Some developments in higher order Renormalisation

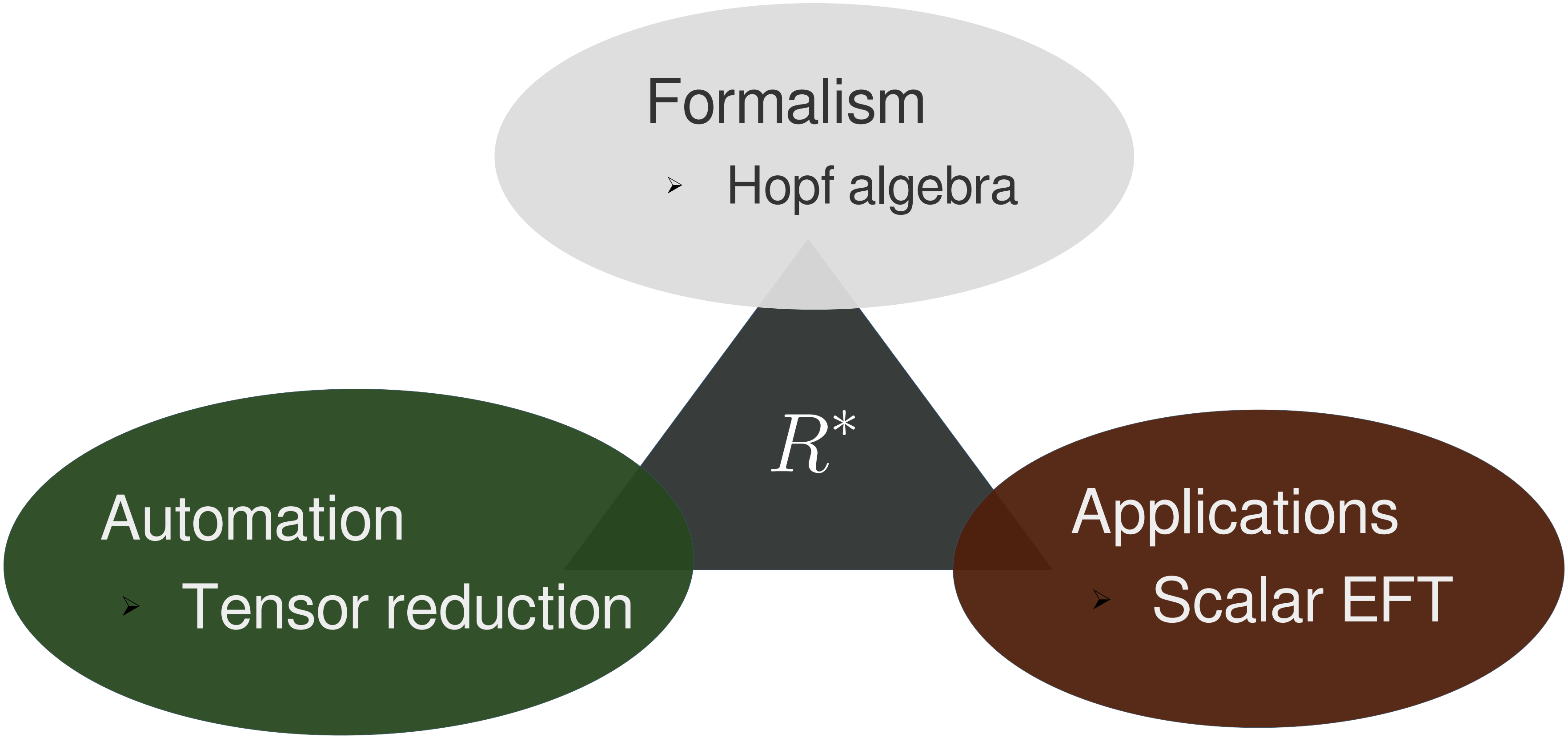
HP2- Newcastle 2022

Franz Herzog

State-of-the-art methods for multi-loop MS counterterms

R* (local&global)	5-loop QCD,...
Massive tadpoles	5-loop QCD,...
Graphical functions	ϕ^4 up to 7 loops

This talk: focus on R* approach



What is R*?

A generalisation of BPHZ to Euclidean IR divergences

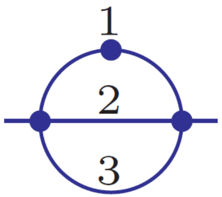
[1982 Chetyrkin, Tkachov; Smirnov]

