P-adic Numbers and Partial Fractions Ansätze for Amplitudes

Ben Page

CERN, Theoretical Physics Department

HP2 Newcastle
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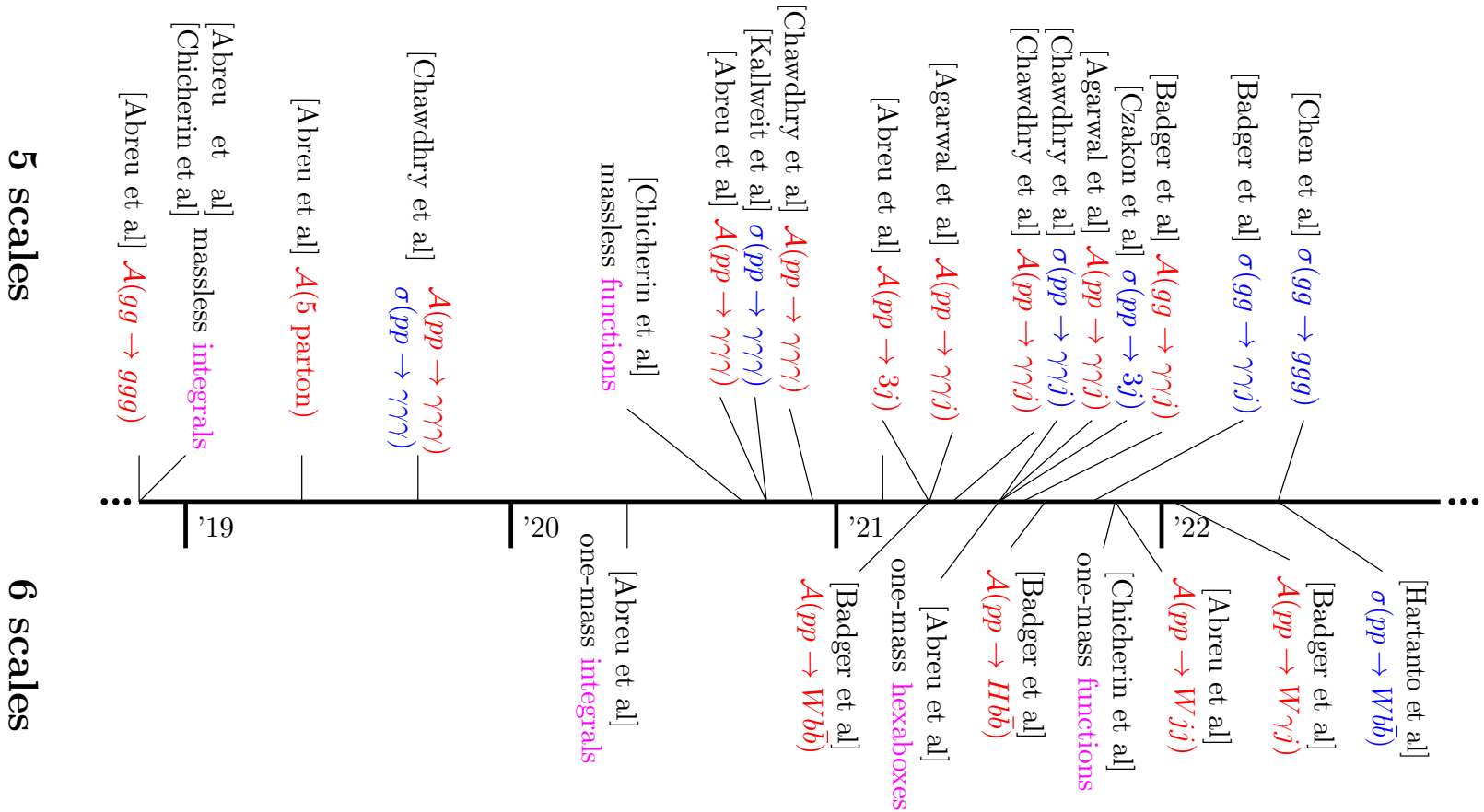
In collaboration with Giuseppe De Laurentis

based on [2203.04269]

