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Speeding up SM Amplitude Calculations with Chirality Flow

HP2 2022 21 SEPTEMBER 2022 - ANDREW LIFSON

BASED ON HEP-PH:2003.05877 (EPJC), HEP-PH:2011.10075 (EPJC), AND HEP-PH:2203.13618 (EPJC)

IN COLLABORATION WITH JOAKIM ALNEFJORD, CHRISTIAN REUSCHLE, MALIN SJÖDAHL, AND ZENNY WETTERSTEN



Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

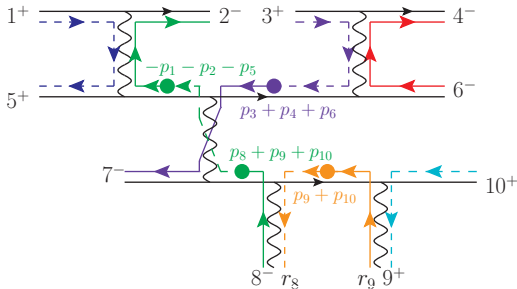
Automation

Aim and method
Results

Conclusions

- 1 Introduction
 - Spinor-helicity recap
 - Colour flow reminder
- 2 Chirality Flow
 - Massless QED
 - Massless QCD
 - Massive Particles
- 3 Automation
 - Aim and method
 - Results
- 4 Conclusions

Our Main Analytical Result



10-particle Feynman Diagram
calculated in single slide

$$\begin{aligned}
 &= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{1\ 2\ s_{3\ 4\ s_{7\ 8\ 9\ 10}}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{1\ 2\ 5\ s_{3\ 4\ 6\ s_{8\ 9\ 10\ s_{9\ 10}}}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8r_8]\langle r_9 9 \rangle}}_{\text{polarization vectors}} [15]\langle 64 \rangle [10\ 9] \\
 &\times \left(\langle r_9 9 \rangle [9r_8] + \langle r_9 10 \rangle [10r_8] \right) \underbrace{\left([33]\langle 37 \rangle + [34]\langle 47 \rangle + [36]\langle 67 \rangle \right)}_0 \\
 &\times \left(-\langle 89 \rangle [91]\langle 12 \rangle - \langle 89 \rangle [95]\langle 52 \rangle - \langle 8\ 10 \rangle [10\ 1]\langle 12 \rangle - \langle 8\ 10 \rangle [10\ 5]\langle 52 \rangle \right)
 \end{aligned}$$



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Our Main Numerical Result (so far) (hep-ph:2203.13618)

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

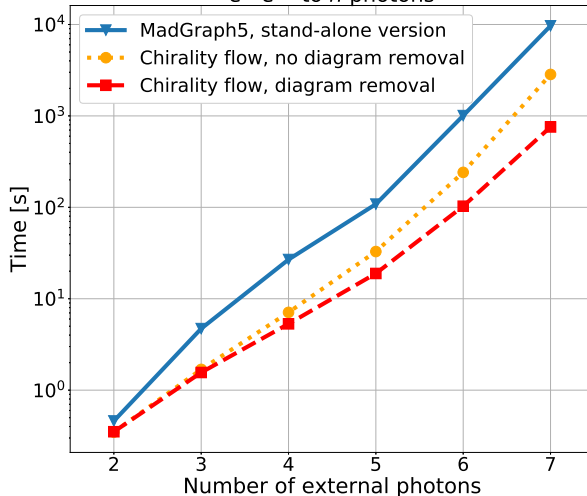
Aim and method
Results

Conclusions



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Evaluation time for 100 000 matrix elements for
 e^+e^- to n photons



Spinor-Helicity: its Building Blocks

Introduction

Spinor-helicity recap

Colour flow reminder

Chirality Flow

Massless QED

Massless QCD

Massive Particles

Automation

Aim and method

Results

Conclusions



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Lorentz algebra $so(3, 1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality \sim helicity

Spinors (use chiral basis):

$$u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix}$$

$$u^-(p) = v^+(p) = \begin{pmatrix} [p] \\ 0 \end{pmatrix}$$

$$\bar{u}^+(p) = \bar{v}^-(p) = ([p] \quad 0)$$

$$\bar{u}^-(p) = \bar{v}^+(p) = (0 \quad \langle p|)$$

- Amplitude written in terms of Lorentz-invariant spinor inner products

$$\langle ij \rangle = -\langle ji \rangle \equiv \langle i||j \rangle \text{ and } [ij] = -[ji] \equiv [i||j]$$

- These are well known complex numbers, $\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$

Spinor-Helicity: Vectors and Removing μ Indices

Introduction

Spinor-helicity recap

Colour flow reminder

Chirality Flow

Massless QED

Massless QCD

Massive Particles

Automation

Aim and method

Results

Conclusions



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$$\text{Lorentz algebra } so(3, 1) \cong su(2) \oplus su(2)$$

Dirac matrices in chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \quad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

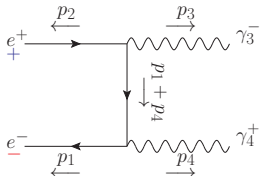
Remove vector indices with e.g.

$$\underbrace{\langle i | \bar{\tau}^\mu | j \rangle [k | \tau_\mu | l \rangle}_{\text{Fierz identity}} = \langle i l \rangle [k j], \quad \underbrace{\sqrt{2} p^\mu \tau_\mu}_{\text{Contraction with Pauli}} \equiv \not{p} = |p\rangle \langle p|$$

Polarisation vectors ($r \equiv$ gauge choice, $r^2 = 0$, $r \cdot p \neq 0$):

$$\not{\epsilon}_+(p, r) = \frac{|p\rangle \langle r|}{\langle r p \rangle}, \quad \not{\epsilon}_-(p, r) = \frac{|r\rangle \langle p|}{[p r]}$$

An Illuminating Example: $e^+ e^- \rightarrow \gamma \gamma$



- $|p\rangle \equiv$ right-chiral spinor
- $|p] \equiv$ left-chiral spinor
- $\tau^\mu, \bar{\tau}^\mu \equiv$ Pauli matrices
- $\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$

Spinor helicity: analytic

$$\begin{aligned}
 & \sim \langle p_1 | \bar{\tau}^\mu \underbrace{(|p_1\rangle\langle p_1| + |p_4\rangle\langle p_4|)}_{\not{p}_1 + \not{p}_4} \bar{\tau}^\nu |p_2] \underbrace{\frac{\langle r_3 | \bar{\tau}_\nu | p_3 \rangle}{\langle r_3 3 \rangle}}_{\epsilon_3^-} \underbrace{\frac{[r_4 | \tau_\mu | p_4 \rangle}{[4 r_4]}}_{\epsilon_4^+} \\
 & = \frac{(\langle p_1 | \bar{\tau}^\mu | p_1 \rangle + \langle p_1 | \bar{\tau}^\mu | p_4 \rangle) [r_4 | \tau_\mu | p_4 \rangle (\langle p_1 | \bar{\tau}^\nu | p_2 \rangle + \langle p_4 | \bar{\tau}^\nu | p_2 \rangle) [p_3 | \tau_\nu | r_3 \rangle]}{\langle r_3 3 \rangle [4 r_4]} \\
 & = \frac{\langle 1 r_4 \rangle ([41] \langle 13 \rangle + [44] \langle 43 \rangle) [r_3 2]}{\langle r_3 3 \rangle [4 r_4]} = \frac{\langle 1 r_4 \rangle [41] \langle 13 \rangle [r_3 2]}{\langle r_3 3 \rangle [4 r_4]} \\
 & \quad \text{Fierz identities like } \langle i | \bar{\tau}^\mu | j \rangle [k | \tau_\mu | l \rangle = \langle il \rangle [kj] \quad [ii] = 0
 \end{aligned}$$