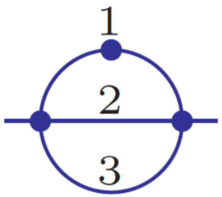
$$R^*(\Gamma) = \text{finite}$$

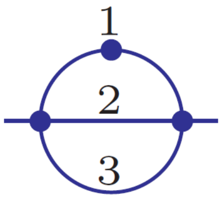
an offshell Feynman diagram

By adding counter-terms associated to UV- or IR- divergent subdiagrams

2-loop example

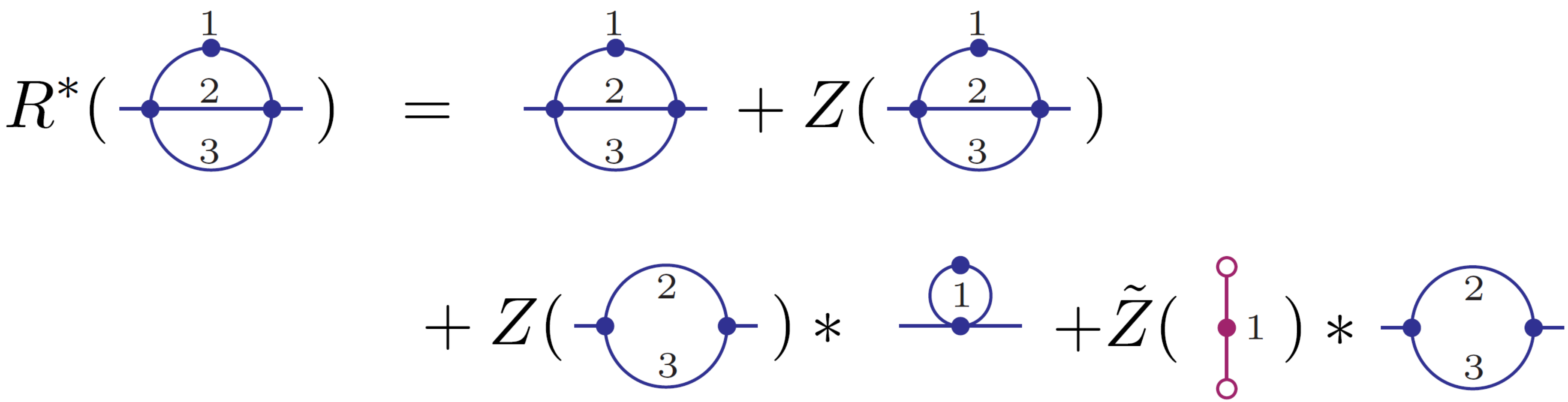
$R^*($ 

$) \quad = \quad$ 

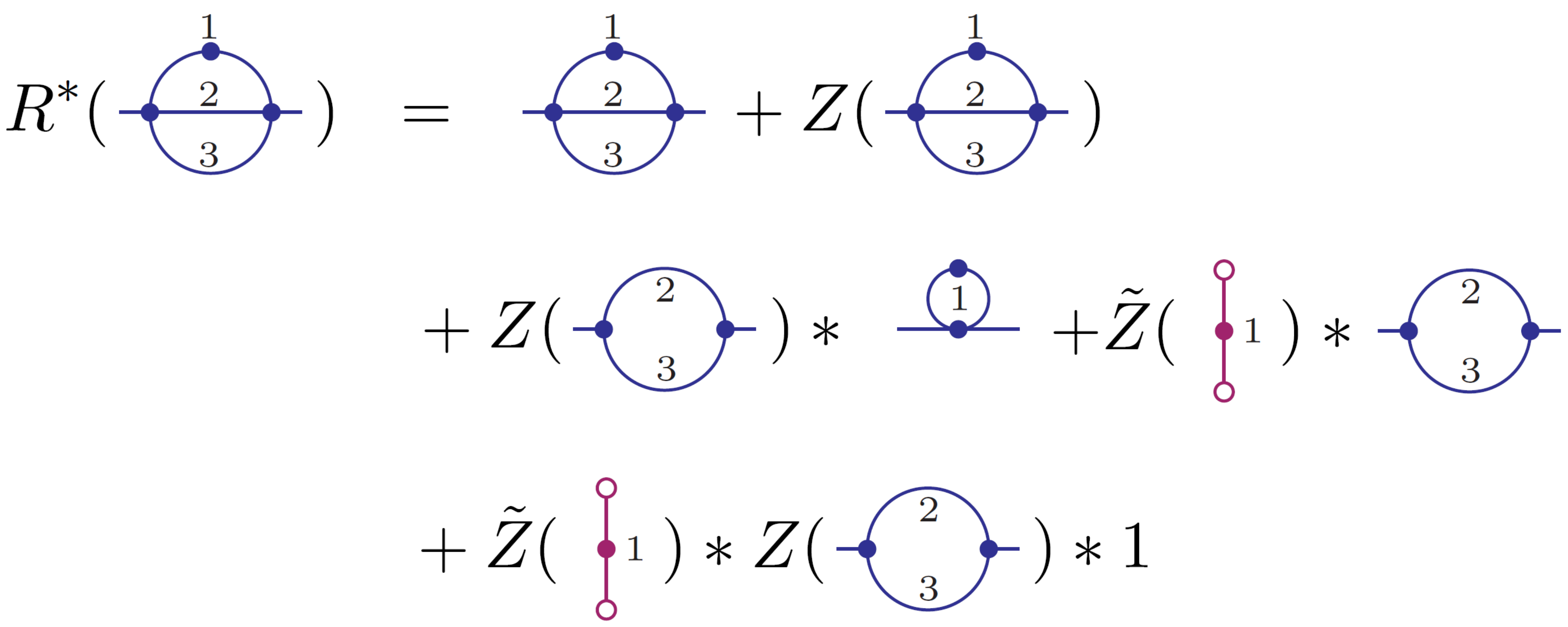
$+ Z($ 

$)$

2-loop example



2-loop example



General structure at L loops

$$R^*(\Gamma) = \sum_{\substack{S\subseteq\Gamma,\tilde{S}\subseteq\Gamma \\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma/S \setminus \tilde{S}$$

