

Recent progress on two-loop amplitude construction using HELAC

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Collaboration with: C. Papadopoulos, G. Bevilacqua & A. Kardos.

Based on: J.Phys.Conf.Ser. 2105 (2021) 5, 012010 & ongoing work

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High Precision for Hard Processes, Discovery Museum (Newcastle upon Tyne),
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Current status @ NNLO (QCD)

- Accurate data (HL-LHC/future colliders) → High-precision theoretical predictions!

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$pp \rightarrow H/V + 2j, H/V'/j + t\bar{t}, V + b\bar{b}, VV' + j, tZj$ with $V' = V, \gamma$ and $V = Z, W$



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- Recent results (at Leading Color):

- 1** $pp \rightarrow \gamma\gamma\gamma$ S. Kallweit, V. Sotnikov and M. Wiesemann [Phys.Lett.B 812 (2021) 136013].
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- At NNLO the hard-part cross-section, $\hat{\sigma}_{ab \rightarrow X}$, receives contributions from

$$\begin{aligned} d\hat{\sigma}_{ab \rightarrow X}^{NNLO} \sim & |\mathcal{A}_{tree}|^2 + \alpha_S \left(2 \operatorname{Re} [\mathcal{A}_{tree} \mathcal{A}_{loop}^*] + |\mathcal{A}_{+1up}|^2 \right) \\ & + \alpha_S^2 \left(|\mathcal{A}_{loop}|^2 + 2 \operatorname{Re} [\mathcal{A}_{tree} \mathcal{A}_{2-loop}^*] + |\mathcal{A}_{+2up}|^2 + 2 \operatorname{Re} [\mathcal{A}_{loop+1up} \mathcal{A}_{+1up}^*] \right) \end{aligned}$$



Final result finite using Renormalization and dimensional Regularization!



DEMOKRITOS

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- $gg \rightarrow ggg$ (FC+all-plus helicity) S. Badger, D. Chicherin, T. Gehrmann, G. Heinrich, J.M. Henn, T. Peraro, P. Wasser, Y. Zhang and S. Zoia [Phys.Rev.Lett. 123 (2019) 7, 071601].



Apologize for missing references herein and from here on!



Workflow for 2-loop scattering amplitude computations

1) Construction of the Amplitude for the process at hand:

- Sum up Feynman graphs (*QGRAF*, *FeynArts*)
- Dyson-Schwinger recursion
- Hybrid approach ←— *HELAC-2LOOP*

$$\mathcal{A}_{2-loop} = \int \frac{d^d k_1 d^d k_2}{(2\pi)^{2d} \mu^{2(d-4)}} A_{2-loop} = \sum_{I \subseteq T} \int \frac{d^d k_1 d^d k_2}{(2\pi)^{2d} \mu^{2(d-4)}} \frac{N_I(k_1, k_2, p_1, \dots, p_{n-1}, \gamma^\mu, \epsilon^\mu, u, v)}{\prod_{\{i_1, i_2, i_3\} \in I} D_{i_1}(k_1) D_{i_2}(k_2) D_{i_3}(k_1, k_2)}$$



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2) Reduction to a set of Master Integrals (or special functions or tensor integrals):

- Integrand Reduction (Numerical Unitarity [*H. Ita, Phys.Rev.D 94 (2016) 11, 116015*], *OpenLoops*¹, *OPP?*)
- IBP Reduction + Finite Fields (*KIRA Johann's talk*, *FIRE*, *Reduze*)
- Hybrid approach (*AID* [*P. Mastrolia, T. Peraro and A. Primo, JHEP 08 (2016) 164*] + *IBP*) *Jonathan's talk*

$$\mathcal{A}_{2\text{-loop}} = \sum_i c_i(\mathbf{s}, \varepsilon) F_i(\mathbf{s}, \varepsilon)$$



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3) Computation of the Master Integrals:

- Analytical (Differential Equations, *SDE* approach, *Feynman Parametrization*)
- Numerical (Sector Decomposition → *pySecDec*, *FIESTA*)
- Semi-Numerical (*DiffExp*, *SeaSyde*, *AMFlow*, Internal reduction, *DiffExp* + *Feynman trick* *Martijn's talk*)



