Triple-collinear one loop splitting functions in QCD

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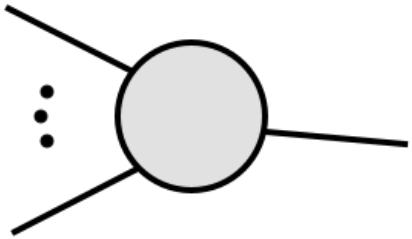
Building blocks of N3LO amplitudes

- Born level



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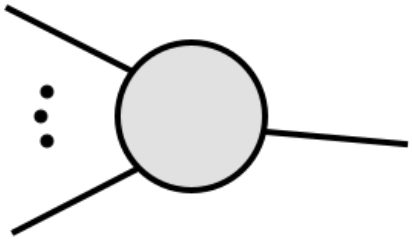
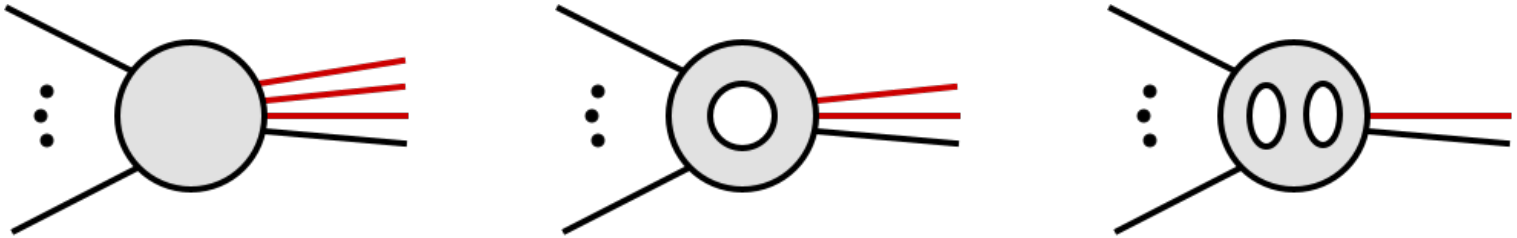


- N3LO



triple collinear limit at one loop

General definitions

The amplitude

$$\mathcal{A} \equiv (\mu^{-\epsilon} g_s^B)^n \Big(\mathcal{A}^{(0)} + \frac{\mu^{-2\epsilon} \alpha_s^B}{(4\pi)^{1-\epsilon}} \mathcal{A}^{(1)} + \mathcal{O}\big(\alpha_s^2\big) \Big) \,, \qquad \alpha_s^B \equiv \frac{(g_s^B)^2}{4\pi} \,,$$

where

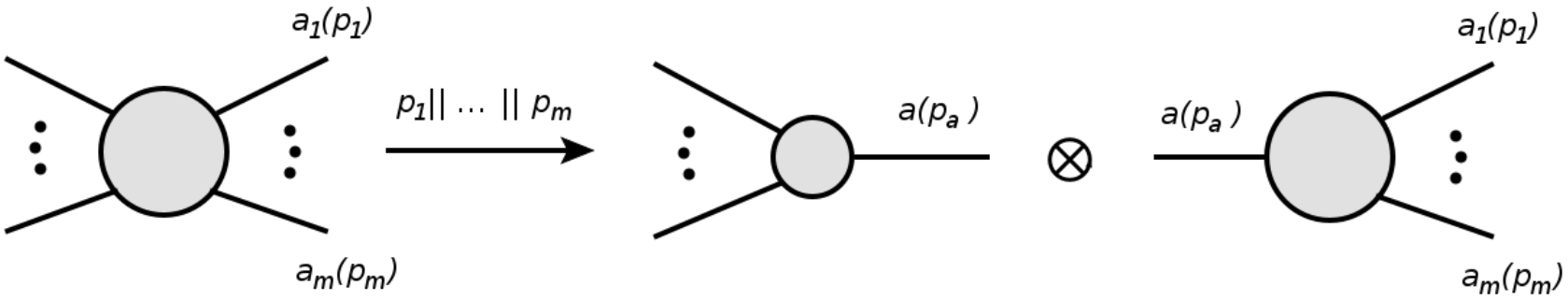
$$g_s^B \quad - \quad \text{bare coupling constant}$$

We work in d dimensions

$$d = 4 - 2\epsilon$$

Our results are not renormalized in UV - not essential - splitting operators renormalize as ordinary amplitudes

Collinear factorization in QCD: tree level



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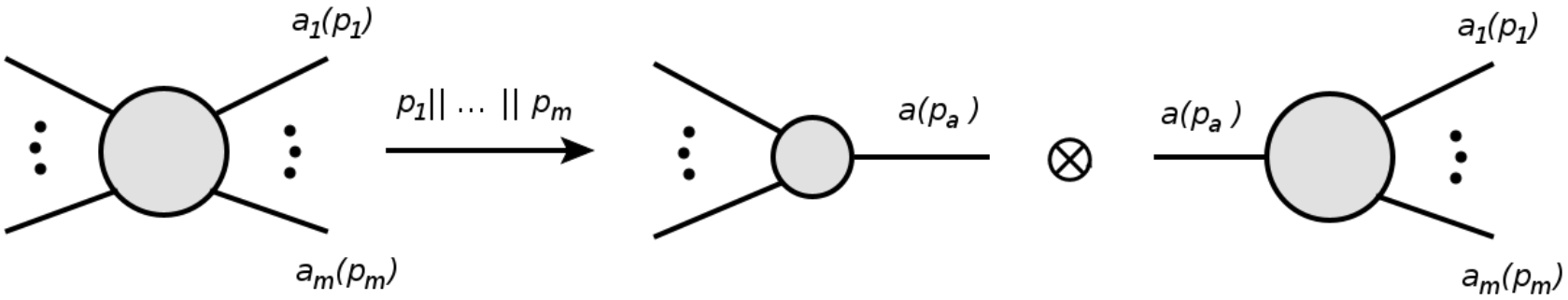