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UV counterterm

$$Z(\Gamma) = -K\Big(\sum_{\substack{S\subsetneq\Gamma,\tilde{S}\subseteq\Gamma \\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma/S \setminus \tilde{S}\Big)$$

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IR counterterm

$$\tilde{Z}(\Gamma_0) = -K\Big(\sum_{\substack{S \subseteq \Gamma_0, \tilde{S} \subsetneq \Gamma_0 \\ S \cap \tilde{S} = \emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma_0/S \setminus \tilde{S} \Big)$$

Hopf algebraic formulation of R*

with M Borinsky and R Beekveldt

$$
\begin{tikzcd}
 & H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & K \arrow[rr, "\eta"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}
$$

Hopf algebraic formulation of R*

with M Borinsky and R Beekveldt

Introduce IR and UV coactions for IR and UV divergent graphs:

$$\Delta^{\mathrm{UV}}_{\mathcal{M}}(\Gamma) = \sum_{\gamma_{\mathrm{UV}} \subseteq \Gamma} \gamma_{\mathrm{UV}} \otimes \Gamma/\gamma_{\mathrm{UV}}$$

$$\Delta^{\mathrm{IR}}_{\mathcal{M}}(\Gamma) = \sum_{\gamma_{\mathrm{mm}} \subseteq \Gamma} \gamma_{\mathrm{mm}} \otimes \underbrace{\Gamma/\gamma_{\mathrm{mm}}}_{= \gamma_{\mathrm{IR}}}$$

Hopf algebraic formulation of R*

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The coactions are *co-associative*:

$$(\Delta_{\mathrm{UV}} \otimes \mathrm{id}) \circ \Delta_{\mathrm{IR}} = (\mathrm{id} \otimes \Delta_{\mathrm{IR}}) \circ \Delta_{\mathrm{UV}}$$

Hopf algebraic formulation of R*

with M Borinsky and R Beekveldt

Allows to rewrite the original R*-operation:

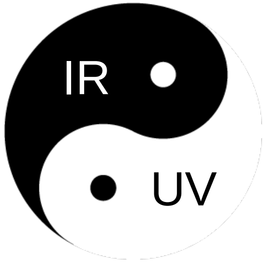
$$
\begin{aligned}
R^*(\Gamma) &= m_{\mathcal{A}}^{(2)} \circ (Z \otimes \phi \otimes \widetilde{Z}) \circ (\mathrm{id} \otimes \Delta_{\mathcal{M}}^{IR}) \circ \Delta_{\mathcal{M}}^{UV}(\Gamma) \\
&= \sum_{\gamma_{\mathrm{UV}} \subseteq \Gamma} \; \sum_{\gamma_{mm} \subseteq \Gamma/\gamma_{\mathrm{UV}}} Z(\gamma_{\mathrm{UV}})\, \phi(\gamma_{\mathrm{mm}})\, \widetilde{Z}(\Gamma/\gamma_{\mathrm{UV}}/\gamma_{\mathrm{mm}})
\end{aligned}
$$

Hopf algebraic formulation of R*

with M Borinsky and R Beekveldt

- UV and IR counterterms are related via the *antipode*

$$\tilde{Z} = Z \circ S$$



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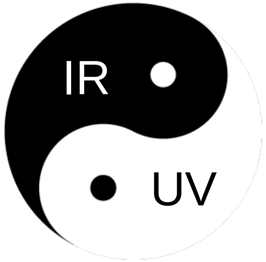
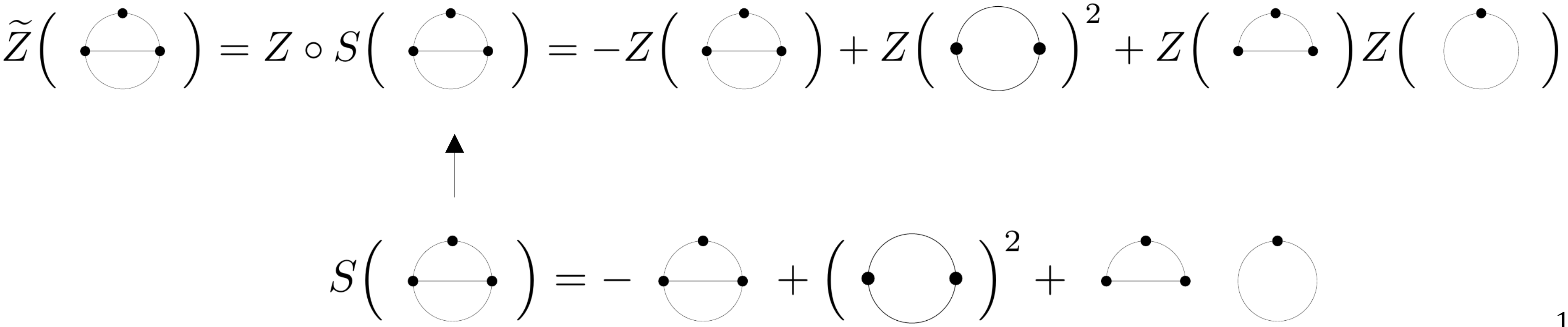
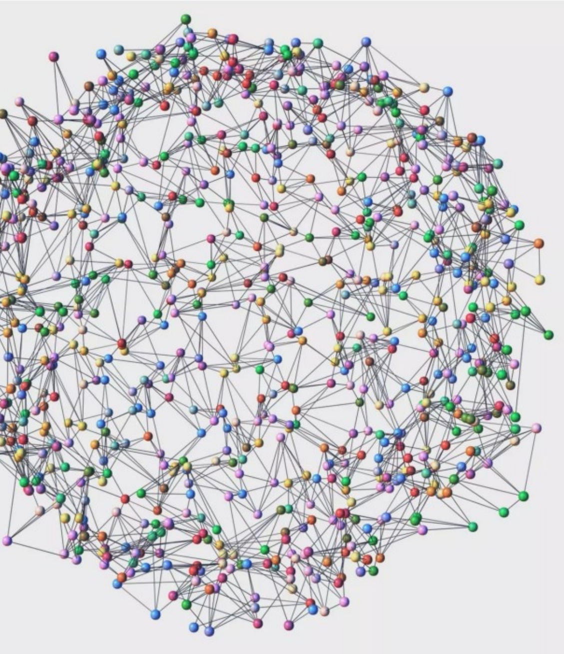