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HELAC-2LOOP for amplitude construction: The algorithm

n – particle, 2 – loop Amplitude → *(n + 2) – particle, 1 – loop Amplitude*



HELAC-2LOOP for amplitude construction: The algorithm

n – particle, 2 – loop Amplitude → (n + 2) – particle, 1 – loop Amplitude

- 1) Definition of the flavor of the $n + 1$ and $n + 2$ particles.
- 2) Generation of the $n + 2$ color-states (($n + 2$)!, Color-Flow Representation).
- 3) Generation of Blob-Topologies.
- 4) Cut of the topologies in the k_3 -line (middle-line) → the 2 extra particles.
- 5) Flavor-Color Dressing of the 1-loop loop-particles.
- 6) Second cut of the blob-topology → tree-level graph ($n + 4$ color-states).
- 7) Creation of currents contributing to the configuration (Dyson-Schwinger to blobs).
- 8) Reduction of the $n + 4$ color-states to n and identification of N_C power.
- 9) Storing of the numerator information to the Skeleton.



Two-loop blob-topologies

- Binary representation for the particles



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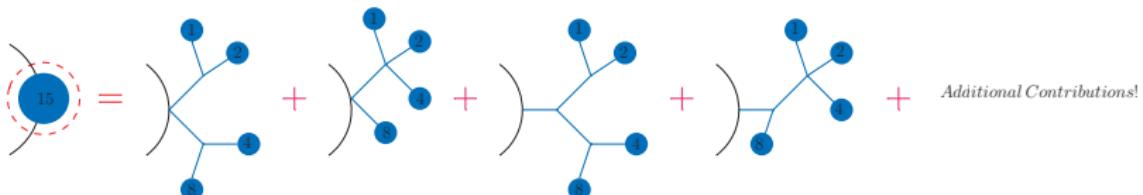
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- What a blob and its level are?



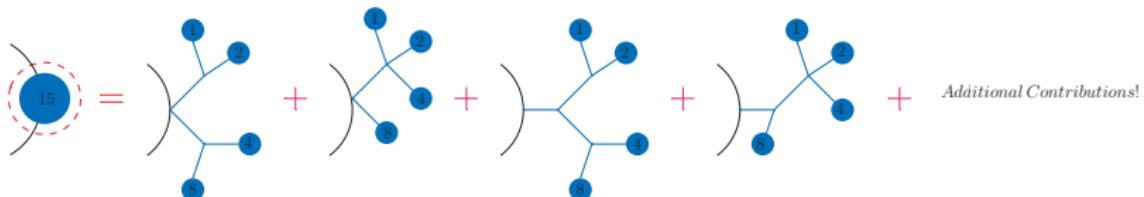
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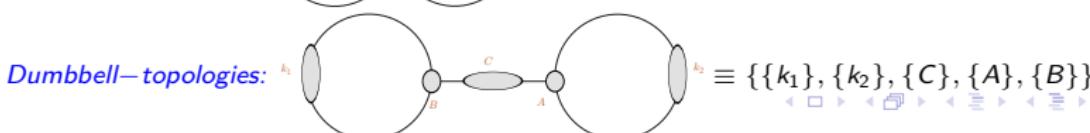
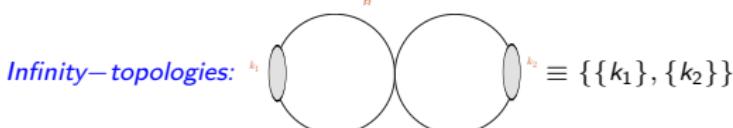
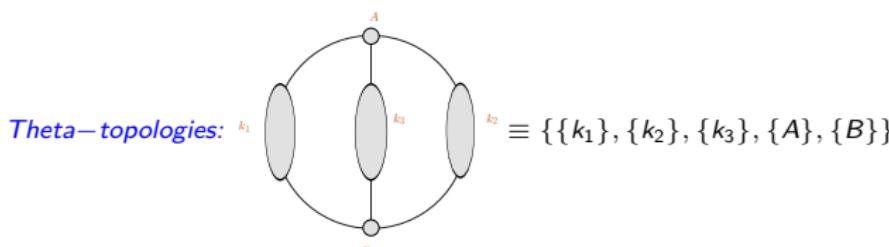