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Introduction

Spinor-helicity recap

Colour flow reminder

Chirality Flow

Massless QED

Massless QCD

Massive Particles

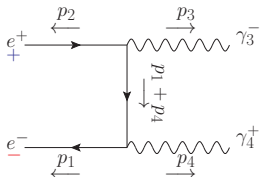
Automation

Aim and method

Results

Conclusions

An Illuminating Example: $e^+ e^- \rightarrow \gamma \gamma$



- $|p\rangle \equiv$ right-chiral spinor
- $[p] \equiv$ left-chiral spinor
- $\tau^\mu, \bar{\tau}^\mu \equiv$ Pauli matrices
- $\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$

Spinor helicity: explicit matrix multiplication

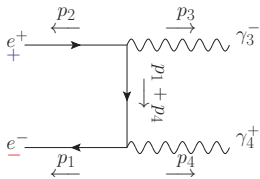
$$\sim [\bar{u}^-(p_1) \gamma^\mu \epsilon_\mu^+(p_4) (p_1^\nu + p_4^\nu) \gamma_\nu \gamma^\rho \epsilon_\rho^-(p_3) v^+(p_2)]$$

- Also cache and recycle various components
- Most common numerical method



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An Illuminating Example: $e^+ e^- \rightarrow \gamma\gamma$



- $|p\rangle \equiv$ right-chiral spinor
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- $\tau^\mu, \bar{\tau}^\mu \equiv$ Pauli matrices
- $\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$

Spinor helicity: explicit matrix multiplication

$$\sim [\bar{u}^-(p_1) \gamma^\mu \epsilon_\mu^+(p_4) (p_1^\nu + p_4^\nu) \gamma_\nu \gamma^\rho \epsilon_\rho^-(p_3) v^+(p_2)]$$

- Also cache and recycle various components
- Most common numerical method

Can we systematically remove need for algebra or matrix multiplication?



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Colour Flow: a Quick Introduction

Introduction

Spinor-helicity recap

Colour flow reminder

Chirality Flow

Massless QED

Massless QCD

Massive Particles

Automation

Aim and method

Results

Conclusions



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Standard method in $SU(N)$ -colour calculations:

Write all objects in terms of $\delta_{i\bar{j}} \equiv$ flows of colour (for simplicity $T_R = 1$)
Calculations done pictorially, not via indices

$$\underbrace{\begin{array}{c} i \quad \bar{l} \\ \swarrow \quad \searrow \\ \text{gluon} \\ \swarrow \quad \searrow \\ \bar{j} \quad k \end{array}}_{t_{i\bar{j}}^a t_{k\bar{l}}^a} = \underbrace{\begin{array}{c} i \quad \bar{l} \\ \swarrow \quad \searrow \\ \text{quark} \\ \swarrow \quad \searrow \\ \bar{j} \quad k \end{array}}_{\delta_{i\bar{l}} \delta_{k\bar{j}}} - \frac{1}{N} \underbrace{\begin{array}{c} i \quad \bar{l} \\ \swarrow \quad \searrow \\ \text{quark} \\ \swarrow \quad \searrow \\ \bar{j} \quad k \end{array}}_{\delta_{i\bar{j}} \delta_{k\bar{l}}}$$

Chirality Flow Building Blocks

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions



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Key idea (hep-ph:2003.05877)

Draw & connect lines to directly obtain inner products $\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$
Removes need to do algebra or matrix multiplication

Define spinors as lines

$$\bar{u}_i^- = \bar{v}_i^+ = \langle i |^\alpha = \text{circle} \leftarrow i, \quad u_j^+ = v_j^- = |j\rangle_\alpha = \text{circle} \rightarrow j$$

$$\bar{u}_i^+ = \bar{v}_i^- = [i]_{\dot{\beta}} = \text{circle} \leftarrow \text{dashed } i, \quad u_j^- = v_j^+ = [j]^{\dot{\beta}} = \text{circle} \rightarrow \text{dashed } j$$

Spinor inner products follow

$$\langle i |^\alpha |j\rangle_\alpha \equiv \langle ij \rangle = -\langle ji \rangle = i \rightarrow j$$

$$[i]_{\dot{\beta}} [j]^{\dot{\beta}} \equiv [ij] = -[ji] = i \leftarrow j$$

Define slashed momentum as dot

$$\not{p} \equiv \sqrt{2} p^\mu \tau_\mu^{\dot{\alpha}\beta} = \text{dashed} \rightarrow \text{circle} \rightarrow \text{solid}, \quad \bar{\not{p}} \equiv \sqrt{2} p_\mu \bar{\tau}^\mu_{\alpha\dot{\beta}} = \text{solid} \rightarrow \text{circle} \leftarrow \text{dashed}$$

The Massless QED Flow Rules: External Particles

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions

Species	Feynman	Flow
$\bar{u}^-(p_i)$		
$v^-(p_j)$		
$v^+(p_j)$		
$\bar{u}^+(p_i)$		
$\epsilon_-^\mu(p_i, r)$		
$\epsilon_+^\mu(p_i, r)$		

Left-chiral \equiv dotted lines

right-chiral \equiv solid lines



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The QED Flow Rules: Vertices and Propagators

Introduction

Spinor-helicity recap
Colour flow reminder

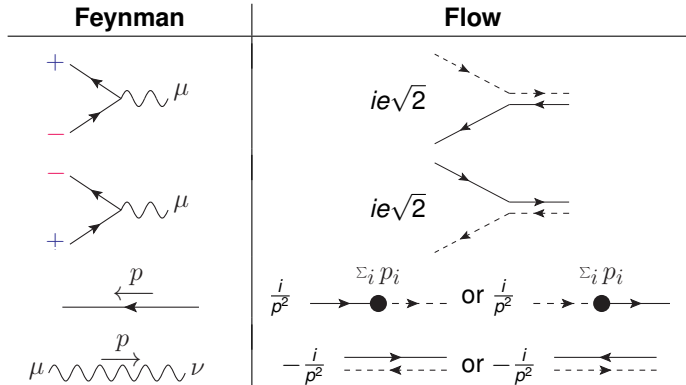
Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions



Left-chiral \equiv dotted lines

right-chiral \equiv solid lines



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An Illuminating Example: $e^+ e^- \rightarrow \gamma \gamma$