$$\mathcal{A}^{(0)}_{a_1\dots a_m\dots}(p_1,\dots,p_m,\dots) \xrightarrow{p_1||p_2||\dots||p_m} \mathbf{Split}^{(0)}_{a\to a_1\dots a_m}(p_1,\dots,p_m)\,\mathcal{A}_{a\dots}(p_a,\dots)$$

$$\sim \left(\frac{1}{\sqrt{s_{1\dots m}}}\right)^{m-1} \quad \text{when} \quad s_{1\dots m}\to 0$$

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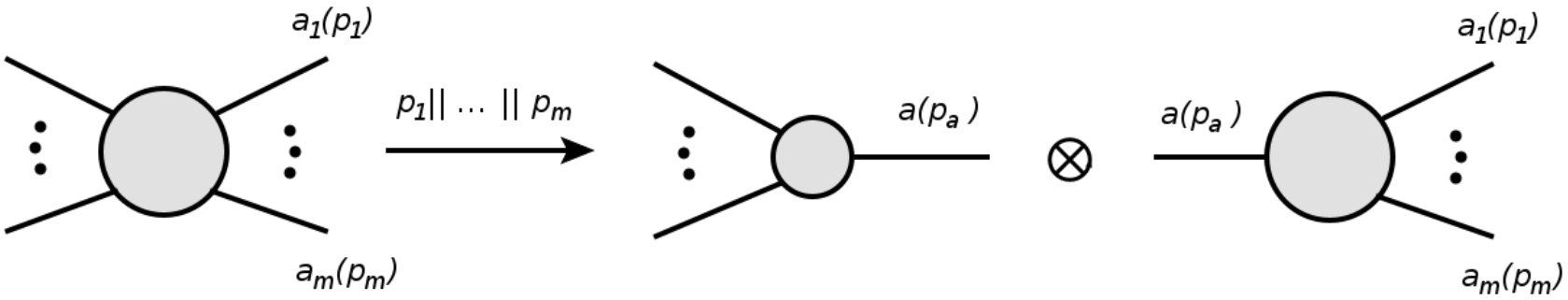