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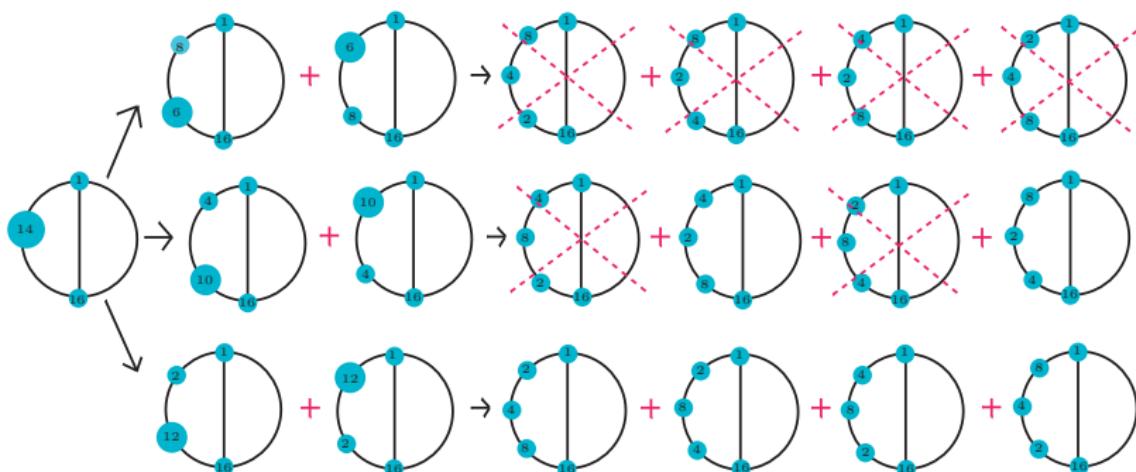


- List representation for the 3 grand blob-topologies:



Blob-Topology generation

Creation of a fortran-based generator → GENTOOLS

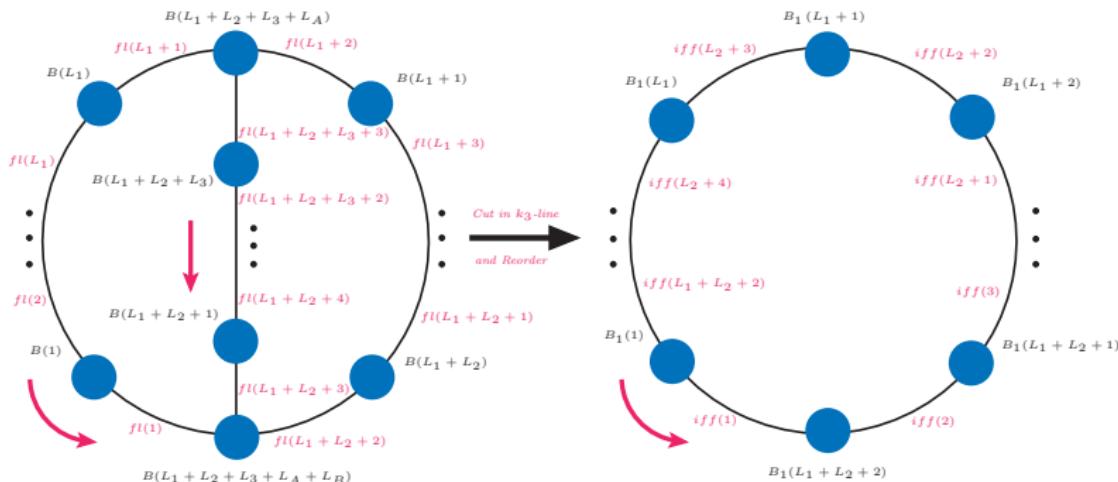


- Generation of all possible blob-topologies: From higher to lower level blobs.
- Order on the Lengths: $L_1 \geq L_2 \geq L_3$ and $L_A \geq L_B$.
- Remove of identical topoes using symmetries: 1) up-down (reversion), 2) loop-line swapping.