Introduction

Spinor-helicity recap
Colour flow reminder

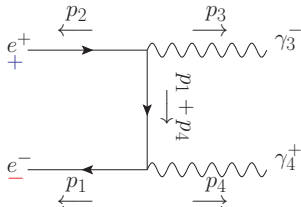
Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions



Spinor helicity:

$$\begin{aligned}
 & \sim \langle p_1 | \bar{\tau}^\mu \underbrace{(|p_1\rangle\langle p_1| + |p_4\rangle\langle p_4|)}_{\not{p}_1 + \not{p}_4} \bar{\tau}^\nu | p_2 \rangle \underbrace{\frac{\langle r_3 | \bar{\tau}_\nu | p_3 \rangle}{\langle r_3 3 \rangle}}_{\epsilon_3^-} \underbrace{\frac{[r_4 | \tau_\mu | p_4 \rangle}{[4 r_4]}}_{\epsilon_4^+} \\
 & = \frac{(\langle p_1 | \bar{\tau}^\mu | p_1 \rangle + \langle p_1 | \bar{\tau}^\mu | p_4 \rangle) [r_4 | \tau_\mu | p_4 \rangle (\langle p_1 | \bar{\tau}^\nu | p_2 \rangle + \langle p_4 | \bar{\tau}^\nu | p_2 \rangle) [p_3 | \tau_\nu | r_3 \rangle]}{\langle r_3 3 \rangle [4 r_4]} \\
 & = \frac{\langle 1 r_4 \rangle ([41] \langle 13 \rangle + [44] \langle 43 \rangle) [r_3 2]}{\langle r_3 3 \rangle [4 r_4]} = \frac{\langle 1 r_4 \rangle [41] \langle 13 \rangle [r_3 2]}{\langle r_3 3 \rangle [4 r_4]} \\
 & \quad \text{Fierz identities like } \langle i | \bar{\tau}^\mu | j \rangle [k | \tau_\mu | l \rangle = \langle i l \rangle [k j] \quad [ii] = 0
 \end{aligned}$$



An Illuminating Example: $e^+e^- \rightarrow \gamma\gamma$

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

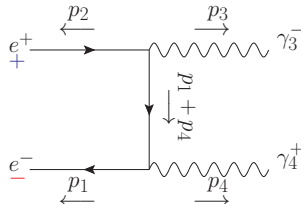
Automation

Aim and method
Results

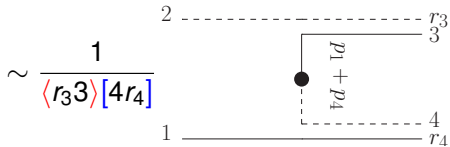
Conclusions



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Chirality flow:



An Illuminating Example: $e^+ e^- \rightarrow \gamma\gamma$

Introduction

Spinor-helicity recap

Colour flow reminder

Chirality Flow

Massless QED

Massless QCD

Massive Particles

Automation

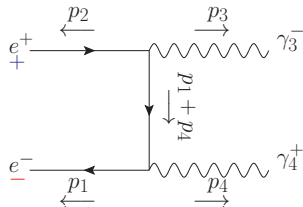
Aim and method

Results

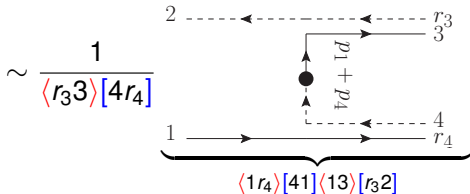
Conclusions



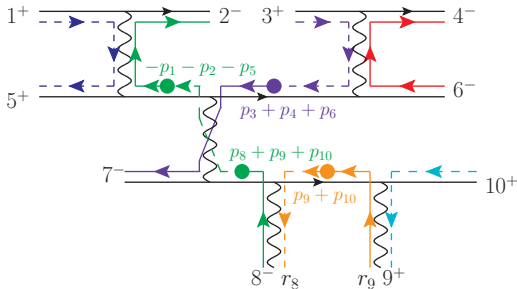
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Chirality flow:



A complicated QED Example



Spinor-helicity analytic:

- 5 charge conjugation/Fierz + rearranging
- Not possible to fit on single slide!

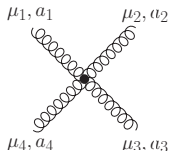
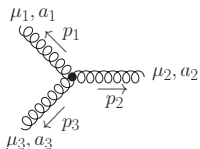
$$\begin{aligned}
 &= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{1\,2\,s_{3\,4\,s_{7\,8\,9\,10}}}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{1\,2\,5\,s_{3\,4\,6\,s_{8\,9\,10\,s_{9\,10}}}}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8r_8]\langle r_9 9 \rangle}}_{\text{polarization vectors}} [15]\langle 64 \rangle [10\,9] \\
 &\times \left(\langle r_9 9 \rangle [9r_8] + \langle r_9 10 \rangle [10r_8] \right) \left(\underbrace{[33]\langle 37 \rangle + [34]\langle 47 \rangle + [36]\langle 67 \rangle}_0 \right) \\
 &\times \left(-\langle 89 \rangle [91]\langle 12 \rangle - \langle 89 \rangle [95]\langle 52 \rangle - \langle 8\,10 \rangle [10\,1]\langle 12 \rangle - \langle 8\,10 \rangle [10\,5]\langle 52 \rangle \right)
 \end{aligned}$$



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The Non-abelian Massless QCD Flow Vertices

Feynman



Flow

$$-\frac{g_s f^{abc}}{2} \left(\underbrace{\text{Diagram 1}}_{g_{12}(p_1 - p_2)_3} + \underbrace{\text{Diagram 2}}_{g_{23}(p_2 - p_3)_1} + \underbrace{\text{Diagram 3}}_{g_{13}(p_3 - p_1)_2} \right)$$

$$ig_s^2 \sum_{Z(2,3,4)} f^{a_1 a_2 b} f^{b a_4 a_3} \left[\underbrace{\text{Diagram 4}}_{g_{14} g_{23}} - \underbrace{\text{Diagram 5}}_{g_{13} g_{24}} \right]$$

Arrow directions only consistently set within full diagram

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions



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QCD Example: $q_1 \bar{q}_1 \rightarrow q_2 \bar{q}_2 g$