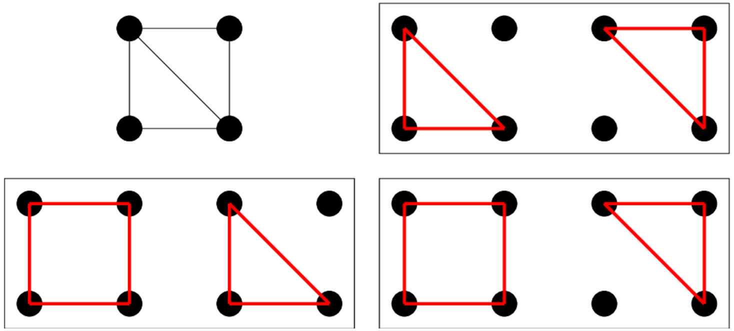
- Example:



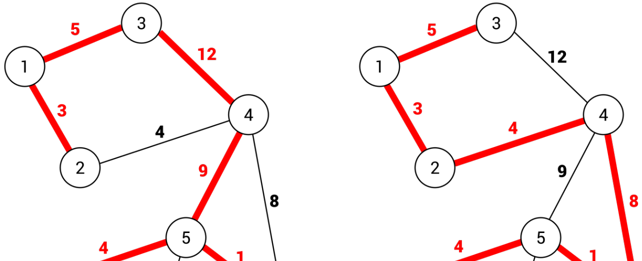
R* Automation

Its mostly just recursion and basic graph theory algorithms





$$Z(\Gamma) = -K\Bigg(\sum_{\substack{S\subsetneq\Gamma,\tilde{S}\subseteq\Gamma\\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma/S \setminus \tilde{S}\Bigg)$$



R* Automation

And Taylor expansion

$$Z(\Gamma) = \frac{1}{n!} \sum_i \Big(\prod_{j=1}^{\omega} Q^{\mu_j}_{i(j)} \Big) Z\big(\prod_{j=1}^{\omega} \partial^{\mu_j}_{i(j)} \Gamma \big)$$

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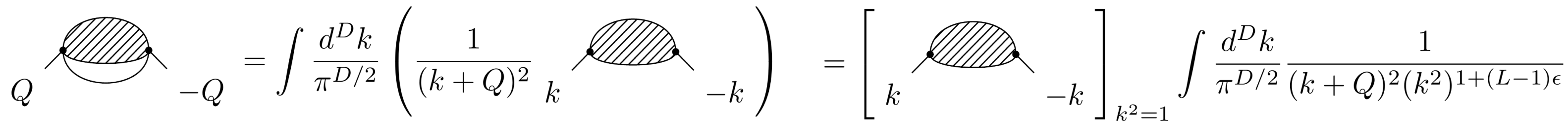
Pros:

- Reduce higher degree divergences to logarithmic ones
- log counterterms are independent of external kinematics, allows for arbitrary IRR



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Cons:

- Many terms+and many indices are created in the process.
- Tough tensor reductions of vacuum integrals!
- For the 5-loop beta function required up to rank ~18

Tensor Reduction: basic problem

$$I^{\mu_1 \dots \mu_n} = \sum_{\sigma \in S_2^n} g^{\mu_{\sigma(1)}\mu_{\sigma(2)}} .. g^{\mu_{\sigma(n-1)}\mu_{\sigma(n)}} I_{\sigma(1)\dots\sigma(n)}$$

- Number of scalar coefficients: $|S_2^n| = \frac{n!}{(n/2)!2^{n/2}} = (n-1)!!$

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n	2	4	6	8	10	12	14	16	18	20
$\lvert S_2^n\rvert$	1	3	15	105	945	10,395	135,135	2,027,025	34,459,425	654,729,075

- i.e. Passarino-Veltman reduction would lead to solving a *dense* 135,135x135,135 linear system at rank 14!