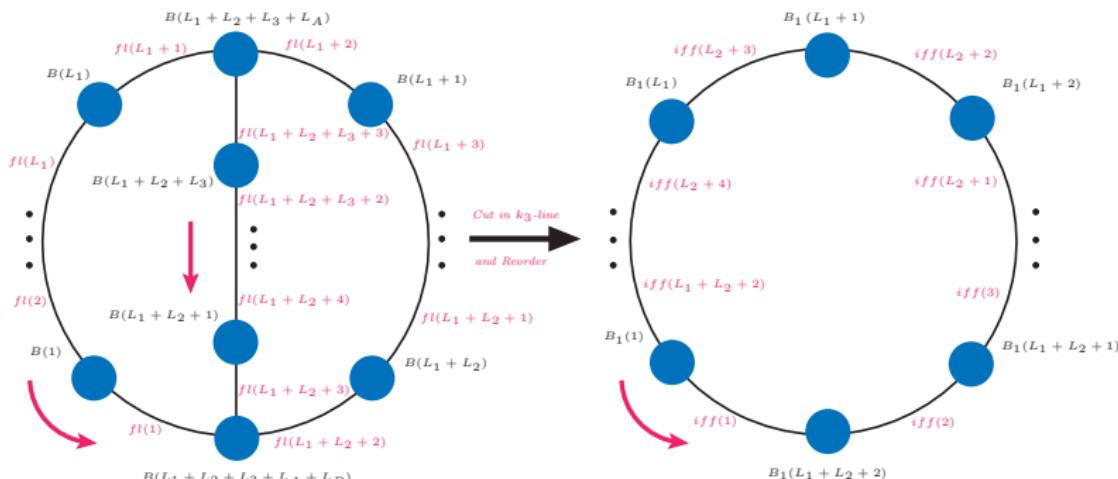
Color-Flavor dressing

QCD particles on the loop + Color-Flow representation!



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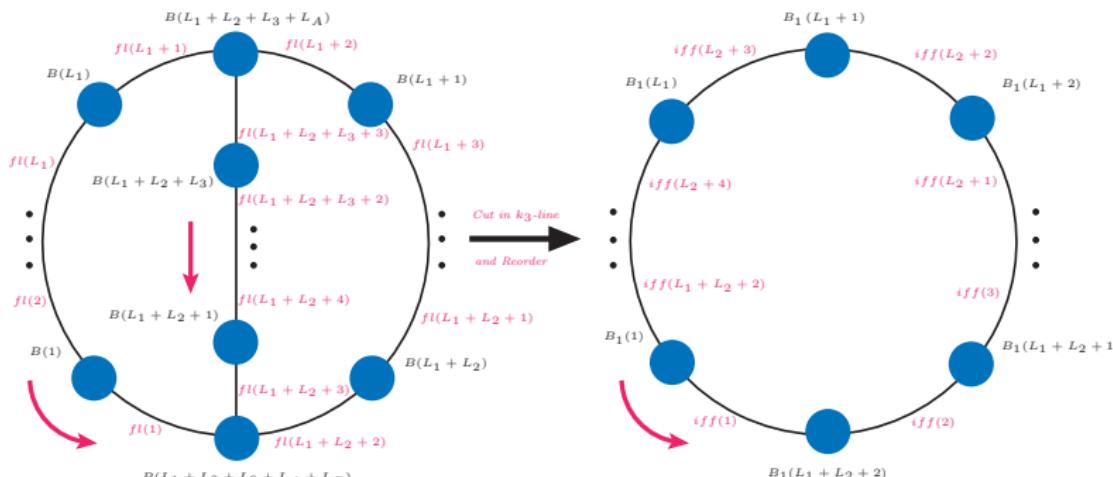
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- Identification of blob-flavors
- Assign flavor to the first propagator.
- QCD Feynman rules in vertices.