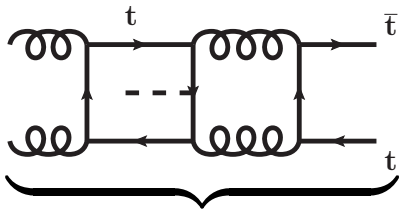
Large Progress in Multi-Scale NNLO Calculations



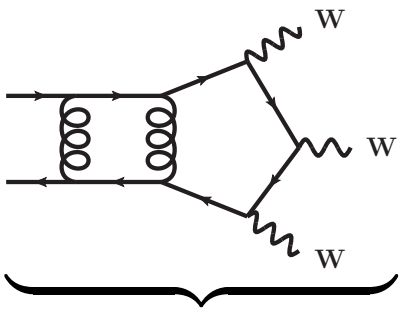
- See talks by [Bayu, Giuseppe, Jakub, Rene, Ryan].
- Collection of $2 \to 3$ calculations facilitated by appropriate techniques.

What About More Scales?

- Many important processes involve multiple massive particles.



7 scales



8 scales

- Integrals are becoming elliptic (or worse).

- Rational functions are becoming exponentially more complicated.

This talk: Focus on rational functions.

Analytic Reconstruction as it Stands

- Paradigm-shift insight: use finite-field evaluations to determine $\mathcal{C}_k$.

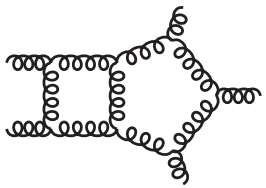
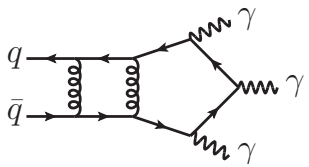
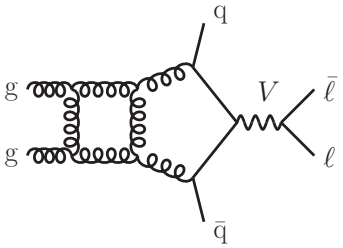
$$\left\{\mathcal{C}_k(p_1^{(1)},\ldots,p_n^{(1)}),\ldots,\mathcal{C}_k(p_1^{(N)},\ldots,p_n^{(N)})\right\} \xrightarrow{\text{reconstruct}} \mathcal{C}_k.$$

[von Manteuffel, Schabinger '14; Peraro '16],

FiniteFlow [Peraro '19], Firefly [Klappert, Lange, Yannick '19, '20]

- Reconstruction complexity dominated by sampling.

- Evaluation count for (selected) recent two-loop five-point amplitudes:

	Three-jet	Three-photon	W + two-jets*
Process			
# Samples	$\sim 10^5$	$\sim 10^5$	$\sim 10^6$.

* After simplification via [Badger et al '20]

Simplifications from Partial Fractions

- Common observation: partial fractions simplifies $\mathcal{C}_k$. Toy example:

$$\frac{\mathcal{N}}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3} \to \cancel{\frac{\Delta}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3}} + \frac{\Delta_1}{\mathcal{D}_2\mathcal{D}_3} + \frac{\Delta_2}{\mathcal{D}_1\mathcal{D}_3} + \cancel{\frac{\Delta_3}{\mathcal{D}_1\mathcal{D}_2}} + \frac{\Delta_{23}}{\mathcal{D}_1} + \cancel{\frac{\Delta_{13}}{\mathcal{D}_2}} + \frac{\Delta_{12}}{\mathcal{D}_3}.$$

- Comes in multiple flavours:

Approach	Analytic Algorithm?	Reconstruction compatible?	Avoids spurious singularities?
Univariate	✓	✓†	X
Multivariate	✓*	X	X

*[Pak '11; Abreu, Dormans, Febres Cordero, Ita, BP, Sotnikov '19]
implementations: [Boehm, Wittmann, Wu, Xu, Zhang '20; Heller, von Manteuffel '21]

†[Badger, Hartanto, Zoia '21]

Can we avoid spurious singularities and simplify reconstruction?