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions

$$\begin{aligned}
 & q_1^+ \quad \bar{q}_1^- \quad q_2^+ \quad \bar{q}_2^- \quad 1^+ \\
 & = \frac{ig_s^3}{2s_{q_1 \bar{q}_1} s_{q_2 \bar{q}_2} \langle r1 \rangle} \left[\begin{aligned} & \text{Diagram 1: } q_1, \bar{q}_1, q_2, \bar{q}_2 \text{ with gluon line } 2(q_1 + p_1) \text{ and } r \\ & \text{Diagram 2: } q_1, \bar{q}_1, q_2, \bar{q}_2 \text{ with gluon line } -2p_1 \text{ and } r \\ & \text{Diagram 3: } q_1, \bar{q}_1, q_2, \bar{q}_2 \text{ with gluon line } 2p_1 \text{ and } r \end{aligned} \right]
 \end{aligned}$$

$$\begin{aligned}
 \left[\dots \right] & \equiv \left\{ 2[q_1 \bar{q}_2] \langle q_2 \bar{q}_1 \rangle ([1 q_1] \langle q_1 r \rangle + [1 \bar{q}_1] \langle 1 r \rangle) \right. \\
 & \quad \left. - 2[q_1 1] \langle 1 \bar{q}_1 \rangle \langle q_2 r \rangle [1 \bar{q}_2] + 2[q_1 1] \langle r \bar{q}_1 \rangle \langle q_2 1 \rangle [1 q_2] \right\}
 \end{aligned}$$



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Massive Chirality Flow (hep-ph:2011.10075)

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions

Decompose massive momentum into massless ones

$$p^\mu = p^b, \mu + \alpha q^\mu, \quad (p^b)^2 = q^2 = 0, \quad \alpha = \frac{p^2}{2p^b \cdot q}$$

- Spinors contain both chiralities, e.g.

$$\bar{v}^-(p) = \text{diagram} = \left(\text{diagram}, \frac{m}{\langle qp^b \rangle} \text{diagram} \right)$$

The diagram shows a fermion line with momentum p entering a vertex from the right. This is equal to the sum of two terms in parentheses. The first term is a fermion line with momentum p^b entering a vertex from the right. The second term is a scalar line with momentum q entering a vertex from the right, multiplied by the factor $\frac{m}{\langle qp^b \rangle}$.

- Add new polarisation vector $\epsilon_0 = \frac{1}{m\sqrt{2}}$
- Need matrix structure in fermion propagators and vertices, e.g.

$$p^\mu \gamma_\mu - m \sim \left(\begin{array}{cc} m \overset{\dot{\alpha}}{\longrightarrow} \overset{\dot{\beta}}{\longrightarrow} & \overset{\Sigma_i p_i}{\longrightarrow} \bullet \longrightarrow \\ \overset{\Sigma_i p_i}{\longrightarrow} \bullet \longrightarrow & m \overset{\alpha}{\longrightarrow} \overset{\beta}{\longrightarrow} \end{array} \right)$$

The diagram shows a matrix structure in fermion propagators and vertices. The top-left element is a fermion line with momentum m and indices $\dot{\alpha}$ and $\dot{\beta}$. The top-right element is a fermion line with momentum $\Sigma_i p_i$ and a vertex. The bottom-left element is a fermion line with momentum $\Sigma_i p_i$ and a vertex. The bottom-right element is a fermion line with momentum m and indices α and β .



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Massive Chirality Flow (hep-ph:2011.10075)

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions

Main conclusion

Matrix structure unavoidable with massive fermions
Proceed as before to calculate without algebra



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A Massive *Illuminating* Example

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

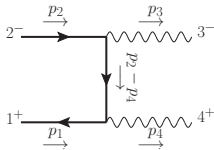
Conclusions



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Consider the same diagram of $f_1^+ \bar{f}_2^- \rightarrow \gamma_3^+ \gamma_4^-$ as before but include mass m_f

- Obtain 3 new terms
- Simplify with choices of q_1, q_2, r_3, r_4
- $e^{i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^b q_i \rangle}$, $e^{-i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^b]}$



$$= \frac{-2ie^2}{(s_{23} - m_f^2) \langle r_3 3 \rangle [4 r_4]} \left\{ \begin{array}{l} \begin{array}{c} p_2^b \text{---} \text{---} r_3 \\ \uparrow \text{---} 3 \\ \bullet \text{---} p_4 - p_1^b - q_1 \\ \uparrow \text{---} 4 \\ p_1^b \text{---} \text{---} r_4 \end{array} - \sqrt{\alpha_1 \alpha_2} e^{i(\varphi_2 - \varphi_1)} \begin{array}{c} q_2 \text{---} \text{---} 3 \\ \uparrow \text{---} r_3 \\ \bullet \text{---} p_4 - p_1^b - q_1 \\ \uparrow \text{---} r_4 \\ q_1 \text{---} \text{---} 4 \end{array} \end{array} \right. \\ + m_f \left(\begin{array}{c} q_2 \text{---} \text{---} 3 \\ \uparrow \text{---} r_3 \\ \bullet \text{---} p_4 - p_1^b - q_1 \\ \uparrow \text{---} 4 \\ p_1^b \text{---} \text{---} r_4 \end{array} \sqrt{\alpha_2} e^{i\varphi_2} - \sqrt{\alpha_1} e^{-i\varphi_2} \begin{array}{c} p_2^b \text{---} \text{---} r_3 \\ \uparrow \text{---} 3 \\ \bullet \text{---} p_4 - p_1^b - q_1 \\ \uparrow \text{---} r_4 \\ q_1 \text{---} \text{---} 4 \end{array} \right) \Bigg\}$$

MadGraph and the Automation of Chirality Flow

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions

Summary

- So far: Numerical calculations use explicit multiplication rather than spin algebra analytically because quicker
- We have made the analytical spin algebra trivial
- Can we use this to make even faster numerics?



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MadGraph and the Automation of Chirality Flow

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions

Summary

- So far: Numerical calculations use explicit multiplication rather than spin algebra analytically because quicker
- We have made the analytical spin algebra trivial
- Can we use this to make even faster numerics?

Use MadGraph5_aMC@NLO (MG5aMC) for proof of concept automation

- Make minimal changes to massless QED in MG5aMC
- Pro: any difference in speed from our changes \Rightarrow sound conclusions
- Con: MG5aMC not designed for chirality flow \Rightarrow not optimal implementation



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Sources of Expect Speed Gains

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions

1 Simplified vertices and propagators

- We minimise matrix multiplication
- Each component of a calculation is simpler



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Sources of Expect Speed Gains

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

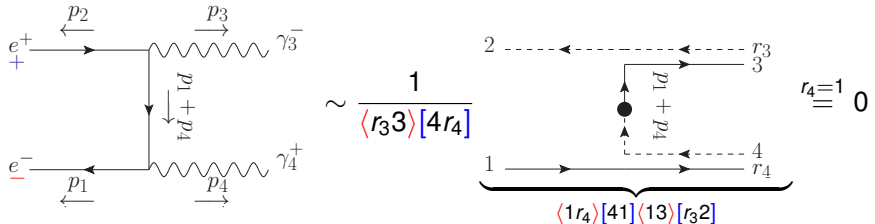
Conclusions

1 Simplified vertices and propagators

- We minimise matrix multiplication
- Each component of a calculation is simpler

2 Gauge-based diagram removal

- Polarisation vectors contain arbitrary gauge-reference spinor of momentum r
- Spinor inner products antisymmetric $\Rightarrow \langle ii \rangle = [jj] = 0$
- Chirality-flow makes optimal choice of r obvious \Rightarrow remove diagrams!