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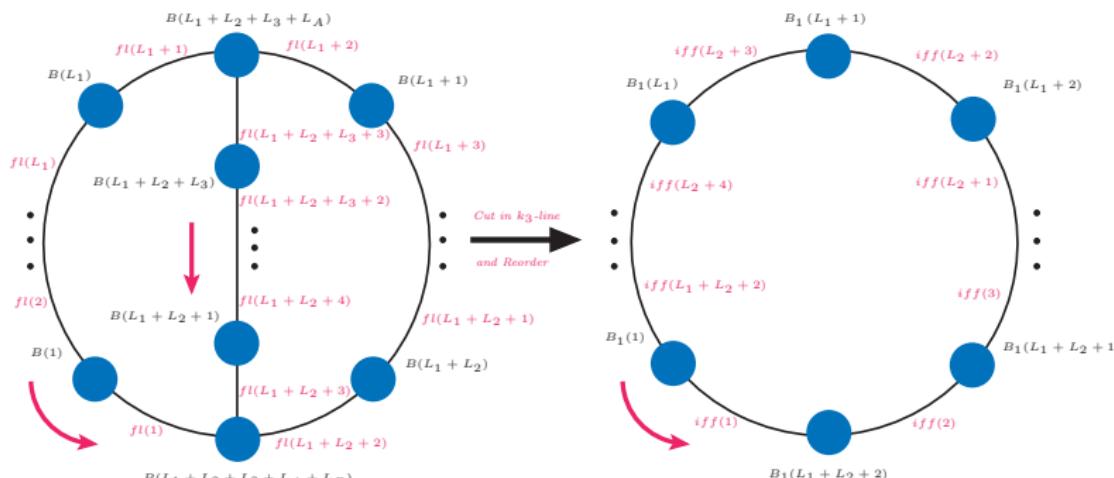
Color Dressing:

- Assign color and anti-color indices to the first propagator.
- Track the color flow at each vertex.



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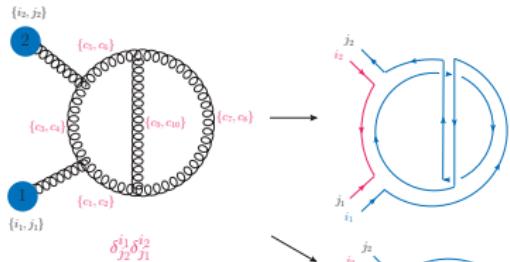
Unique configuration with specific {color, anti-color} and flavor for each loop particle!

Color Dressing:

- Assign color and anti-color indices to the first propagator.
- Track the color flow at each vertex.



Two-loop color-flow dressing → identical configurations for HELAC



$$\{c_1, c_2\} = \{i_1, i_1\}$$

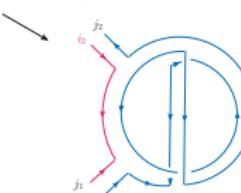
$$\{c_3, c_4\} = \{i_2, i_1\}$$

$$\{c_5, c_6\} = \{i_1, i_1\}$$

$$\{c_7, c_8\} = \{i_1, i_1\}$$

$$\{c_9, c_{10}\} = \{i_1, i_1\}$$

$$C_F = -\delta^{i_1}_{c_1} \delta^{c_1}_{c_7} \delta^{c_1}_{c_{10}} \delta^{c_2}_{c_2} \delta^{c_2}_{c_4} \delta^{c_2}_{c_6} \delta^{c_2}_{c_8} \delta^{c_2}_{c_9} \delta^{c_2}_{c_5} \delta^{c_2}_{j_2} \delta^{c_2}_{c_3} \delta^{c_2}_{j_1}$$



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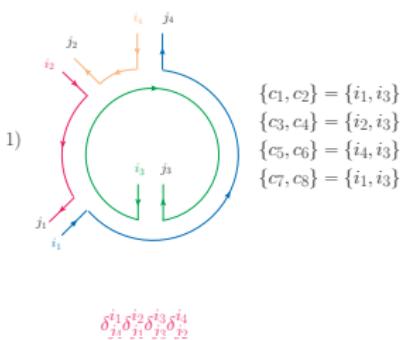
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This is not any more the case after cutting in k_3 -line:

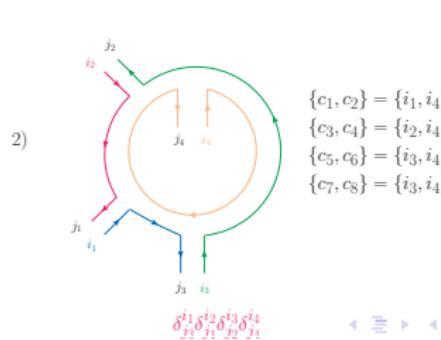


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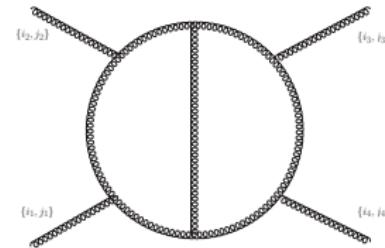
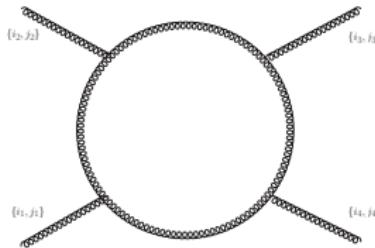
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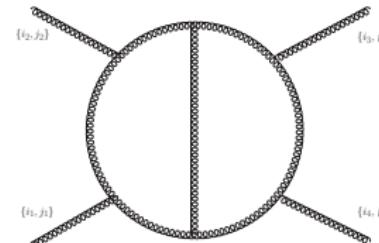
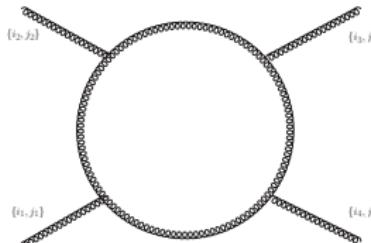
$$\{c_7, c_8\} = \{i_3, i_4\}$$



Comments on 1-loop and 2-loop Color-Flow representation



Comments on 1-loop and 2-loop Color-Flow representation



Box	Double-Box	Color-State ($n!$)
0	0	$\delta^{i_1} j_1 \delta^{i_2} j_2 \delta^{i_3} j_3 \delta^{i_4} j_4$
0	-4	$\delta^{i_1} j_1 \delta^{i_2} j_2 \delta^{i_3} j_4 \delta^{i_4} j_3$
0	2	$\delta^{i_1} j_1 \delta^{i_2} j_3 \delta^{i_3} j_2 \delta^{i_4} j_4$
-1	$-N_c$	$\delta^{i_1} j_1 \delta^{i_2} j_3 \delta^{i_3} j_4 \delta^{i_4} j_2$
-1	$-N_c$	$\delta^{i_1} j_1 \delta^{i_2} j_4 \delta^{i_3} j_2 \delta^{i_4} j_3$
0	-2	$\delta^{i_1} j_1 \delta^{i_2} j_4 \delta^{i_3} j_3 \delta^{i_4} j_2$
0	-4	$\delta^{i_1} j_2 \delta^{i_2} j_1 \delta^{i_3} j_3 \delta^{i_4} j_4$
2	$6N_c$	$\delta^{i_1} j_2 \delta^{i_2} j_1 \delta^{i_3} j_4 \delta^{i_4} j_3$
-1	$-N_c$	$\delta^{i_1} j_2 \delta^{i_2} j_3 \delta^{i_3} j_1 \delta^{i_4} j_4$
N_c	$N_c^2 + 2$	$\delta^{i_1} j_2 \delta^{i_2} j_3 \delta^{i_3} j_4 \delta^{i_4} j_1$
0	2	$\delta^{i_1} j_2 \delta^{i_2} j_4 \delta^{i_3} j_1 \delta^{i_4} j_3$
-1	$-N_c$	$\delta^{i_1} j_2 \delta^{i_2} j_4 \delta^{i_3} j_3 \delta^{i_4} j_1$