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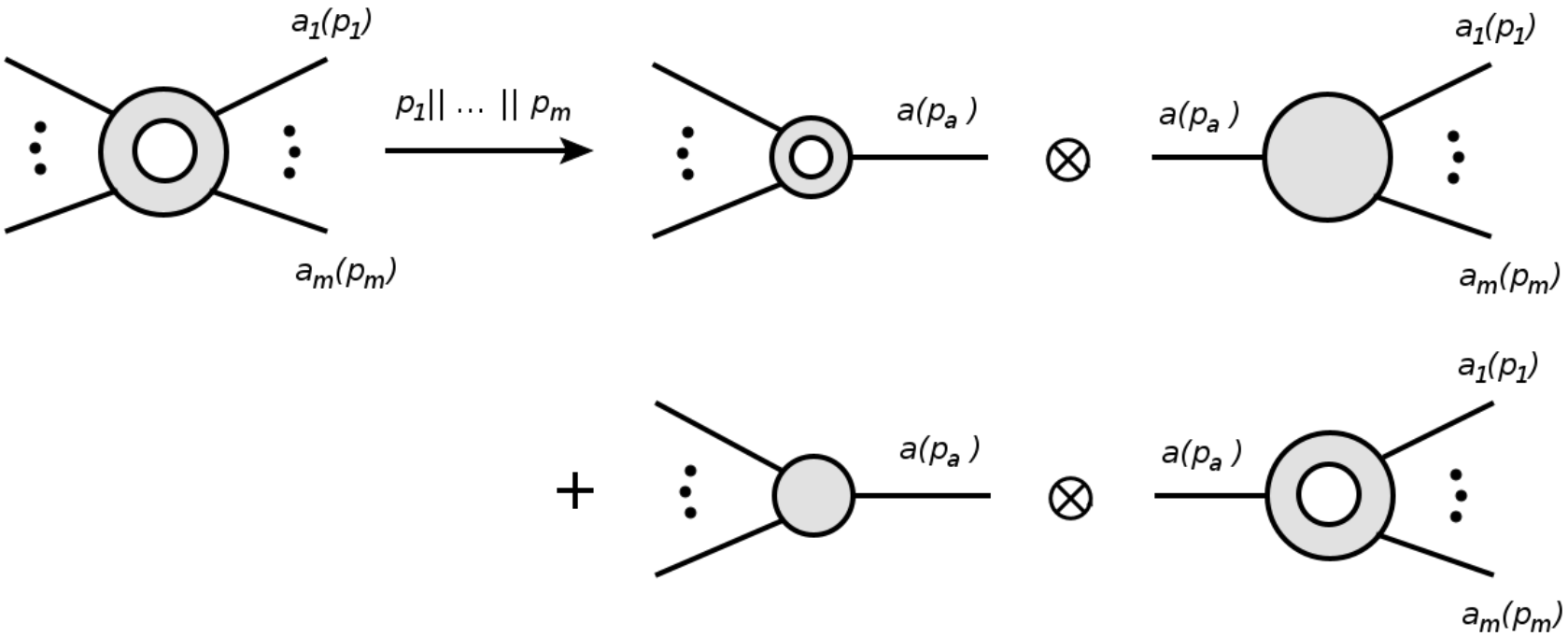
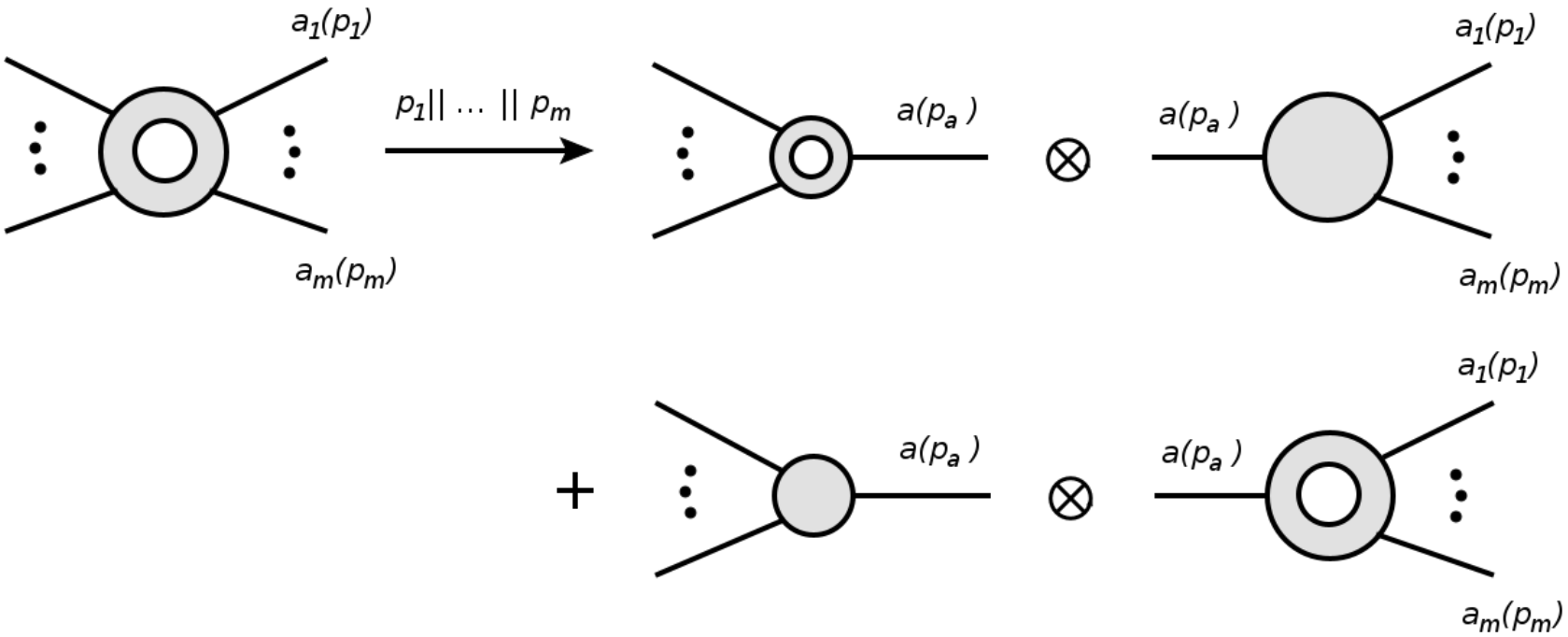
- $\mathbf{Split}^{(0)}_{a\to a_1\dots a_m}(p_1,\dots,p_m)$ is the splitting operator at tree level

Collinear factorization in QCD: one-loop



Collinear factorization in QCD: one-loop



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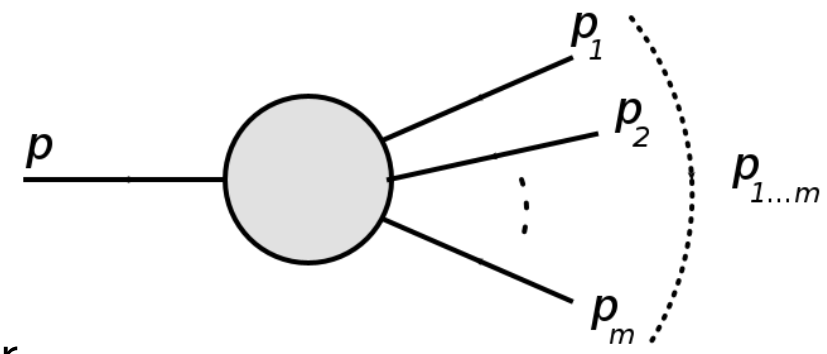
$$+\ \mathbf{Split}^{(1)}_{a\to a_1\dots a_m}(p_1,\dots,p_m)\,\mathcal{A}^{(0)}_{a\dots}(p_a,\dots)$$

$$\sim \left(\frac{1}{\sqrt{s_{1\dots m}}}\right)^{m-1}\left(\frac{s_{1\dots m}}{\mu^2}\right)^{-\epsilon} \text{ when } s_{1\dots m}\to 0$$

Collinear splitting functions

$$p_{1\ldots m} \equiv p + \frac{s_{1\ldots m}}{2\, p_{1\ldots m}\cdot q}\, q\ ,$$

$$p^2 = q^2 = 0\ , \quad p\cdot q \neq 0\ ,$$



where q is an auxiliary light-like vector

In the collinear limit: $\quad p_i \to z_i p\ , \qquad z_i = \dfrac{p_i\cdot n}{p\cdot n}$