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Our Main Result (hep-ph:2203.13618)

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

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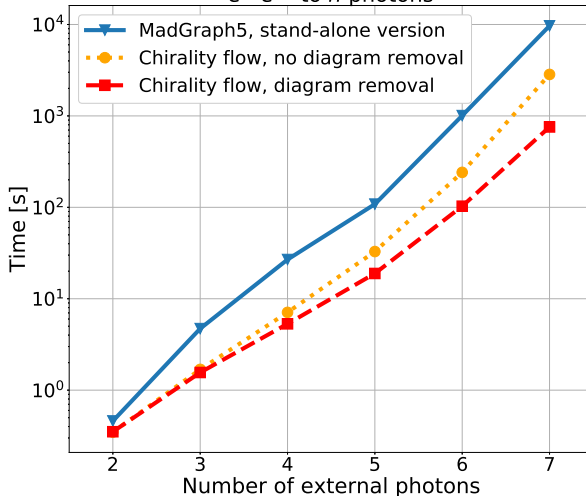
Aim and method
Results

Conclusions



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Evaluation time for 100 000 matrix elements for
 e^+e^- to n photons



Conclusions and Outlook

Introduction

Spinor-helicity recap
Colour flow reminder

Chirality Flow

Massless QED
Massless QCD
Massive Particles

Automation

Aim and method
Results

Conclusions



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Conclusions:

- Chirality flow is the shortest route from Feynman diagram to complex number
- We have flow rules for full SM at tree level
- We automatised it for massless QED, found significant gains in MadGraph

Outlook and other work in this area:

- Simon Plätzer and Malin Sjö Dahl used chirality flow as basis for resummation (hep-ph:2204.03258)
- Use method analytically to calculate loop amplitudes
 - Ongoing work by AL, Simon Plätzer, and Malin Sjö Dahl,
- Automate for rest of (tree-level) Standard Model and tweak algorithm to use all possible features of chirality flow
 - Two current master students working to achieve this

Reminder: Lorentz Group Representations

Backup Slides

Spinor Helicity Reminder
Colour flow reminder
Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

Spinor-hel details

Chirality-Flow Motivation



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Lorentz group elements: $e^{i(\theta_i J_i + \eta_i K_i)}$ $J_i \equiv$ rotations, $K_i \equiv$ boosts

- Lorentz group generators $\simeq 2$ copies of $\mathfrak{su}(2)$ generators

- $\mathfrak{so}(3,1)_{\mathbb{C}} \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$

Group algebra defined by commutator relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k$$

$$N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i), \quad [N_i^-, N_j^+] = 0,$$

$$[N_i^-, N_j^-] = i\epsilon_{ijk} N_k^-, \quad [N_i^+, N_j^+] = i\epsilon_{ijk} N_k^+$$

- Representations

- $(0,0)$ scalar particles
 - $(\frac{1}{2}, 0)$ left-chiral and $(0, \frac{1}{2})$ right-chiral Weyl (2-component) spinors.
 - $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, Dirac (4-component) spinors.
 - $(\frac{1}{2}, \frac{1}{2})$ vectors, e.g. gauge bosons

Spinor-Helicity: Gauge Bosons in Terms of Spinors

Backup Slides

Spinor Helicity Reminder
Colour flow reminder
Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

Spinor-hel details

Chirality-Flow Motivation



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Lorentz algebra $so(3, 1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality \sim helicity

Outgoing polarisation vectors:

$$\epsilon_+^\mu(p, r) = \frac{\langle r | \bar{\tau}^\mu | p \rangle}{\langle rp \rangle}, \quad \epsilon_-^\mu(p, r) = \frac{[r | \tau^\mu | p]}{[pr]}$$

- r is a (massless) arbitrary reference momentum ($p \cdot r \neq 0$)
- Different r choices correspond to different gauges

$$\epsilon_+^\mu(p, r') - \epsilon_+^\mu(p, r) = -p^\mu \frac{\langle r' r \rangle}{\langle r' p \rangle \langle rp \rangle}$$

- Gauge invariant quantities must be r -invariant
 - Choose r as conveniently as possible (remember $\langle ij \rangle = -\langle ji \rangle$ s.t. $\langle ii \rangle = 0$)
(4-gluon amplitude: can make 20/21 terms vanish)
 - Variance under $r \rightarrow r'$ good check of gauge invariance of (partial) amplitude

Spinor-Helicity: Vectors and Removing μ Indices

Backup Slides

Spinor Helicity Reminder

Colour flow reminder

Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

Spinor-hel details

Chirality-Flow Motivation



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Lorentz algebra $so(3, 1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality \sim helicity

Dirac matrices in chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \quad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

Remove $\tau/\bar{\tau}$ matrices in amplitude with

$$\underbrace{\langle i | \bar{\tau}^\mu | j \rangle [k | \tau_\mu | l \rangle}_{\text{Fierz identity}} = \langle il \rangle [kj], \quad \underbrace{\langle i | \bar{\tau}^\mu | j \rangle}_{\text{Charge Conjugation}} = [j | \tau^\mu | i \rangle$$

Express (massless) p^μ in terms of spinors

$$p^\mu = \frac{[p | \tau^\mu | p \rangle}{\sqrt{2}} = \frac{\langle p | \bar{\tau}^\mu | p \rangle}{\sqrt{2}}, \quad \sqrt{2}p^\mu \tau_\mu \equiv \not{p} = [p] \langle p |, \quad \sqrt{2}p^\mu \bar{\tau}_\mu \equiv \bar{\not{p}} = |p \rangle [p|$$

Spinor-Helicity: Gauge Bosons in Terms of Spinors

Backup Slides

Spinor Helicity Reminder

Colour flow reminder

Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

Spinor-hel details

Chirality-Flow Motivation



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Lorentz algebra $so(3, 1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality \sim helicity

Outgoing polarisation vectors ($r \equiv$ gauge choice, $r^2 = 0$, $r \cdot p \neq 0$):

$$\begin{aligned}\epsilon_+^\mu(p, r) &= \frac{\langle r | \bar{\tau}^\mu | p \rangle}{\langle rp \rangle}, & \epsilon_-^\mu(p, r) &= \frac{[r | \tau^\mu | p \rangle}{[pr]} \\ p \cdot \epsilon_+(p, r) &= \underbrace{\frac{\langle r | p^\mu \bar{\tau}_\mu | p \rangle}{\langle rp \rangle}}_{\text{Weyl eq. } p^\mu \bar{\tau}_\mu | p \rangle = 0} = 0, & p \cdot \epsilon_-(p, r) &= \underbrace{\frac{[r | p^\mu \tau_\mu | p \rangle}{[pr]}}_{\text{Weyl eq. } p^\mu \tau_\mu | p \rangle = 0} = 0 \\ \epsilon_+(p, r) \cdot (\epsilon_-)^*(p, r) &= \underbrace{\frac{\langle r | \bar{\tau}^\mu | p \rangle}{\langle rp \rangle} \frac{[r | \tau_\mu | p \rangle}{[pr]}}_{\epsilon_\pm = (\epsilon_\mp)^*} = \frac{\langle rp \rangle [rp]}{\langle rp \rangle [pr]} = \underbrace{-1}_{[pr] = -[rp]}\end{aligned}$$