Tensor reduction: Projectors

- Brute force approach is unfeasible for large n!
  (where !=exclamation mark not factorial)

  Use shorthand

$$I = \sum_{\sigma \in S_2^n} g(\sigma) I(\sigma)$$

Tensor reduction: Projectors

- Brute force approach is unfeasible for large n!
  (where !=exclamation mark not factorial)

  Use shorthand
  $$I = \sum_{\sigma \in S_2^n} g(\sigma) I(\sigma)$$

  Define projector $P(\sigma) = P^{\mu_{\sigma(1)} \cdots \mu_{\sigma(n)}}$ such that $P(\sigma) \cdot g(\sigma') = \delta_{\sigma\sigma'}$

  then
  $$I(\sigma) = P(\sigma) \cdot I$$

Tensor reduction: Projectors

General Ansatz $$P(\sigma) = \sum_{\sigma'} c(\sigma, \sigma')\, g(\sigma')$$

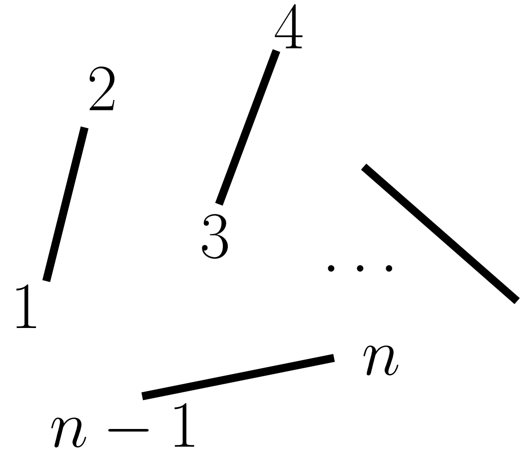
Tensor reduction: Projectors

General Ansatz $$P(\sigma) = \sum_{\sigma'} c(\sigma, \sigma')\, g(\sigma')$$

We constrain the coefficients using symmetry (stabilizer) group of $g(\sigma)$:

$$\boxed{S_2 \times \cdots \times S_2} \times \boxed{S_{n/2}}$$

$$g_{\mu_1\mu_2}\, g_{\mu_3\mu_4} \cdots g_{\mu_{n-1}\mu_n}$$



Tensor reduction: Projectors

The orbit partition formula:

$$P(\sigma) = \sum_k c_k \sum_{\tau \in C_k(\sigma)} g(\tau)$$

***k* sums over orbits**:
C_k : sets of permutations which are related by internal symmetry group.

Tensor reduction: Projectors

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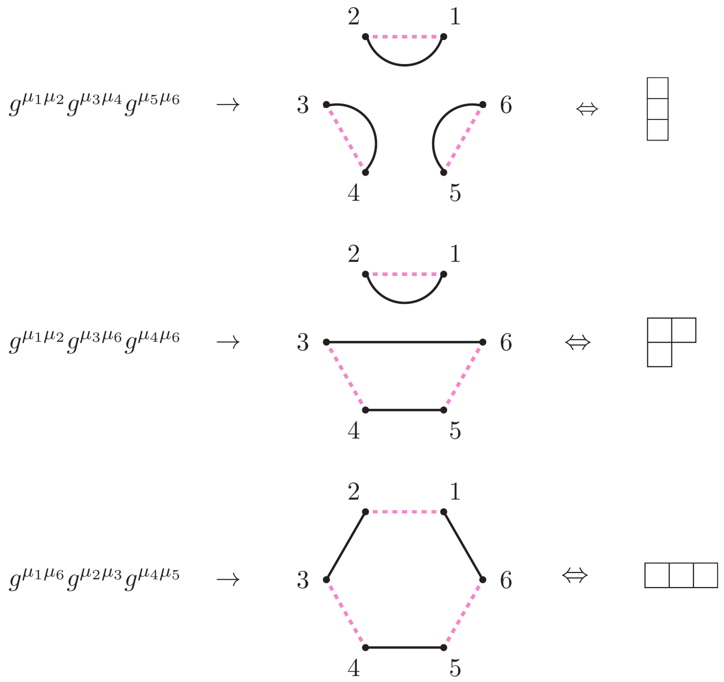
Example: Projector of $g^{\mu_1\mu_2} g^{\mu_3\mu_4}$ (G:1↔2,3↔4,12↔34)

$$P_{\mu_1\mu_2\mu_3\mu_4} = c_{1,1}\, g_{\mu_1\mu_2} g_{\mu_3\mu_4} + c_2 \left(g_{\mu_1\mu_3} g_{\mu_2\mu_4} + g_{\mu_1\mu_4} g_{\mu_2\mu_3} \right)$$

Tensor reduction: Projectors

in preparation with J. Vermaseren, J Goode and Sam Teale

- Can label *orbits* (independent c_ks) by *integer partitions* of n/2
- Integer partitions $\leftrightarrow$ *cycle structure* of bi-chord graphs