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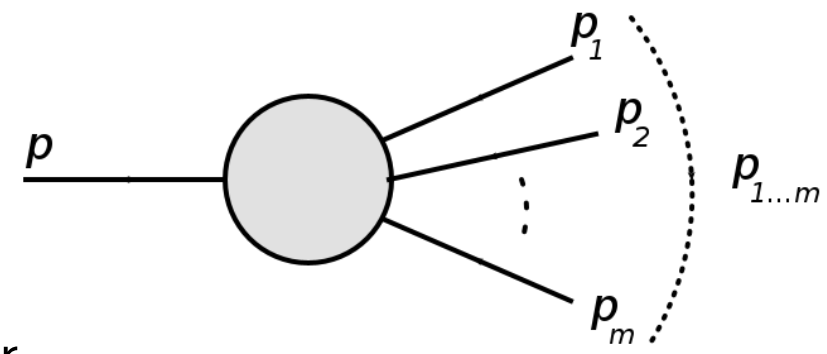

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The splitting functions and the averaged splitting functions are defined as

$$\hat{\boldsymbol{P}}_{a_1\ldots a_m} \equiv \left(\frac{s_{1\ldots m}}{2}\right)^2 \mathbf{Split}^\dagger_{a_1\ldots a_m}\mathbf{Split}_{a_1\ldots a_m}, \quad \langle \hat{P}_{a_1\ldots a_m}\rangle \equiv \frac{1}{n_a^{\rm col}\, n_a^{\rm spin}}\,{\rm Tr}\left[\hat{\boldsymbol{P}}_{a_1\ldots a_m}\right]$$

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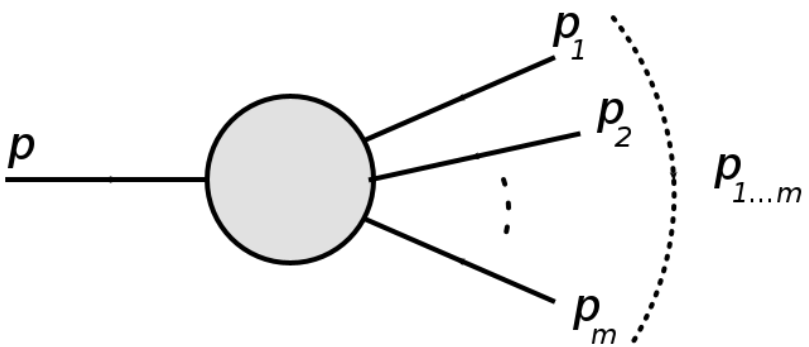

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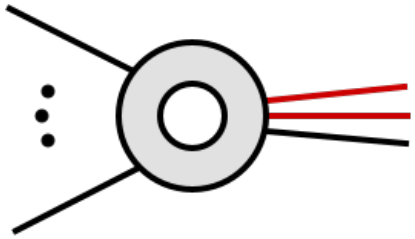
Requirements from a subtraction scheme at N3LO



Based on our discussion so far, we can see that the triple collinear singularities are encoded in the expression

$$-\left(\frac{2}{s_{123}}\right)^2 \left[\left\langle \mathcal{A}^{(0)}_{a\ldots} \middle| \hat{\boldsymbol{P}}^{(1)}_{a_1 a_2 a_3} \middle| \mathcal{A}^{(0)}_{a\ldots} \right\rangle + 2\,\mathrm{Re} \left\langle \mathcal{A}^{(0)}_{a\ldots} \middle| \hat{\boldsymbol{P}}^{(0)}_{a_1 a_2 a_3} \middle| \mathcal{A}^{(1)}_{a\ldots} \right\rangle \right]$$

- The above can be used as a subtraction term when constructing a scheme for N3LO cross section, as it removes singularities from

$$2\,\mathrm{Re} \left\langle \mathcal{A}^{(0)}_{a_1 a_2 a_3 \ldots} \middle| \mathcal{A}^{(1)}_{a_1 a_2 a_3 \ldots} \right\rangle$$

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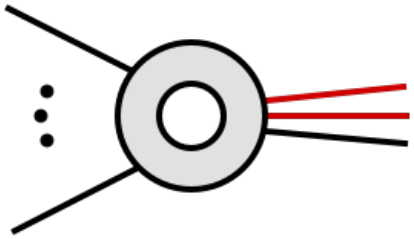

At NLO, we have

$$\int_0^1 \frac{\mathrm{d}\eta}{\eta^{1+\epsilon}} f(\eta) = \Big[\int_0^1 \frac{\mathrm{d}\eta}{\eta^{1+\epsilon}} \big(f(\eta) - f(0) \big) \Big] + \Big[f(0) \int_0^1 \frac{\mathrm{d}\eta}{\eta^{1+\epsilon}} \Big]$$

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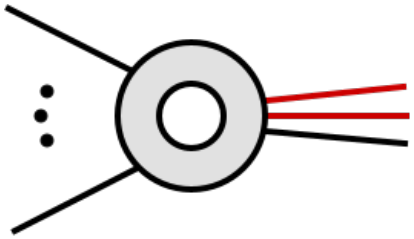

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Hence, we need our triple-collinear splitting function at least to $\mathcal{O}(\epsilon)$.

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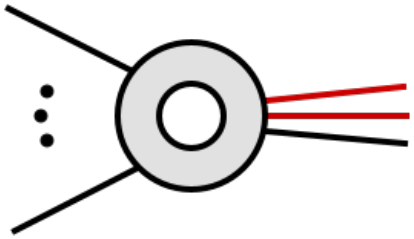

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Hence, we need our triple-collinear splitting function at least to $\mathcal{O}(\epsilon)$.