The Approach of [De Laurentis, BP '22]

- Work in spinor space* to manifest gauge-theory simplifications.

$$\mathcal{C}_k(p_1, \ldots, p_n) \quad \rightarrow \quad \mathcal{C}_k(\lambda, \tilde{\lambda}).$$

*Algorithmic toolkit provided.

- Numerically study $\mathcal{C}_k$ to understand partial-fractions structure.

$$\frac{\mathcal{N}}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{\text{rest}}} = \cancel{\frac{\Delta}{\mathcal{D}_1\mathcal{D}_2\mathcal{D}_{\text{rest}}}} + \frac{\Delta_1}{\mathcal{D}_2\mathcal{D}_{\text{rest}}} + \frac{\Delta_2}{\mathcal{D}_1\mathcal{D}_{\text{rest}}}?$$

See also [De Laurentis, Maître '19].

- Construct Ansatz $\mathfrak{a}_l$ from study. Constrain c_{kl} by finite field sampling.

$$\mathcal{C}_k(\lambda, \tilde{\lambda}) = \sum_{l=1}^{N} c_{kl}\mathfrak{a}_l(\lambda, \tilde{\lambda}), \qquad c_{kl} \in \mathbb{Q}.$$

A First Attempt at Numerical Partial Fractions

- Consider tree-level six-point one-quark line amplitude $\mathcal{A}_{q^+g^+g^+\overline{q}^-g^-g^-}$

$$\mathcal{A} = \frac{\mathcal{N}^*}{\langle 12\rangle\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}}.$$

*$\mathcal{N}$ is a degree 6 polynomial in spinor brackets.

- Can we rewrite without both $\langle 12\rangle$ and $\langle 23\rangle$ poles?

$$\mathcal{A} = \frac{\Delta_{12}}{\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}} + \frac{\Delta_{23}}{\langle 12\rangle\langle 34\rangle[45][56][61]s_{345}}?$$

- [De Laurentis, Maître '19]: Probe $\mathcal{A}$ on points where $\langle 12\rangle$, $\langle 23\rangle$ are small.

$$\lambda_2^\alpha \sim \epsilon \quad \Rightarrow \quad \mathcal{A} \sim \epsilon^{-2}$$

$$\langle 12\rangle \sim \langle 23\rangle \sim \langle 13\rangle \sim \epsilon \quad \Rightarrow \quad \mathcal{A} \sim \epsilon^{-1}.$$

Thinking in Terms of Polynomials

- Let's ask an equivalent question:

$$\mathcal{N} = \Delta_{12}\langle 12\rangle + \Delta_{23}\langle 23\rangle ?$$

- Mathematically, we can ask if $\mathcal{N}$ belongs to an "ideal":

$$\mathcal{N} \in \Big\langle \langle 12\rangle, \langle 23\rangle \Big\rangle ?$$

- Ideal is infinite set of polynomial combinations of generators:

$$\Big\langle \langle 12\rangle, \langle 23\rangle \Big\rangle = \Big\{ a_1\langle 12\rangle + a_2\langle 23\rangle \quad | \quad a_i \text{ are any spinor polynomials} \Big\}.$$

