\mu \sum_{k=1}^{\infty} \sum_{\substack{i_1 + \dots + i_k \\ N-k-1}} \left(\partial^{i_1} A^{a_1} \right) \dots \left(\partial^{i_k} A^{a_k} \right) \sum_{\sigma \in Z_k} C_{i_{\sigma(1)} \dots i_{\sigma(k)}}^{a; a_{\sigma(1)} \dots a_{\sigma(k)}} \\ &\quad + g \Delta_\mu \sum_{k=1}^{\infty} \sum_{\substack{i_1 + \dots + i_{k+1} \\ N-k-2}} \left(\partial^{i_1} A^{a_1} \right) \dots \left(\partial^{i_k} A^{a_k} \right) \left(\partial^{i_{k+1}+1} \omega^{a_{k+1}} \right) \sum_{m=1}^k \binom{i_m + i_{k+1} + 1}{i_m} \\ &\quad \times C_{i_1 \dots i_m + i_{k+1} + 1 \dots i_k}^{a; a_1 \dots a_{m-1} b a_{m+1} \dots a_k} f^{b a_m a_{k+1}} \\ &= -\Delta_\mu \sum_{k=1}^{\infty} g^{k-1} \sum_{\substack{i_1 + \dots + i_k \\ = N-k-1}} \tilde{C}_{i_1 \dots i_k}^{a; a_1 \dots a_k} \left(\partial^{i_1} A^{a_1} \right) \dots \left(\partial^{i_{k-1}} A^{a_{k-1}} \right) \left(\partial^{i_k+1} \omega^{a_k} \right) \end{aligned}$$

Generalised Gauge transformation (II)

Explicit coefficients up to 4 loops

$$\begin{aligned}\widetilde{C}_{i_1 i_2}^{a; a_1 a_2} &= \eta_{i_1 i_2}^{(1)} f^{a; a_1 a_2}, \\ \widetilde{C}_{i_1 i_2 i_3}^{a; a_1 a_2 a_3} &= \eta_{i_1 i_2 i_3}^{(1)} (ff)^{aa_1 a_2 a_3} + \eta_{i_1 i_2 i_3}^{(2)} d^{aa_1 a_2 a_3} + \eta_{i_1 i_2 i_3}^{(3)} d_{4\bar{f}}^{aa_1 a_2 a_3}, \\ \widetilde{C}_{i_1 i_2 i_3 i_4}^{a; a_1 a_2 a_3 a_4} &= \eta_{i_1 i_2 i_3 i_4}^{(1)} (fff)^{aa_1 a_2 a_3 a_4} + \eta_{i_1 i_2 i_3 i_4}^{(2a)} d_{4f}^{aa_1 a_2 a_3 a_4} + \eta_{i_1 i_2 i_3 i_4}^{(2b)} d_{4\bar{f}}^{aa_4 a_1 a_2 a_3}.\end{aligned}$$

The η coefficients are given by

$$\begin{aligned}\eta_{i_1 i_2 i_3}^{(1)} &= 2\kappa_{i_1(i_2+i_3+1)} \binom{i_2+i_3+1}{i_2} + 2 \left[\kappa_{i_1 i_2 i_3}^{(1)} + \kappa_{i_3 i_2 i_1}^{(1)} \right], & \eta_{i_1 i_2}^{(1)} &= 2\kappa_{i_1 i_2} + \eta \binom{i_1+i_2+1}{i_1}, \\ \eta_{i_1 i_2 i_3 i_4}^{(1)} &= 2 \left[\kappa_{i_1 i_2 (i_3+i_4+1)}^{(1)} + \kappa_{(i_3+i_4+1) i_2 i_1}^{(1)} \right] \binom{i_3+i_4+1}{i_3} & \eta_{i_1 i_2 i_3}^{(2)} &= 3\kappa_{i_1 i_2 i_3}^{(2)}, \\ &+ 2 \left[\kappa_{i_1 i_2 i_3 i_4}^{(1)} + \kappa_{i_1 i_4 i_3 i_2}^{(1)} + \kappa_{i_4 i_1 i_3 i_2}^{(1)} + \kappa_{i_4 i_3 i_1 i_2}^{(1)} \right], & \eta_{i_1 i_2 i_3}^{(3)} &= 2 \left[\kappa_{i_1 i_2 i_3}^{(3)} - \kappa_{i_3 i_2 i_1}^{(3)} \right], \\ \eta_{i_1 i_2 i_3 i_4}^{(2a)} &= 3\kappa_{i_1 i_2 (i_3+i_4+1)}^{(2)} \binom{i_3+i_4+1}{i_3} + 2\kappa_{i_1 i_2 i_3 i_4}^{(2)}, & \eta_{i_1 i_2 i_3 i_4}^{(2b)} &= 2\kappa_{i_4 i_1 i_2 i_3}^{(2)}.\end{aligned}$$