Colour Flow: a Quick Introduction

Standard method in $SU(N)$ -colour calculations:

Write all objects in terms of $\delta_{i\bar{j}} \equiv$ flows of colour (for simplicity $T_R = 1$)
 Calculations done pictorially, not via indices

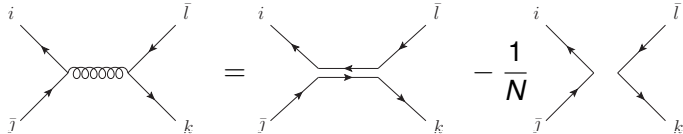
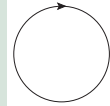
$$\begin{aligned}
 \delta_{i\bar{j}} &= \bar{j} \longrightarrow i, & \sum_i \delta_{ii} &= N = \text{circle}, & t_{i\bar{j}}^a &= \begin{array}{c} i \\ \searrow \\ \text{circle} \text{---} a \\ \nearrow \\ \bar{j} \end{array} \\
 if^{abc} &= \begin{array}{c} b \\ \text{circle} \\ \swarrow \quad \searrow \\ a \quad c \end{array} = \begin{array}{c} b \\ \text{circle} \\ \swarrow \quad \searrow \\ a \quad c \end{array} - \begin{array}{c} b \\ \text{circle} \\ \swarrow \quad \searrow \\ a \quad c \end{array} = \text{Tr}(t^a[t^b, t^c]) \\
 \underbrace{\begin{array}{c} i \quad \bar{l} \\ \swarrow \quad \searrow \\ \text{circle} \\ \nearrow \quad \nwarrow \\ \bar{j} \quad k \end{array}}_{t_{i\bar{j}}^a t_{k\bar{l}}^a} &= \underbrace{\begin{array}{c} i \quad \bar{l} \\ \swarrow \quad \searrow \\ \text{circle} \\ \nearrow \quad \nwarrow \\ \bar{j} \quad k \end{array}}_{\delta_{i\bar{l}} \delta_{k\bar{j}}} - \frac{1}{N} \underbrace{\begin{array}{c} i \quad \bar{l} \\ \swarrow \quad \searrow \\ \text{circle} \\ \nearrow \quad \nwarrow \\ \bar{j} \quad k \end{array}}_{\delta_{i\bar{j}} \delta_{k\bar{l}}}
 \end{aligned}$$



Colour Flow: a Quick Introduction

Standard method in $SU(N)$ -colour calculations:

Calculations done pictorially, not via indices $\sum_i \delta_{ii} = N =$



$$\text{Tr}(t^a t^a) = \underbrace{\text{Diagram 1}}_{\text{Tr}(t^a t^a)} = \underbrace{\text{Diagram 2}}_{N^2} - \frac{1}{N} \underbrace{\text{Diagram 3}}_N = N^2 - 1$$

Backup Slides

Spinor Helicity Reminder

Colour flow reminder

Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

Spinor-hel details

Chirality-Flow
Motivation



The Non-abelian Massless QCD Flow Vertices

Backup Slides

Spinor Helicity Reminder

Colour flow reminder

Massless QCD

Massive Chirality Flow

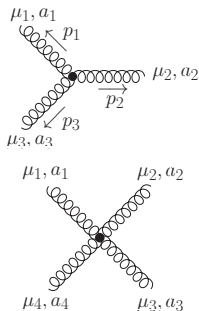
Massive Examples

Lorentz Group Details

Spinor-hel details

Chirality-Flow Motivation

Feynman



Flow

$$-\frac{g_s f^{abc}}{2} \left(\begin{array}{c} 1 \\ \text{---} \text{---} \text{---} 2 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} + \begin{array}{c} 2-3 \\ \text{---} \text{---} \text{---} 2 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} + \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} 3 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right)$$

$$ig_s^2 \sum_{Z(2,3,4)} f_{a_1 a_2 b} f_{b a_4 a_3} \left[\begin{array}{c} 1 \\ \text{---} \text{---} \text{---} 4 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \quad \begin{array}{c} 2 \\ \text{---} \text{---} \text{---} 3 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} - \begin{array}{c} 1 \\ \text{---} \text{---} \text{---} 2 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \right]$$

Arrow directions only consistently set within full diagram

Double line $\equiv g_{\mu\nu}$, momentum dot $\equiv p_\mu$



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QCD Example: $q_1 \bar{q}_1 \rightarrow q_2 \bar{q}_2 g$

Backup Slides

Spinor Helicity Reminder

Colour flow reminder

Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

Spinor-hel details

Chirality-Flow Motivation

$$\begin{aligned}
 & \text{Diagram 1: } q_1^+ \bar{q}_1^- \rightarrow q_2^+ \bar{q}_2^- \text{ via } t\text{-channel gluon exchange (gluon line } 1^+) \\
 & \text{Diagram 2: } q_1^+ \bar{q}_1^- \rightarrow q_2^+ \bar{q}_2^- \text{ via } u\text{-channel gluon exchange (gluon line } 1) \\
 & \text{Diagram 3: } q_1^+ \bar{q}_1^- \rightarrow q_2^+ \bar{q}_2^- \text{ via } s\text{-channel gluon exchange (gluon line } 1) \\
 & \text{Sum of diagrams multiplied by } \frac{ig_s^3}{2s_{q_1 \bar{q}_1} s_{q_2 \bar{q}_2} \langle r1 \rangle}
 \end{aligned}$$

$$\begin{aligned}
 \left[\dots \right] & \equiv \left\{ 2[q_1 \bar{q}_2] \langle q_2 \bar{q}_1 \rangle ([1 q_1] \langle q_1 r \rangle + [1 \bar{q}_1] \langle 1 r \rangle) \right. \\
 & \quad \left. - 2[q_1 1] \langle 1 \bar{q}_1 \rangle \langle q_2 r \rangle [1 \bar{q}_2] + 2[q_1 1] \langle r \bar{q}_1 \rangle \langle q_2 1 \rangle [1 q_2] \right\}
 \end{aligned}$$



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Incoming Massive Spinors in Chirality Flow

Backup Slides

Spinor Helicity Reminder

Colour flow reminder

Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

Spinor-hel details

Chirality-Flow Motivation



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$$p^\mu = p^b, \mu + \alpha q^\mu, \quad (p^b)^2 = q^2 = 0, \quad e^{i\varphi} \sqrt{\alpha} = \frac{m}{\langle p^b q \rangle}, \quad e^{-i\varphi} \sqrt{\alpha} = \frac{m}{[qp^b]}$$

$$\text{Spin operator } -\frac{\Sigma^\mu s_\mu}{2} = \frac{\gamma^5 s^\mu \gamma_\mu}{2}, \quad s^\mu = \frac{1}{m}(p^b, \mu - \alpha q^\mu)$$