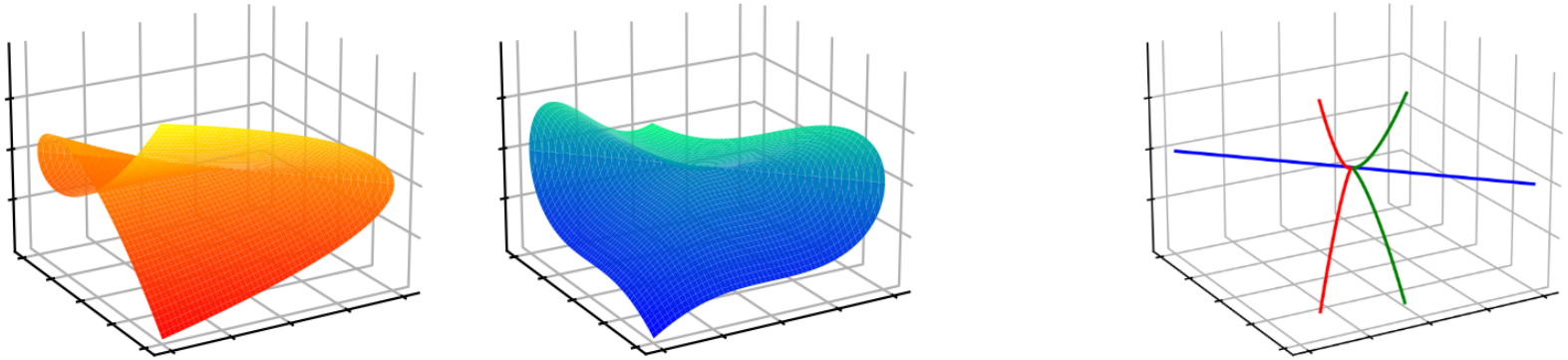
Zariski Nagata Theorem[**]

If $\mathcal{N}$ vanishes everywhere where $\langle 12\rangle = \langle 23\rangle = 0$[*] then $\mathcal{N} \in \Big\langle \langle 12\rangle, \langle 23\rangle \Big\rangle$.

[*] and $\Big\langle \langle 12\rangle, \langle 23\rangle \Big\rangle$ is radical. [**] Higher order vanishing also handled.

Branching of Surfaces Defined by Polynomials

- When we intersect surfaces, we may have multiple branches.



$xy^2 + y^3 - z^2 = 0$ $\quad$ $x^3 + y^3 - z^2 = 0,$ $\quad$ $xy^2 + y^3 - z^2 = x^3 + y^3 - z^2 = 0.$

- Our double denominator zero surface has two branches:

$$\langle 12 \rangle = \langle 23 \rangle = 0 \quad \Leftrightarrow \quad \langle 12 \rangle = \langle 23 \rangle = \langle 13 \rangle = 0 \quad \text{or} \quad \lambda_2^\alpha = 0.$$

- We compute branchings with primary decomposition techniques.

[De Laurentis, BP '22], see also [Zhang '12].

Ansatz Construction Algorithm, Sketched

1. Construct branches of surfaces where two denominators vanish.

$$\mathcal{D}_i = \mathcal{D}_j = 0 \qquad \longrightarrow \qquad \mathcal{V} = \{U_1, U_2, \ldots\}.$$

2. Sample near surface to determine degree of divergence.

$$U: \quad \mathcal{D}_i \sim \mathcal{D}_j \sim \epsilon \qquad \Rightarrow \qquad \mathcal{C}_k \sim \frac{1}{\epsilon^{\kappa_U}}.$$

3. Ansatz is basis of intersection of associated ideals of vanishing polynomials. Ansatz constructed using Gröbner basis techniques.

$$\mathcal{N}_k \in \bigcap_{U \in \mathcal{V}} I(U)^{\langle \kappa_U \rangle}.$$

How To Perform Numerical Investigations?

- Need to find phase-space points $(\lambda^\epsilon, \tilde{\lambda}^\epsilon)$ where $\mathcal{D}_i$ are small.

$$\mathcal{D}_i(\lambda^\epsilon, \tilde{\lambda}^\epsilon) \quad \sim \quad \mathcal{D}_j(\lambda^\epsilon, \tilde{\lambda}^\epsilon) \quad \sim \quad \epsilon.$$

- Conflict with modern techniques: no small elements in a finite field.

$$|0|_{\mathbb{F}_p} = 0, \qquad \text{and} \qquad a \neq 0 \;\Rightarrow\; |a|_{\mathbb{F}_p} = 1.$$

- Approaching with complex numbers would be plagued by instabilities.