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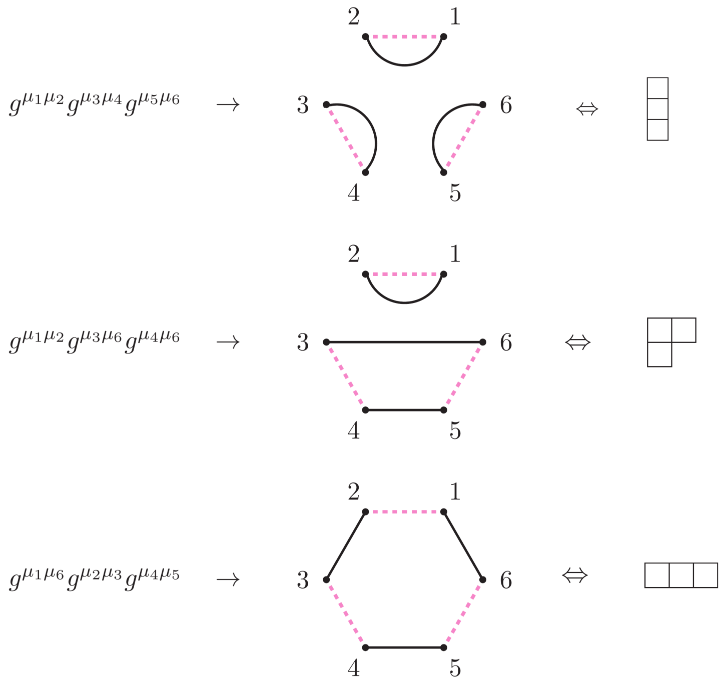
- Integer partitions $\leftrightarrow$ *cycle structure* of bi-chord graphs
- Orbit partition tames growth of system size:



n	2	4	6	8	10	12	14	16	18	20
$\lvert S_2^n \rvert$	1	3	15	105	945	10,395	135,135	2,027,025	34,459,425	654,729,075
$p(\frac{n}{2})$	1	2	3	5	7	11	15	22	30	42

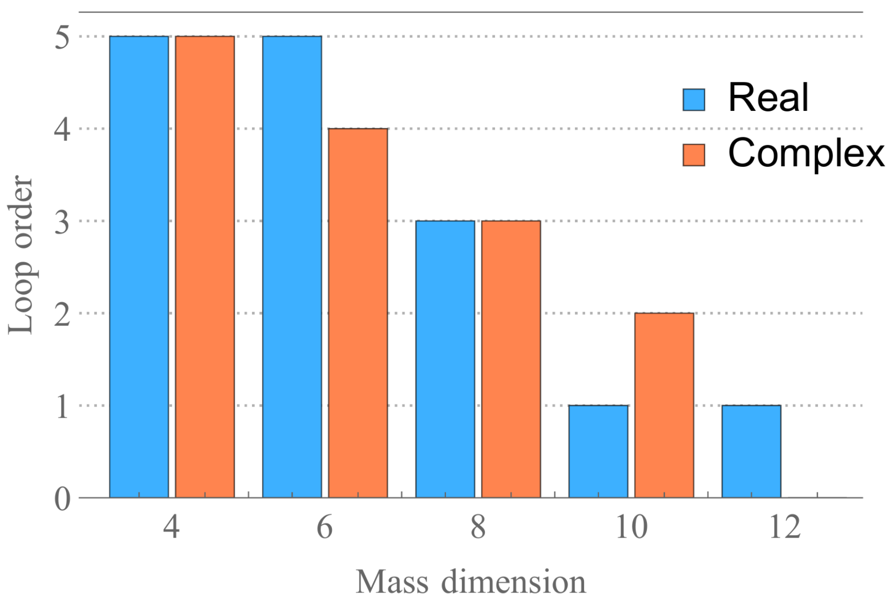
Extension to spinors in progress

Application: Scalar EFT

arxiv:2105.12742 with W Cao, T Melia and J Nepveu

Compute higher orders both in loop and EFT expansion with R*:

- Great testing ground for method
- Investigate non-renormalisation theorems



Scalar EFT Basics

- Lagrangian

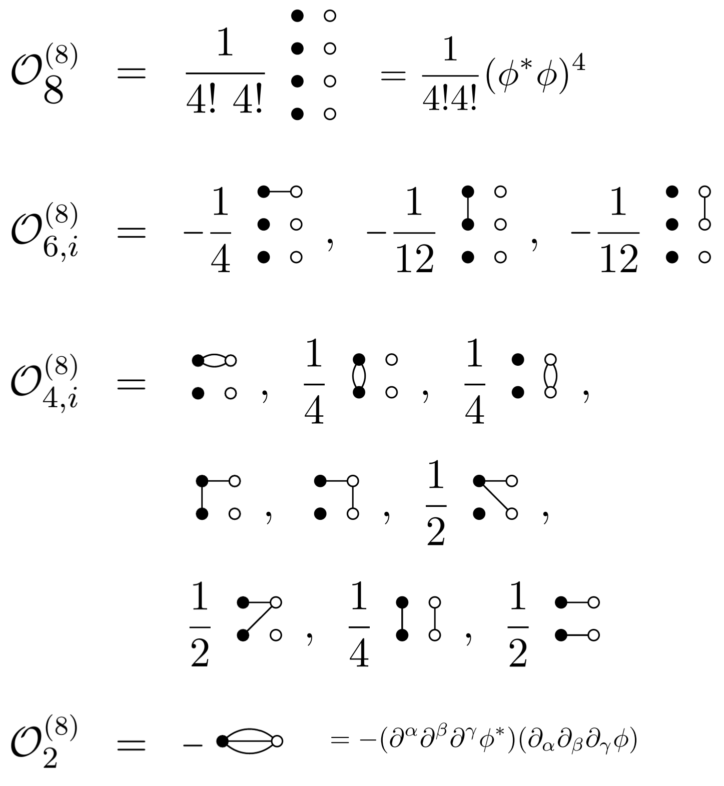
$$\mathcal{L} = \mathcal{L}^{(4)}(\phi, \partial_\mu \phi) + \sum_{n>4} \frac{1}{\Lambda^{n-4}} \sum_i \tilde{c}_i^{\,b(n)}\, \tilde{\mathcal{O}}_i^{b(n)}$$