- We also need approximations up to $\mathcal{O}(\epsilon^4)$ to the triple-collinear one-loop splitting functions, valid in various additional limits (iterated single-collinear, soft, etc.).

Triple-collinear splitting functions - state of the art

- $q \to q\, q'\, \bar{q}'$ — asymmetric part only, $\mathcal{O}\left(\epsilon^0\right)$
  [Catani, de Florian, Rodrigo '04]
- $q \to q\, q\, \bar{q}$ — missing
- $q \to q\, g\, g$ — missing
- $g \to g\, q\, \bar{q}$ — $\mathcal{O}\left(\epsilon^0\right)$
  [Badger, Buciuni, Peraro '15]
- $g \to g\, g\, g$ — $\mathcal{O}\left(\epsilon^0\right)$
  [Badger, Buciuni, Peraro '15]

Our aim is to get all the above splitting functions to $\mathcal{O}(\epsilon)$

Two approaches

- top-down

  Use ordinary Feynman rules, calculate the matrix element for the process

  $$\gamma^*/H \to 4 \text{ partons}\,,$$

  and take the collinear limit.

- bottom-up

  Use modified Feynman rules and calculate the amplitude for the process

  $$q^*/g^* \to 3 \text{ partons}\,.$$

Derivation of bottom-up approach

[Bern, Del Duca, Kilgore, Schmidt '99]

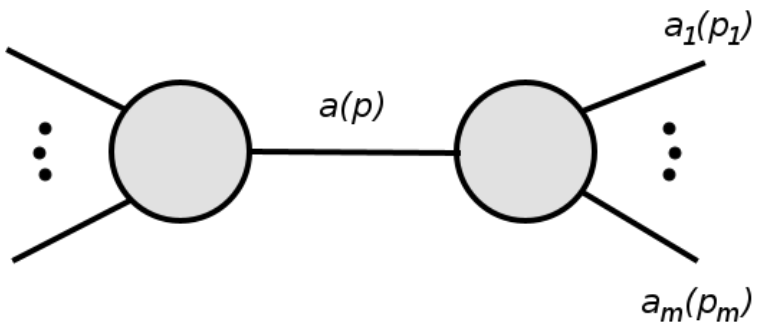
$$(\cdots)\,\frac{D(p)}{p^2}\,(\cdots)$$

$$\downarrow$$

$$(\cdots)\,\frac{\sum_{\text{pol}} a(p)\bar{a}(p)}{s_{1\ldots m}}\,(\cdots)$$

$$\sum_{\text{pol}}(\cdots)\,a(p)\,\frac{\bar{a}(p)}{s_{1\ldots m}}\,(\cdots)$$

$$\sum_{\text{pol}}\mathcal{A}_{n-m}(\ldots,p,\ldots)\mathbf{Split}_{a\to a_1\ldots a_m}(p_1,\ldots,p_m)$$



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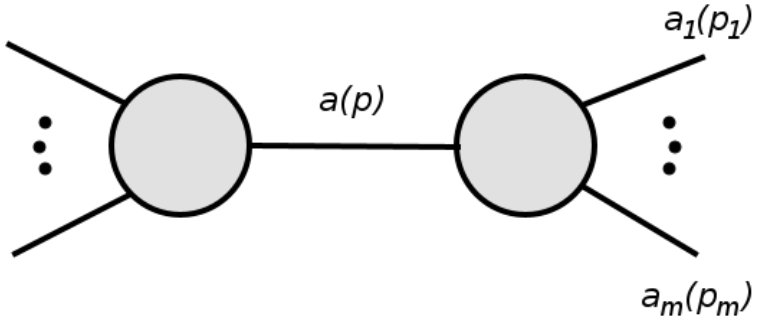
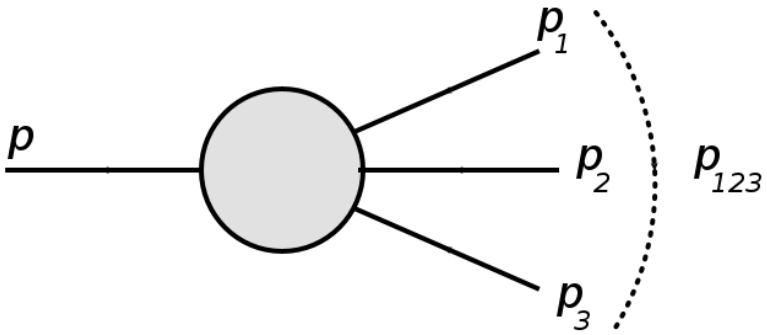


- Splitting function is derived by contracting the incoming off-shell line of the splitting parton with massless spinor (for a quark) or a massless transverse polarization vector (for a gluon)

$$\mathbf{Split}_{a\to a_1\ldots a_m} = \frac{\bar{a}(p)}{s_{1\ldots m}}\,A(a^*\to a_1,\ldots,a_m)$$

Triple-collinear splitting functions - kinematic variables



Triple-collinear splitting functions depend on

$$x_1 \equiv \frac{s_{23}}{s_{123}}\,, \quad x_2 \equiv \frac{s_{13}}{s_{123}}\,, \quad x_3 \equiv \frac{s_{12}}{s_{123}}\,, \qquad z_i \equiv \frac{p_i \cdot q}{p_{123} \cdot q}\,,$$

where

$$x_i \in (0,1)\,, \qquad z_i \in (0,1)\,,$$

and

$$\sum_{i=1}^{3} x_i = \sum_{i=1}^{3} z_i = 1$$

- $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$ singular terms known

[Catani, Dittmaier, Trocsanyi '01; Catani, de Florian, Rodrigo '04]