Enter the p-adic numbers – a middle ground between finite fields and $\mathbb{C}$.

Introduction to the p-adic Numbers

- The p-adic numbers roughly correspond to Laurent series in p.

$$x = \sum_{i=\nu}^{\infty} a_i p^i = a_\nu p^\nu + a_{\nu+1} p^{\nu+1} + \cdots, \qquad \left(\begin{smallmatrix} a_i \in [0, p-1], \\ a_\nu \neq 0. \end{smallmatrix} \right).$$

- The p-adic numbers form a field. $x, y \in \mathbb{Q}_p \Rightarrow$

$$x + y \in \mathbb{Q}_p, \quad -x \in \mathbb{Q}_p, \quad x \times y \in \mathbb{Q}_p, \quad \frac{1}{x} \in \mathbb{Q}_p \text{ (if } x \neq 0).$$

- The p-adic absolute value allows for small numbers ($p \sim \epsilon$).

$$|x|_p = p^{-\nu}, \quad \Rightarrow \quad |p|_p < |1|_p.$$

Computing with p-adic Numbers

- For computing purposes* we truncate to finite order.

$$x = p^{\nu(x)}\Big(\underbrace{\tilde{x}}_{\text{mantissa}} + \mathcal{O}(p^k)\Big).$$

*Try [https://github.com/GDeLaurentis/pyadic] to investigate yourselves.

- Truncation reduces to finite field case for $\nu = 0, k = 1$.

- Arithmetic $(+ - /*)$ is essentially performed modulo p^k, e.g.

$$x \times y = p^{\nu(x)+\nu(y)}\left(\tilde{x}\tilde{y} + \mathcal{O}(p^k)\right).$$

- Mantissa inverse computed with extended euclidean algorithm.

Studying the Six-Point Tree [De Laurentis, BP; 2207.10125]

$$\mathcal{A} = \frac{\mathcal{N}}{\langle 12\rangle\langle 23\rangle\langle 34\rangle[45][56][61]s_{345}}.$$

- Probe 108 surfaces where pairs of $\langle ij\rangle, [ij], s_{ijk}$ are p-adically small.

$$\text{e.g.} \qquad [12] \sim [13] \sim [23] \sim \mathcal{O}(p) \qquad \Rightarrow \qquad \mathcal{A} \sim \mathcal{O}(p^2).$$

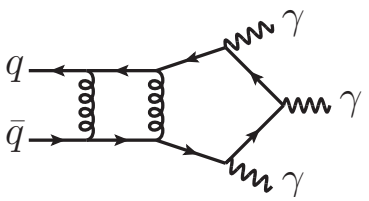
- $\mathcal{N}$ vanishes non-trivially on 28 surfaces. Many ideal memberships:

$$\mathcal{N} \quad \in \quad \Big\langle [12],[13],[23]\Big\rangle^2 \cap \Big\langle \langle 12\rangle, \langle 34\rangle\Big\rangle \cap \Big\langle \langle 12\rangle, [16]\Big\rangle \cap (25 \text{ more}).$$

- Imposing that $\mathcal{N}$ is a degree six polynomial gives one term Ansatz:

$$\mathcal{N} = c_0\Big(\langle 12\rangle[21]\langle 45\rangle[54]\langle 4|2{+}3|1]\rangle + [16]\langle 6|1{+}2|3]\langle 34\rangle s_{123}\Big), \quad c_0 \in \mathbb{Q}.$$

Proof-of-Concept Remainders for $q\bar{q} \to \gamma\gamma\gamma$

$\mathcal{A} =$ 

$+ \cdots .$

(Simulated evaluations using analytics from [Abreu, BP, Pascual, Sotnikov '20]).