- Basis of operators not unique
  - Redundancies IBP and Field redefinitions
- Couplings mix under renormalisation

$$\mu \frac{d}{d\mu} c_i^{(n)} = \sum_j \gamma_{ij}^{(n)} c_j^{(n)}$$

Complex scalar operators

at mass dimension 8



- dots = complex fields
- Circles= complex conjugate fields
- Lines= contracted pair of derivatives
- Conformal Primaries:

$$\mathcal{O}_8^{(8)c} = \mathcal{O}_8^{(8)}$$

$$\mathcal{O}_6^{(8)c} = -\frac{2}{3}\mathcal{O}_{6,1}^{(8)} + \left(\mathcal{O}_{6,2}^{(8)} + \mathcal{O}_{6,3}^{(8)}\right)$$

$$\begin{aligned}\mathcal{O}_4^{(8)c}(x,y) = {} & (5x-y)\mathcal{O}_{4,1}^{(8)} + 2x\mathcal{O}_{4,2}^{(8)} + 2x\mathcal{O}_{4,3}^{(8)} - 4x\mathcal{O}_{4,4}^{(8)} - 4x\mathcal{O}_{4,5}^{(8)} \\ & + (4y-16x)\mathcal{O}_{4,6}^{(8)} + (4y-16x)\mathcal{O}_{4,7}^{(8)} + 4y\mathcal{O}_{4,8}^{(8)} + (36x-8y)\mathcal{O}_{4,9}^{(8)}\end{aligned}$$

Anomalous dimension matrix

at mass dimension 8

- Most zeroes were expected by a selection rule by Bern et al.
- Two unexpected zeroes were found!

$$\gamma_{\mathrm{C}}^{(8)} = \begin{array}{c} \begin{array}{cccc} g^3\,\mathcal{O}_8^{(8)c} & g^2\,\mathcal{O}_6^{(8)c} & g\,\mathcal{O}_4^{(8)c}{}_{(1,0)} & g\,\mathcal{O}_4^{(8)c}{}_{(0,1)} \end{array} \\ \begin{pmatrix} 29g - 409g^2 + \left(2352\zeta_3 + \frac{57765}{8}\right)g^3 & \boxed{0g} - 240g^2 + \left(2304\zeta_3 + 5882\right)g^3 & \frac{54216g}{5} - \frac{958314g^2}{5} + \left(\frac{4083264\zeta_3}{5} + \frac{55731313}{15}\right)g^3 & -\frac{8856g}{5} + \frac{159894g^2}{5} - \left(\frac{713664\zeta_3}{5} + \frac{9281093}{15}\right)g^3 \\ 0g+0g^2+\boxed{0g^3} & 4g - \frac{122g^2}{3} + \left(216\zeta_3 + \frac{4559}{12}\right)g^3 & -\frac{679g}{5} + \frac{14209g^2}{12} - \left(\frac{29216\zeta_3}{5} + \frac{3248605}{324}\right)g^3 & \frac{164g}{5} - \frac{9233g^2}{36} + \left(\frac{6056\zeta_3}{5} + \frac{2740291}{1296}\right)g^3 \\ 0g + 0g^2 + 0g^3 & 0g+0g^2 + \frac{5g^3}{108} & \frac{11g}{3} - \frac{29g^2}{180} - \left(\frac{32\zeta_3}{3} + \frac{97091}{9720}\right)g^3 & -\frac{4g}{3} + \frac{1057g^2}{540} - \left(\frac{16\zeta_3}{3} + \frac{10868}{1215}\right)g^3 \\ 0g + 0g^2 + 0g^3 & 0g+0g^2 + \frac{115g^3}{324} & \frac{46g}{3} - \frac{14557g^2}{540} + \left(96\zeta_3 + \frac{198001}{2430}\right)g^3 & -\frac{19g}{3} + \frac{2809g^2}{180} - \left(\frac{176\zeta_3}{3} + \frac{718739}{9720}\right)g^3 \end{pmatrix} \end{array}$$

which multiplies the couplings of

$$\left\{\mathcal{O}_8^{(8)c}\,,\ \mathcal{O}_6^{(8)c}\,,\ \mathcal{O}_4^{(8)c}{}_{(1,0)}\,,\ \mathcal{O}_4^{(8)c}{}_{(0,1)}\right\}.$$