Bottom-up procedure

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- Simplifications of expressions (FORM)
- Passarino-Veltman reduction (FERMAT)
- Integration By Parts (IBP) reduction (KIRA)
  - bubbles, triangles, boxes and "pentagon" (only at $\mathcal{O}(\epsilon)$)
  - 34 master integrals, most of which related by permutations of external momenta
  - at the end: 9 master integrals

Top-down procedure

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- Taking the collinear limit of the final expression

$$s_{123} \to 0\,, \quad s_{12} \to 0\,, \quad s_{13} \to 0\,, \quad s_{23} \to 0\,,$$

with the finite ratios: $\dfrac{s_{12}}{s_{123}}\,,\ \dfrac{s_{13}}{s_{123}}\,,\ \dfrac{s_{23}}{s_{123}}\,,\ \dfrac{s_{12}}{s_{13}}\,,\ \dfrac{s_{12}}{s_{23}}\,,\ \dfrac{s_{13}}{s_{23}}$

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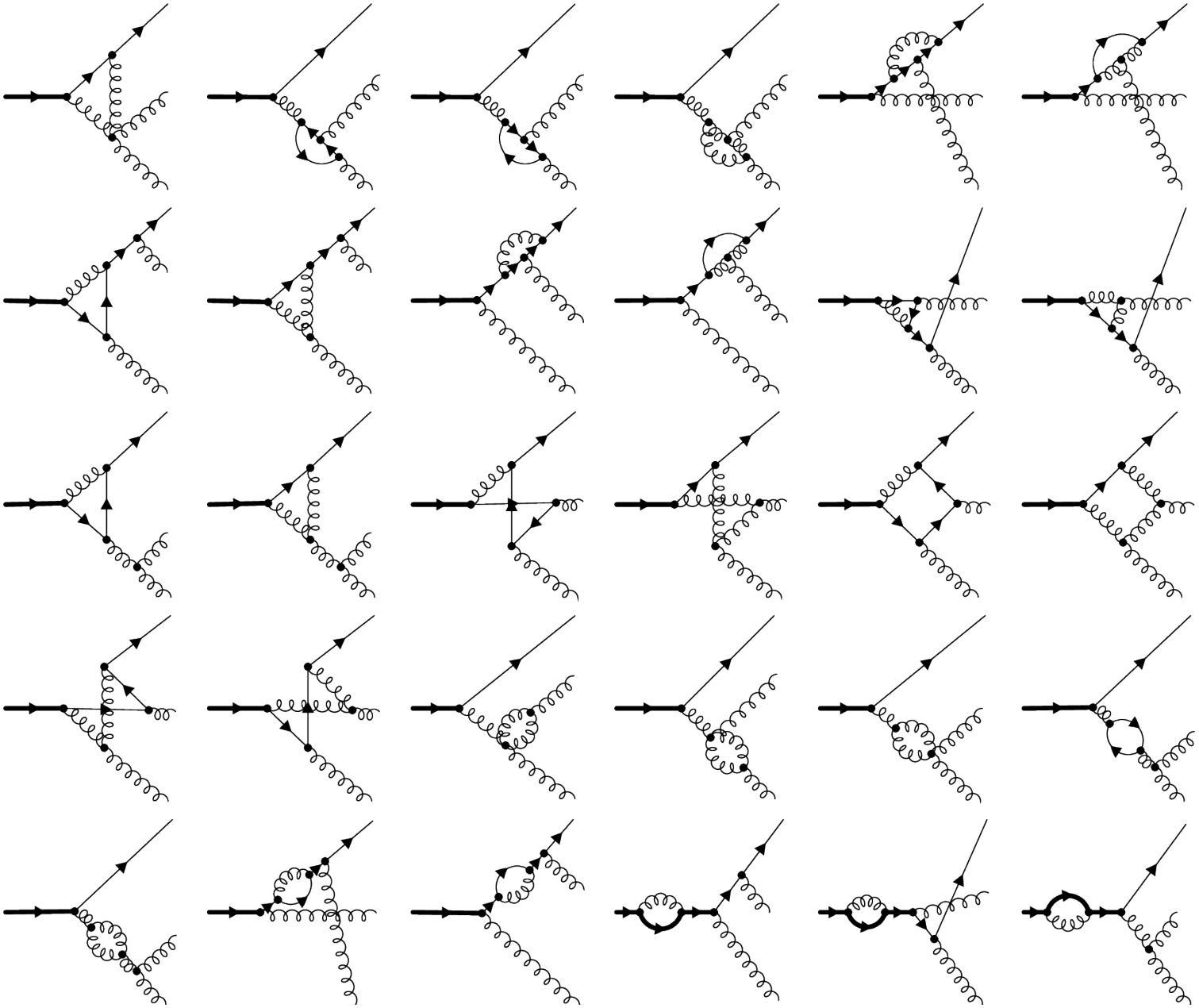
- The masters above available from [Bern, Dixon, Kosower '94] up to box at $\mathcal{O}(\epsilon^0)$.