- Analyze remainder, reconstruct pentagon function coefficients.
- Fitting Ansatz now requires at most 566 $\mathbb{F}_p$ samples.

Amplitude	$R^{(2,0)}_{-++}$	$R^{(2,N_f)}_{-++}$	$R^{(2,0)}_{+++}$	$R^{(2,N_f)}_{+++}$
Ansatz Dim [Abreu et al '20]	41301	2821	7905	1045
Ansatz Dim [De Laurentis, BP '22]	566	20	18	6

Summary

- Rational functions in amplitudes have poorly understood structure.
- We study that structure with p-adic evaluations in singular limits. This behavior is interpreted in terms of ideals to build Ansätze.
- Approach shows great promise for amplitudes involving more scales.

Backup

Lorentz Invariance

- Coefficients are Lorentz invariant functions of spinor brackets.

$$\mathcal{C}(\lambda, \tilde{\lambda}) = \mathcal{C}(\langle\rangle, [\,]).$$

- Relevant ring is Lorentz invariant subring of S_n.

$$\mathcal{S}_n = \mathbb{F}\Big[\langle 12\rangle, \ldots, \langle (n-1)n\rangle, [12], \ldots [(n-1)n]\Big].$$

- Variables are brackets, now have "Schouten identities".

$$\mathcal{J}_{\Lambda_n} = \left\langle \sum_{j=1}^{n} \langle ij\rangle[jk], \langle ij\rangle\langle kl\rangle - \langle ik\rangle\langle jl\rangle - \langle il\rangle\langle kj\rangle, \langle\rangle \leftrightarrow [\,] \right\rangle.$$

- Physical spinor bracket functions also form a quotient ring.

$$\mathcal{R}_n = \mathcal{S}_n / \mathcal{J}_{\Lambda_n}.$$

Bases of Spinor Space and Polynomial Reduction

- Numerators are $\mathbb{Q}-$linear combinations of spinor monomials.

$$m_\alpha = \prod_i v_i^{\alpha_i} \qquad \text{where} \quad \vec{v} = \{\langle 12\rangle, \langle 23\rangle, \ldots [12], [23], \ldots\}.$$

- Polynomial reduction writes p in terms of generators g_i.

$$p = \Delta_{\{g_1,\ldots,g_k\}}(p) + \sum_{i=1}^{k} c_i g_i.$$

- Polynomial in ideal if and only if Groebner remainder is 0.

$$\Delta_{\mathcal{G}(J)}(p) = 0 \qquad \Leftrightarrow \qquad p \in J.$$

- Monomials irreducible by $\mathcal{G}(\mathcal{J}_{\Lambda_n})$ form basis. Related [Zhang '12]

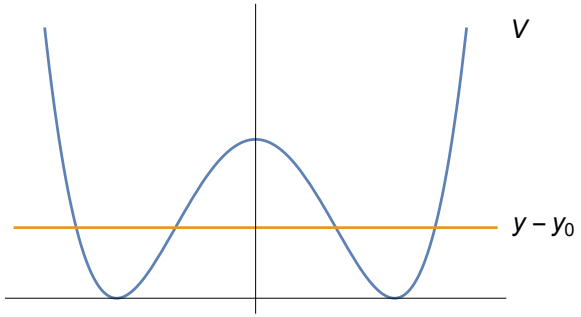
$$\text{basis} = \{m_\alpha \text{ such that } \Delta_{\mathcal{G}(\mathcal{J}_{\Lambda_n})}(m_\alpha) = m_\alpha\}.$$

P-adic (Integer) Points Near an Irreducible Variety

- Want to find $(\lambda^{(\epsilon)}, \tilde{\lambda}^{(\epsilon)})$ "close" to $U = V(\langle q_1, \ldots, q_m \rangle_{R_n})$:

$$q_i\left(\lambda^{(\epsilon)}, \tilde{\lambda}^{(\epsilon)}\right) = pc_i + \mathcal{O}(p^k), \qquad \sum_{i=1}^{n} \lambda_{i\alpha}^{(\epsilon)} \tilde{\lambda}_{i\dot{\alpha}}^{(\epsilon)} = 0 + \mathcal{O}(p^k).$$

- First, find finite field $x \in U$ by intersecting with random plane.