Spinor	Feynman	Flow
$\bar{v}^-(p)$		$\left(\text{grey circle} \xleftarrow{\text{dashed}} p^b, \sqrt{\alpha} e^{i\varphi} \text{grey circle} \xleftarrow{\text{solid}} q \right)$
$\bar{v}^+(p)$		$\left(-\sqrt{\alpha} e^{-i\varphi} \text{grey circle} \xleftarrow{\text{dashed}} q, \text{grey circle} \xleftarrow{\text{solid}} p^b \right)$
$u^-(p)$		$\left(\begin{array}{c} \text{grey circle} \xrightarrow{\text{dashed}} p^b \\ \sqrt{\alpha} e^{i\varphi} \text{grey circle} \xrightarrow{\text{solid}} q \end{array} \right)$
$u^+(p)$		$\left(\begin{array}{c} -\sqrt{\alpha} e^{-i\varphi} \text{grey circle} \xrightarrow{\text{dashed}} q \\ \text{grey circle} \xrightarrow{\text{solid}} p^b \end{array} \right)$

Some Fermion Flow Rules

Backup Slides

Spinor Helicity Reminder

Colour flow reminder

Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

Spinor-hel details

Chirality-Flow Motivation



$$p^\mu = p^b, \mu + \alpha q^\mu, \quad (p^b)^2 = q^2 = 0, \quad \alpha = \frac{p^2}{2p \cdot q} \neq 0$$

Fermion-vector vertex

$$\begin{array}{c} \text{Feynman diagram: two solid lines (fermions) meeting at a vertex with a wavy line (vector).} \end{array} = ie(P_L C_L + P_R C_R) \gamma^\mu = ie\sqrt{2} \left(\begin{array}{cc} 0 & C_R \\ C_L & 0 \end{array} \right)$$

The matrix elements are represented by small Feynman diagrams: C_L shows a fermion line splitting into a solid and a dashed line; C_R shows a fermion line splitting into a dashed and a solid line.

Fermion propagator

$$\frac{i}{p^2 - m_f^2} \begin{pmatrix} m_f \delta_{\dot{\alpha}\dot{\beta}} & \sqrt{2} p^{\dot{\alpha}\beta} \\ \sqrt{2} \bar{p}_{\alpha\dot{\beta}} & m_f \delta_{\alpha\beta} \end{pmatrix} = \frac{i}{p^2 - m_f^2} \begin{pmatrix} m_f \overset{\dot{\alpha}}{\dashrightarrow} \overset{\dot{\beta}}{\dashrightarrow} & \overset{\Sigma_i p_i}{\bullet} \dashrightarrow \\ \overset{\Sigma_i p_i}{\dashrightarrow} \bullet \dashrightarrow & m_f \overset{\alpha}{\rightarrow} \overset{\beta}{\rightarrow} \end{pmatrix}$$

The matrix elements are represented by small Feynman diagrams: the top-left shows two dashed lines with arrows; the top-right shows a dashed line with an arrow entering a vertex and a solid line with an arrow exiting; the bottom-left shows a solid line with an arrow entering a vertex and a dashed line with an arrow exiting; the bottom-right shows two solid lines with arrows.

Left and right chiral couplings may differ

A Massive *Illuminating* Example

Backup Slides

Spinor Helicity Reminder

Colour flow reminder

Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

Spinor-hel details

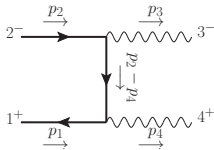
Chirality-Flow Motivation



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Consider the same diagram of $f_1^+ \bar{f}_2^- \rightarrow \gamma_3^+ \gamma_4^-$ as before but include mass m_f

- Obtain 3 new terms
- Simplify with choices of q_1, q_2, r_3, r_4
- $e^{i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^b q_i \rangle}$, $e^{-i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^b]}$



$$= \frac{-2ie^2}{(s_{23} - m_f^2) \langle r_3 3 \rangle [4 r_4]} \left\{ \begin{array}{l} \text{Diagram 1: } p_2^b \text{ (dashed) to } r_3 \text{ (dashed), } p_1^b \text{ (dashed) to } r_4 \text{ (dashed), } p_4 - p_1^b - q_1 \text{ (dashed) to } 3 \text{ (solid), } p_4 - p_1^b - q_1 \text{ (dashed) to } 4 \text{ (solid)} \\ - \sqrt{\alpha_1 \alpha_2} e^{i(\varphi_2 - \varphi_1)} \text{Diagram 2: } q_2 \text{ (solid) to } r_3 \text{ (dashed), } q_1 \text{ (dashed) to } r_4 \text{ (dashed), } p_4 - p_1^b - q_1 \text{ (dashed) to } 3 \text{ (solid), } p_4 - p_1^b - q_1 \text{ (dashed) to } 4 \text{ (solid)} \end{array} \right.$$

$$+ m_f \left(\sqrt{\alpha_2} e^{i\varphi_2} \text{Diagram 3: } q_2 \text{ (solid) to } r_3 \text{ (dashed), } p_1^b \text{ (dashed) to } r_4 \text{ (dashed), } p_4 - p_1^b - q_1 \text{ (dashed) to } 3 \text{ (solid), } p_4 - p_1^b - q_1 \text{ (dashed) to } 4 \text{ (solid)} \right. \\ \left. - \sqrt{\alpha_1} e^{-i\varphi_2} \text{Diagram 4: } p_2^b \text{ (dashed) to } r_3 \text{ (dashed), } q_1 \text{ (dashed) to } r_4 \text{ (dashed), } p_4 - p_1^b - q_1 \text{ (dashed) to } 3 \text{ (solid), } p_4 - p_1^b - q_1 \text{ (dashed) to } 4 \text{ (solid)} \right) \Bigg\}$$

A Second Massive Example: $f_1 \bar{f}_2 \rightarrow W \rightarrow f_3 \bar{f}_4 h_5$

Backup Slides

Spinor Helicity Reminder
Colour flow reminder
Massless QCD

Massive Chirality Flow

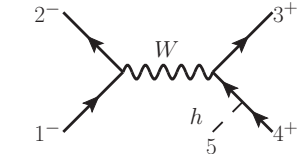
Massive Examples

Lorentz Group Details

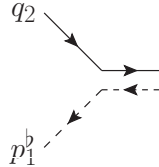
Spinor-hel details

Chirality-Flow Motivation

- W bosons simplifies ($C_R = 0$)
- Simplify with choices of q_1, \dots, q_5
- $e^{i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^b q_i \rangle}$, $e^{-i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^b]}$
- Scalar has no flow line



Step 1: Draw fermion lines: $\sim C_{L,12} \sqrt{\alpha_2} e^{i\varphi_2}$



$$\times C_{L,34} \sqrt{\alpha_3} (-e^{i\varphi_3}) \left[\sqrt{\alpha_4} (-e^{i\varphi_4}) \text{ (diagram with q3, q4, 4-5, p4^b)} + m_4 \text{ (diagram with q3, 4-5, p4^b)} \right]$$