Channels

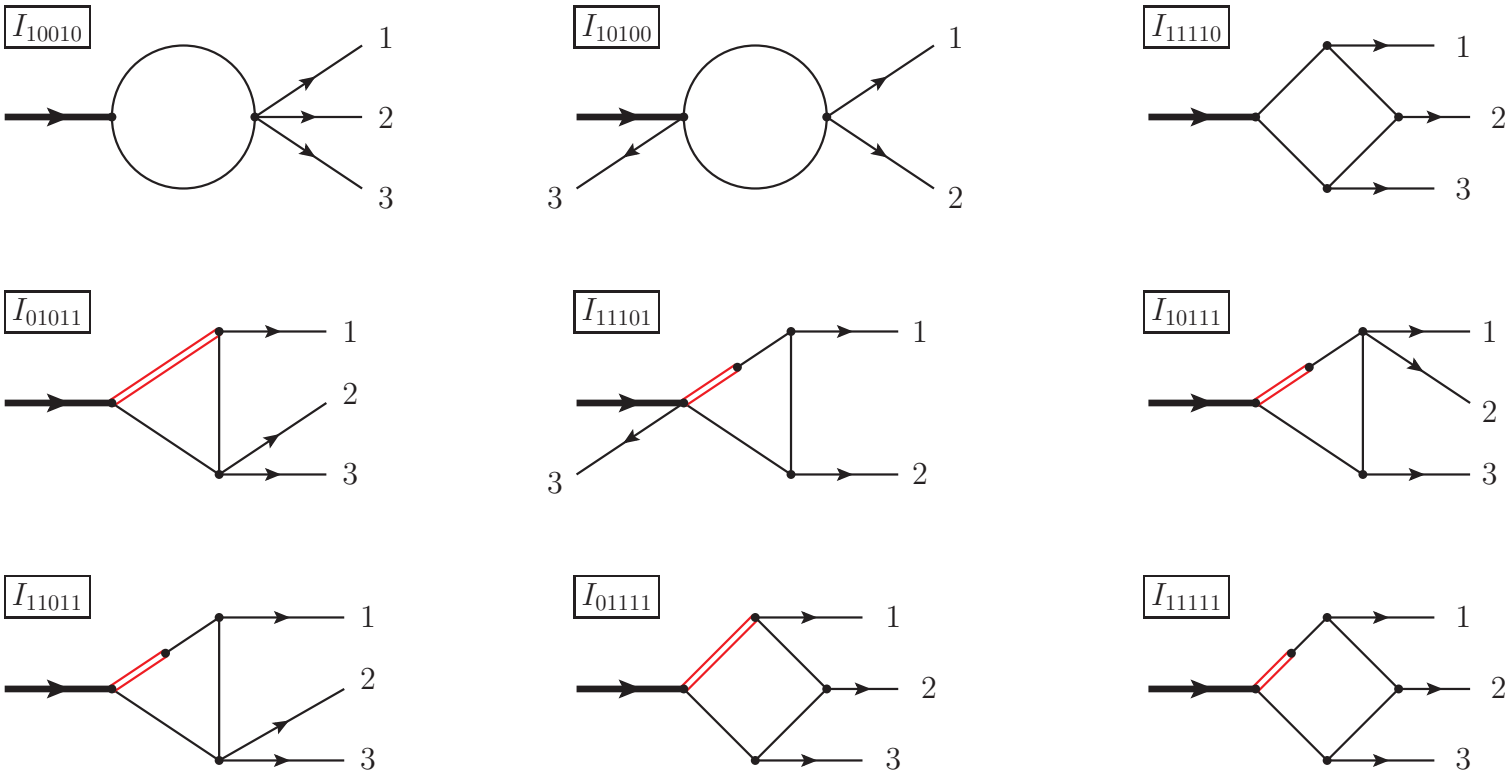
- $q \to q\, q'\, \bar{q}'$ (9 diagrams)
- $q \to q\, q\, \bar{q}$ (18 diagrams)
- $q \to q\, g\, g$ (30 diagrams)

- $g \to g\, q\, \bar{q}$ (33 diagrams)
- $g \to g\, g\, g$ (68 diagrams)

Example set: $q \to qgg$



Master integrals

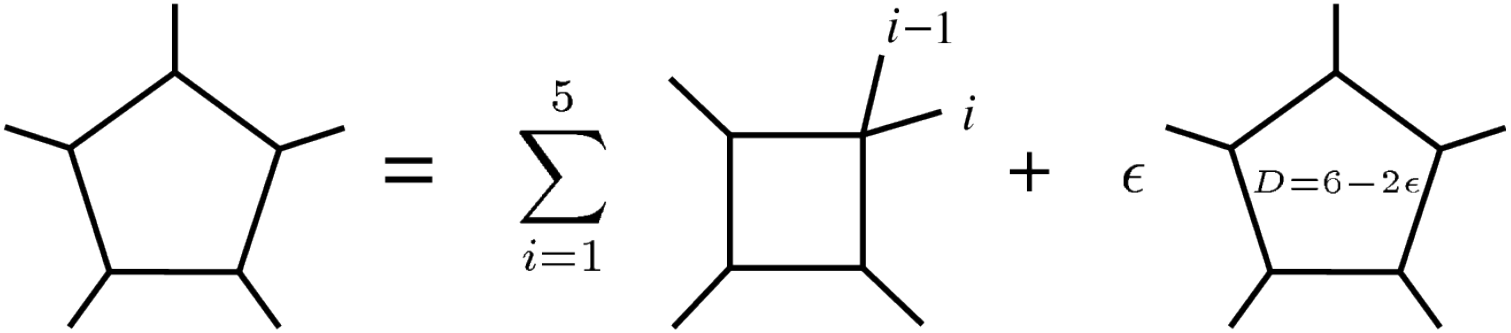


$$I^{(d)}_{a_1 a_2 a_3 a_4 a_5} \equiv$$

$$\mu^{2\epsilon} \int \frac{\mathrm{d}^d l}{i\pi^{d/2}} \frac{1}{\left(l^2\right)^{a_1} \left((l+p_1)^2\right)^{a_2} \left((l+p_1+p_2)^2\right)^{a_3} \left((l+p_1+p_2+p_3)^2\right)^{a_4} \left(l\cdot q\right)^{a_5}}$$