- Arbitrarily extend $\mathbb{F}_p$ point $(\lambda, \tilde{\lambda})$ to k digits. Trivially near U.

- To satisfy momentum conservation, perturb by $(p\delta, p\tilde{\delta})$.

$$(\lambda^{(\epsilon)}, \tilde{\lambda}^{(\epsilon)}) = (\lambda + p\delta, \tilde{\lambda} + p\tilde{\delta}).$$

Polynomials that Vanish on a Variety

- Polynomials that vanish on all points of U form an ideal

$$I(U) = \Big\{ q \in S_n \quad \text{where} \quad q(x) = 0 \quad \text{for all } x \in U \Big\}.$$

- Consider if $\mathcal{N}_i$ vanishes to order k_U on U,

$$\mathcal{N}_i(x^{(\epsilon)}) = \mathcal{O}(\epsilon^{k_U}), \quad \text{where} \quad |x - x^{(\epsilon)}| \leq \epsilon \quad \text{and} \quad x \in U.$$

- It turns out that $\mathcal{N}_i$ still belongs to an ideal!

Zariski-Nagata Theorem

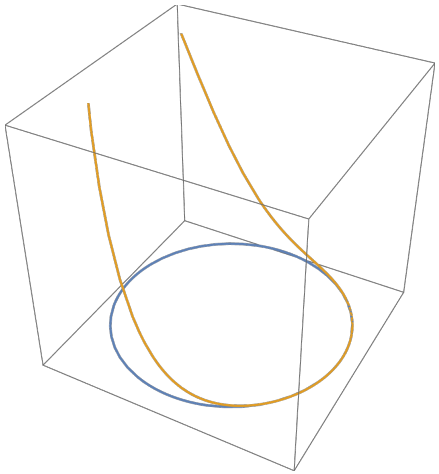
Polynomials vanishing to $\mathcal{O}(k_U)$ on U belong to $I(U)^{\langle k_U \rangle}$ – the k_Uth "symbolic power" of $I(U)$.

- Computed from primary decomposition of ideal power $I(U)^{k_U}$.

Examples of Symbolic Powers

- A function vanishing to fourth order at a point on the circle:

$$\langle x-1\rangle^{\langle 4\rangle}_{\mathbb{F}[x,y]/\langle x^2+y^2-1\rangle} \sim$$



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- Often the symbolic power coincides with standard power, e.g.

$$\langle\langle 12\rangle,[12]\rangle^{\langle 2\rangle}_{R_5} = \langle\langle 12\rangle,[12]\rangle^{2}_{R_5} = \langle\langle 12\rangle^2,\langle 12\rangle[12],[12]^2\rangle_{R_5}.$$

- Symbolic/standard power may not coincide. E.g. in $\mathbb{F}[x,y,z]$

$$\langle xy,xz,yz\rangle^{\langle 2\rangle} = \langle x^2y^2,x^2z^2,y^2z^2,xyz\rangle \neq \langle xy,xz,yz\rangle^2$$

The p-adic Logarithm

- Over the $p-adic$ numbers, one can define converging power series.
- The power series for a logarithm converges for $|x|_p < 1$.

$$\log_p(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$$

- To map to radius of convergence, use Fermat's little theorem.

$$w^{p-1} = 1 \mod p \qquad \Rightarrow \qquad |w^{p-1} - 1|_p < 1$$

- Logarithm relations then p-adically analytically continue $\log_p$.

$$\log_p(w) = \frac{1}{p-1} \log(w^{p-1})$$