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A Second Massive Example: $f_1 \bar{f}_2 \rightarrow W \rightarrow f_3 \bar{f}_4 h_5$

Backup Slides

Spinor Helicity Reminder
Colour flow reminder
Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

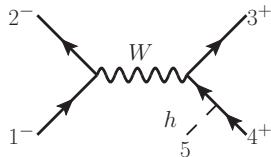
Spinor-hel details

Chirality-Flow Motivation



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- W bosons simplifies ($C_R = 0$)
- Simplify with choices of q_1, \dots, q_5
- $e^{i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{\langle p_i^b q_i \rangle}$, $e^{-i\varphi_i} \sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^b]}$
- Scalar has no flow line



Step 2: Flip arrows and connect: $C_{L,12} C_{L,34} \sqrt{\alpha_2 \alpha_3} e^{i(\varphi_2 + \varphi_3)}$

$$\times \left[\begin{array}{c} \sqrt{\alpha_4} e^{i\varphi_4} \\ \begin{array}{c} \text{Diagram 1: } q_2 \text{ and } q_3 \text{ are solid lines meeting at a vertex. A dashed line connects this vertex to another vertex where } n_1^b \text{ (dashed) and } q_4 \text{ (solid) meet. A black dot is on the dashed line between the vertices.} \\ \text{Diagram 2: } q_2 \text{ and } q_3 \text{ are solid lines meeting at a vertex. A dashed line connects this vertex to another vertex where } n_1^b \text{ (dashed) and } n_4^b \text{ (dashed) meet.} \end{array} \end{array} \right] - m_4$$

Lorentz Group Representations

Backup Slides

Spinor Helicity Reminder
Colour flow reminder
Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

Spinor-hel details

Chirality-Flow Motivation



Lorentz group elements: $e^{i(\theta_i J_i + \eta_i K_i)}$ $J_i \equiv$ rotations, $K_i \equiv$ boosts

- Lorentz group generators \simeq 2 copies of $\mathfrak{su}(2)$ generators

- $\mathfrak{so}(3,1)_{\mathbb{C}} \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$

Group algebra defined by commutator relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k$$

$$N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i), \quad [N_i^-, N_j^+] = 0,$$

$$[N_i^-, N_j^-] = i\epsilon_{ijk} N_k^-, \quad [N_i^+, N_j^+] = i\epsilon_{ijk} N_k^+$$

- Representations (i.e. realisations of N_i^{\pm})

- $(0,0)$ scalar particles
 - $(\frac{1}{2}, 0)$ left-chiral and $(0, \frac{1}{2})$ right-chiral Weyl (2-component) spinors.
 - $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, Dirac (4-component) spinors.
 - $(\frac{1}{2}, \frac{1}{2})$ vectors, e.g. gauge bosons

How to Calculate? Spinor-Helicity

Give each particle a defined helicity \Rightarrow amplitude now a number!

Spinors (in chiral basis):

$$u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix}$$

$$u^-(p) = v^+(p) = \begin{pmatrix} [p] \\ 0 \end{pmatrix}$$

$$\bar{u}^+(p) = \bar{v}^-(p) = ([p] \ 0)$$

$$\bar{u}^-(p) = \bar{v}^+(p) = (0 \ \langle p|)$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix}$$

$$\sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

- Amplitude written in terms of Lorentz-invariant spinor inner products

$$\langle ij \rangle = -\langle ji \rangle \equiv \langle i||j \rangle \text{ and } [ij] = -[ji] \equiv [i||j]$$

- These are well known complex numbers, $\langle ij \rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$

- Remove $\tau/\bar{\tau}$ matrices in amplitude with

$$\langle i|\bar{\tau}^\mu[j][k|\tau_\mu|l\rangle = \langle il\rangle[kj], \quad \langle i|\bar{\tau}^\mu[j] = [j|\tau^\mu|i\rangle$$



How to Calculate a Process

Backup Slides

Spinor Helicity Reminder
Colour flow reminder
Massless QCD

Massive Chirality Flow

Massive Examples

Lorentz Group Details

Spinor-hel details

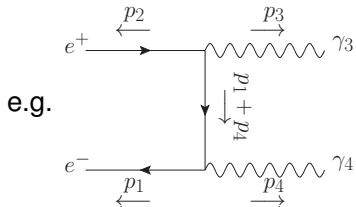
Chirality-Flow Motivation



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Sum all Feynman diagrams, square, and integrate

Often spin structure is non-trivial



$$\sim \underbrace{[\bar{u}(p_1) \gamma^\mu (p_1^\nu + p_4^\nu) \gamma_\nu \gamma^\rho v(p_2)] \epsilon_\rho(p_3) \epsilon_\mu(p_4)}$$

A mathematical expression we have simplify and square

Most common method: use helicity basis

Each diagram is a complex number, easy to square

Can use algebra to simplify first, or brute force matrix multiplication

Define Problem

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Kinematic part of amplitude slowed by spin and vector structures

- Can we still improve on this?
 - Deriving spinor inner products $\langle ij \rangle$, $[kl]$ requires at least 2 steps
 - Re-write every object as spinors
 - Use Fierz identity $\bar{\tau}_{\alpha\dot{\beta}}^{\mu} \tau_{\mu}^{\dot{\alpha}\beta} = \delta_{\alpha}^{\dot{\alpha}} \delta_{\dot{\beta}}^{\beta}$
 - Not intuitive which inner products we obtain
- In SU(N) use graphical reps for calculations
 - E.g. using the colour-flow method
 - (Also birdtracks etc.)
- Spinor-helicity $\equiv su(2) \oplus su(2)$
 - Can we use graphical reps?

Creating Chirality Flow: Building Blocks

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Chirality-Flow Motivation

A flow is a directed line from one object to another

$su(2)$ objects have dotted indices and $su(2)$ objects undotted indices

- First step: Ansatz for spinor inner products (only possible Lorentz invariant)

$$\langle i |^{\alpha} | j \rangle_{\alpha} \equiv \langle ij \rangle = -\langle ji \rangle = i \longrightarrow j$$

$$[i]_{\dot{\beta}} [j]^{\dot{\beta}} \equiv [ij] = -[ji] = i \dashrightarrow j$$

- Spinors and Kronecker deltas follow

$$\langle i |^{\alpha} = \text{circle} \longleftarrow i \quad ,$$

$$| j \rangle_{\alpha} = \text{circle} \longrightarrow j$$

$$[i]_{\dot{\beta}} = \text{circle} \dashleftarrow i \quad ,$$

$$[j]^{\dot{\beta}} = \text{circle} \dashrightarrow j$$

$$\delta_{\alpha}^{\beta} \equiv \mathbb{1}_{\alpha}^{\beta} = \alpha \longrightarrow \beta \quad ,$$

$$\delta_{\dot{\alpha}}^{\dot{\beta}} \equiv \mathbb{1}_{\dot{\alpha}}^{\dot{\beta}} = \dot{\beta} \dashrightarrow \dot{\alpha}$$



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