Similar at mass dimension 6

$$\gamma_{\mathrm{C}}^{(6)} = \begin{array}{c} \begin{array}{cc} g^2\,\mathcal{O}_6^{(6)c} \qquad\qquad\qquad\qquad & \qquad\qquad\qquad\qquad g\,\mathcal{O}_4^{(6)c} \end{array} \\ \begin{pmatrix} \begin{array}{c} 14g - \frac{297g^2}{2} + \left(816\zeta_3 + \frac{14981}{8}\right)g^3 \\ -\left(\frac{32087\zeta_3}{2} - 2892\zeta_4 + 23320\zeta_5 + \frac{888983}{32}\right)g^4 \end{array} & \begin{array}{c} \boxed{0g} - \frac{45g^2}{2} + \left(216\zeta_3 + \frac{6153}{16}\right)g^3 \\ -\left(783\zeta_3 + 8100\zeta_5 + \frac{74079}{16}\right)g^4 \end{array} \\ \\ 0g + 0g^2 \boxed{+0g^3} + \frac{5g^4}{6} & \begin{array}{c} -g + \frac{13g^2}{2} - \left(36\zeta_3 + \frac{383}{12}\right)g^3 \\ +\left(\frac{769\zeta_3}{2} - 123\zeta_4 + 560\zeta_5 + \frac{7893}{32}\right)g^4 \end{array} \end{pmatrix} \end{array}$$

- The existence of upper right zero appears only in the conformal primary basis – we have little understanding why
- The lower one seems basis independent and we have no idea why it appears.

and mass dimension 10

$$
\gamma_{\mathrm{C}}^{(10)} =
\begin{array}{c}
\begin{array}{cccccccc}
g^4\,\mathcal{O}_{10}^{(10)c} & g^3\,\mathcal{O}_{8}^{(10)c} & g^2\,\mathcal{O}_{6}^{(10)c}(1,0,0,0) & g^2\,\mathcal{O}_{6}^{(10)c}(0,1,0,0) & g^2\,\mathcal{O}_{6}^{(10)c}(0,0,1,0) & g^2\,\mathcal{O}_{6}^{(10)c}(0,0,0,i) & g\,\mathcal{O}_{4}^{(10)c}(1,0) & g\,\mathcal{O}_{4}^{(10)c}(0,1)
\end{array} \\
\begin{pmatrix}
50g - \frac{1725g^2}{2} & \boxed{0g} - 1125g^2 & \frac{412500g}{7} - \frac{61613665g^2}{42} & \frac{3300g}{7} - \frac{19655g^2}{6} & \frac{141900g}{7} - \frac{6734025g^2}{14} & 0 & -\frac{500460g}{7} + \frac{31376515g^2}{21} & -64740g + \frac{16819700g^2}{21} \\
0g \boxed{+0g^2} & 15g - \frac{385g^2}{2} & -\frac{1474g}{7} + \frac{83689g^2}{45} & -\frac{234g}{7} + \frac{386467g^2}{315} & \frac{326g}{7} - \frac{4577g^2}{7} & 0 & -\frac{43732g}{35} + \frac{5863532g^2}{315} & \frac{14272g}{5} - \frac{15354406g^2}{315} \\
0g + 0g^2 & 0g+0g^2 & \frac{35g}{6} - \frac{678389g^2}{15120} & -\frac{g}{6} + \frac{24379g^2}{15120} & \frac{7g}{6} - \frac{28265g^2}{3024} & 0 & \frac{1006g}{105} + \frac{1249571g^2}{15120} & -\frac{967g}{30} + \frac{1903001g^2}{3780} \\
0g + 0g^2 & 0g+0g^2 & \frac{4g}{3} - \frac{331g^2}{630} & -\frac{2g}{3} - \frac{5401g^2}{945} & 2g - \frac{14965g^2}{756} & 0 & -\frac{4159g}{105} + \frac{3146033g^2}{7560} & \frac{914g}{15} - \frac{103037g^2}{315} \\
0g + 0g^2 & 0g+0g^2 & \frac{15g}{2} - \frac{247109g^2}{5040} & -\frac{g}{2} - \frac{4127g^2}{1680} & \frac{7g}{2} - \frac{31333g^2}{1008} & 0 & -\frac{1769g}{35} + \frac{243011g^2}{336} & \frac{573g}{10} + \frac{37049g^2}{1260} \\
0 & 0 & 0 & 0 & 0 & \frac{10g}{3} - \frac{770g^2}{27} & 0 & 0 \\
0g + 0g^2 & 0g + 0g^2 & 0g+0g^2 & 0g+0g^2 & 0g+0g^2 & 0 & -2g + \frac{223g^2}{24} & -g + \frac{7g^2}{24} \\
0g + 0g^2 & 0g + 0g^2 & 0g+0g^2 & 0g+0g^2 & 0g+0g^2 & 0 & -g + \frac{7g^2}{12} & -3g + \frac{251g^2}{24}
\end{pmatrix}
\end{array}
$$

Summary & Outlook

- R* formalism
  - Discussed Hopf algebra framework to streamline operations, e.g. IR counterterm via antipode relation
- Discussed high rank tensor problem with R*
  - Presented an orbit partition formula for the projector
  - In progress: extend to spinor indices
- Finally discussed application in Scalar EFT
  - Two new zeros were found in mixing matrix
  - In progress: Extending non-renormalisation thms to multi-linear mixing