Anti-BRST identities

Consistency with anti-BRST symmetry requires

$$-\bar{c}^a \partial^\mu (s_\Delta A_\mu^a) = c^a \partial^\mu (\bar{s}_\Delta A_\mu^a)$$

which implies constraints on the η coefficients in $s_\Delta A_\mu^a$ and $\bar{s}_\Delta A_\mu^a$

$$\eta_{i_1 i_2}^{(1)} = - \sum_{s_1=0}^{i_1} (-1)^{s_1+i_2} \binom{s_1 + i_2}{s_1} \eta_{(i_1-s_1)(i_2+s_1)}^{(1)},$$

$$\eta_{i_1 i_2 i_3}^{(1)} = \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \frac{(s_1 + s_2 + i_3)!}{s_1! s_2! i_3!} (-1)^{s_1+s_2+i_3} \eta_{(i_2-s_2)(i_1-s_1)(i_3+s_1+s_2)}^{(1)},$$

$$\eta_{i_1 i_2 i_3}^{(2)} = \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \frac{(s_1 + s_2 + i_3)!}{s_1! s_2! i_3!} (-1)^{s_1+s_2+i_3} \eta_{(i_1-s_1)(i_2-s_2)(i_3+s_1+s_2)}^{(2)},$$

$$\eta_{i_1 i_2 i_3}^{(3)} = \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \frac{(s_1 + s_2 + i_3)!}{s_1! s_2! i_3!} (-1)^{s_1+s_2+i_3} \eta_{(i_2-s_2)(i_1-s_1)(i_3+s_1+s_2)}^{(3)}$$

Anti-BRST identities (continued)

$$\begin{aligned}\eta_{i_1 i_2 i_3 i_4}^{(1)} &= - \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \sum_{s_3=0}^{i_3} \frac{(s_1 + s_2 + s_3 + i_4)!}{s_1! s_2! s_3! i_4!} \\ &\quad \times (-1)^{s_1+s_2+s_3+i_4} \eta_{(i_3-s_3)(i_2-s_2)(i_1-s_1)(i_4+s_1+s_2+s_3)}^{(3)}, \\ \eta_{i_1 i_2 i_3 i_4}^{(2a)} &= - \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \sum_{s_3=0}^{i_3} \frac{(s_1 + s_2 + s_3 + i_4)!}{s_1! s_2! s_3! i_4!} \\ &\quad \times (-1)^{s_1+s_2+s_3+i_4} \eta_{(i_1-s_1)(i_2-s_2)(i_3-s_3)(i_4+s_1+s_2+s_3)}^{(2a)}, \\ \eta_{i_1 i_2 i_3 i_4}^{(2b)} &= \eta_{i_1 i_3 i_2 i_4}^{(2a)} - \eta_{i_1 i_2 i_3 i_4}^{(2a)} + \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \sum_{s_3=0}^{i_3} \frac{(s_1 + s_2 + s_3 + i_4)!}{s_1! s_2! s_3! i_4!} \\ &\quad \times (-1)^{s_1+s_2+s_3+i_4} \eta_{(i_1-s_1)(i_2-s_2)(i_3-s_3)(i_4+s_1+s_2+s_4)}^{(2b)}.\end{aligned}$$

Calculation in the background field method

$$\Gamma_{i;BB}^{(N)}(g, \xi) = \text{---} \circlearrowleft^i \text{---} = \begin{cases} \Gamma_{i;BB}^{(N),0} + \delta\Gamma_{i;BB}^{(N)}(g, \xi) & i = 1 \\ \delta\Gamma_{i;BB}^{(N)}(g, \xi) & i \neq 1 \end{cases}$$

Extraction of the physical renormalisation constant $Z_{11}^{(N)} = 1 + \delta Z_{11}^{(N)}$

$$\delta Z_{11}^{(N)} = -\frac{1}{Z_B \Gamma_{1;BB}^{(N),0}} K_\epsilon \left[Z_B \sum_{i \geq 1} Z_{1i}^{(N)} \delta\Gamma_{i;BB}^{(N)}(g_B, \xi_B) \right]$$

- $Z_B = Z_a^{-1}$ B-field renormalisation

K_ϵ extracts poles in ϵ

- $Z_{1i}^{(N)}$ cancel subdivergences

→ 2-, 3-pt subgraphs up to 3 loops

→ 4-pt subgraphs up to 2 loops

→ 5-pt subgraphs up to 1 loop