The pentagon

It is well known [Bern, Dixon, Kosower '94] that for the standard pentagon



- Follows from 4-dimensional relations between spin structures
- The same happens for our "pentagon" integral, with four ordinary and one linear propagator

Master results - 8 integrals with full ϵ dependence

- Ordinary Feynman integrals [Bern, Dixon, Kosower '94]

$$\text{bubbles:} \quad I_{10010}^{(4-2\epsilon)}, \quad I_{10100}^{(4-2\epsilon)}$$

$$\text{one-external-mass box:} \quad I_{11110}^{(4-2\epsilon)}$$

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- Integrals with linear propagator (known) [Sborlini '14]

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$$I_{01011}^{(4-2\epsilon)}, \quad I_{11101}^{(4-2\epsilon)}, \quad I_{10111}^{(4-2\epsilon)}$$

- Integrals with linear propagator (new)

$$I_{11011}^{(4-2\epsilon)}, \quad I_{01111}^{(4-2\epsilon)}$$

Master results - the 9th integral

- one-external-mass box with linear propagator

$$I_{11111}^{(4-2\epsilon)}$$

Master results - the 9th integral

- one-external-mass box with linear propagator

$$I_{11111}^{(4-2\epsilon)}$$

- we derive the following dimension-shift relation

$$
\begin{aligned}
2s_{123}\, I_{11111}^{(d)} = {} & \frac{\left(x_1 z_1 - x_2 z_2 + x_3 z_3\right)^2 - 4 x_1 x_3 z_1 z_3}{x_1 x_3 z_1 \left(1 - x_3 - z_3\right)} \left(d-4\right) I_{11111}^{(d+2)} \\
& + \frac{\left(x_1 z_1 - x_2 z_2 + x_3 z_3\right)\left(z_1 + z_2\right) - 2 x_3 z_1 z_3}{x_3 z_1 \left(1 - x_3 - z_3\right)} I_{01111}^{(d)} \\
& - \frac{x_1 z_1 - x_2 z_2 + x_3 z_3}{x_1 x_3 z_1} I_{10111}^{(d)} \\
& + \frac{\left(x_1 z_1 - x_2 z_2 + x_3 z_3\right) - 2 x_1 x_3}{x_1 x_3 \left(1 - x_3 - z_3\right)} I_{11011}^{(d)} \\
& - \frac{\left(x_1 z_1 - x_2 z_2 + x_3 z_3\right) - 2 x_1 \left(z_1 + z_2\right)}{x_1 \left(1 - x_3 - z_3\right)} I_{11101}^{(d)} \\
& + \frac{\left(x_1 z_1 - x_2 z_2 + x_3 z_3\right) - 2 z_1 \left(x_1 + x_2\right)}{z_1 \left(1 - x_3 - z_3\right)} \left(\frac{s_{123}}{p_{123} \cdot q}\right) I_{11110}^{(d)}
\end{aligned}
$$

Master results - the 9th integral

Hence, we need to evaluate

$$I_{11111}^{(6-2\epsilon)}$$

- Feynman representation
- Rescalings Feynman parameters

$$\alpha_3 \;\to\; \alpha_3\, y_1 \;, \qquad \alpha_4 \;\to\; \alpha_4\, z_1$$

- Defining

$$y_1 \equiv \frac{z_1}{z_1 + z_2} \in (0,1) \;, \qquad u_3 \equiv \frac{x_3}{1 - z_3} \in (0,1)$$

- Integration with PolyLogTools in the order $\alpha_3, \alpha_2, \alpha_4$
- The result up to $\mathcal{O}(\epsilon^0)$ and up to $\mathcal{O}(\epsilon)$ in double-soft limit, is expressed in terms of multiple polylogarithms

$$G(a_1, \ldots, a_n, z) \equiv \int_0^z \frac{\mathrm{d}t}{t - a_1}\, G(a_2, \ldots, a_n, t) \;, \quad G(\underbrace{0, \ldots, 0}_{n}, z) \equiv \frac{1}{n!} \ln^n(z)$$

Checks

1. comparison of the predicted singularity structure of the splitting operators [Catani et al.] with that obtained from our direct calculation

2. comparison of the anti-symmetric part of the splitting function for $q \to qq'\bar{q}'$ with the result given in [Catani, de Florian, Rodrigo '04]

3. comparison of the splitting functions for $q \to qq'\bar{q}'$ and $q \to qq\bar{q}$ expanded to $\mathcal{O}\left(\epsilon^0\right)$ between the top-down and the bottom-up approaches

4. numerical comparison of the triple-collinear limits of one-loop matrix-elements squared at $\mathcal{O}\left(\epsilon^0\right)$ for the processes $V \to q\bar{q}gg$, $H \to q\bar{q}gg$ and $H \to gggg$ with the predicted asymptotics

5. comparison of the values of the master integrals obtained from analytic formulae and from Mellin-Barnes representations up to the provided orders of ϵ-expansion

Conclusions and outlook

- We have completed the study of one-loop triple-collinear splitting functions in QCD
- We used two strategies of calculations and performed extensive analytic and numerical checks
- Our results are sufficient in ϵ expansion in order to be used for construction of a N3LO subtraction scheme
- The complete set of the splitting functions and splitting operators is provided in the form of MATHEMATICA files