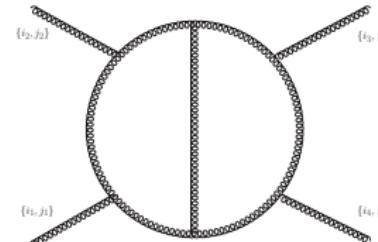
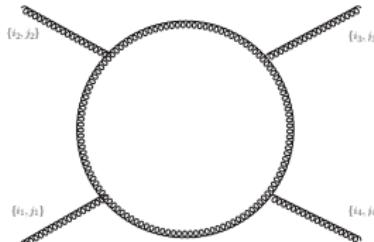


Box	Double-Box	Color-State ($n!$)
-1	$-N_c$	$\delta^{i_1} j_3 \delta^{i_2} j_1 \delta^{i_3} j_2 \delta^{i_4} j_4$
0	2	$\delta^{i_1} j_3 \delta^{i_2} j_1 \delta^{i_3} j_4 \delta^{i_4} j_2$
0	2	$\delta^{i_1} j_3 \delta^{i_2} j_2 \delta^{i_3} j_1 \delta^{i_4} j_4$
-1	$-N_c$	$\delta^{i_1} j_3 \delta^{i_2} j_2 \delta^{i_3} j_4 \delta^{i_4} j_1$
2	0	$\delta^{i_1} j_3 \delta^{i_2} j_4 \delta^{i_3} j_1 \delta^{i_4} j_2$
0	-4	$\delta^{i_1} j_3 \delta^{i_2} j_4 \delta^{i_3} j_2 \delta^{i_4} j_1$
N_c	$N_c^2 + 2$	$\delta^{i_1} j_4 \delta^{i_2} j_1 \delta^{i_3} j_2 \delta^{i_4} j_3$
-1	$-N_c$	$\delta^{i_1} j_4 \delta^{i_2} j_1 \delta^{i_3} j_3 \delta^{i_4} j_2$
-1	$-N_c$	$\delta^{i_1} j_4 \delta^{i_2} j_2 \delta^{i_3} j_1 \delta^{i_4} j_3$
0	2	$\delta^{i_1} j_4 \delta^{i_2} j_2 \delta^{i_3} j_3 \delta^{i_4} j_1$
0	-4	$\delta^{i_1} j_4 \delta^{i_2} j_3 \delta^{i_3} j_1 \delta^{i_4} j_2$
2	0	$\delta^{i_1} j_4 \delta^{i_2} j_3 \delta^{i_3} j_2 \delta^{i_4} j_1$



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Comments on 1-loop and 2-loop Color-Flow representation



Box	Double-Box	Color-State ($n!$)
0	0	$\delta^{i_1} j_1 \delta^{i_2} j_2 \delta^{i_3} j_3 \delta^{i_4} j_4$
0	-4	$\delta^{i_1} j_1 \delta^{i_2} j_2 \delta^{i_3} j_4 \delta^{i_4} j_3$
0	2	$\delta^{i_1} j_1 \delta^{i_2} j_3 \delta^{i_3} j_2 \delta^{i_4} j_4$
-1	$-N_c$	$\delta^{i_1} j_1 \delta^{i_2} j_3 \delta^{i_3} j_4 \delta^{i_4} j_2$
-1	$-N_c$	$\delta^{i_1} j_1 \delta^{i_2} j_4 \delta^{i_3} j_2 \delta^{i_4} j_3$
0	-2	$\delta^{i_1} j_1 \delta^{i_2} j_4 \delta^{i_3} j_3 \delta^{i_4} j_2$
0	-4	$\delta^{i_1} j_2 \delta^{i_2} j_1 \delta^{i_3} j_3 \delta^{i_4} j_4$
2	$6N_c$	$\delta^{i_1} j_2 \delta^{i_2} j_1 \delta^{i_3} j_4 \delta^{i_4} j_3$
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N_c	$N_c^2 + 2$	$\delta^{i_1} j_2 \delta^{i_2} j_3 \delta^{i_3} j_4 \delta^{i_4} j_1$
0	2	$\delta^{i_1} j_2 \delta^{i_2} j_4 \delta^{i_3} j_1 \delta^{i_4} j_3$
-1	$-N_c$	$\delta^{i_1} j_2 \delta^{i_2} j_4 \delta^{i_3} j_3 \delta^{i_4} j_1$

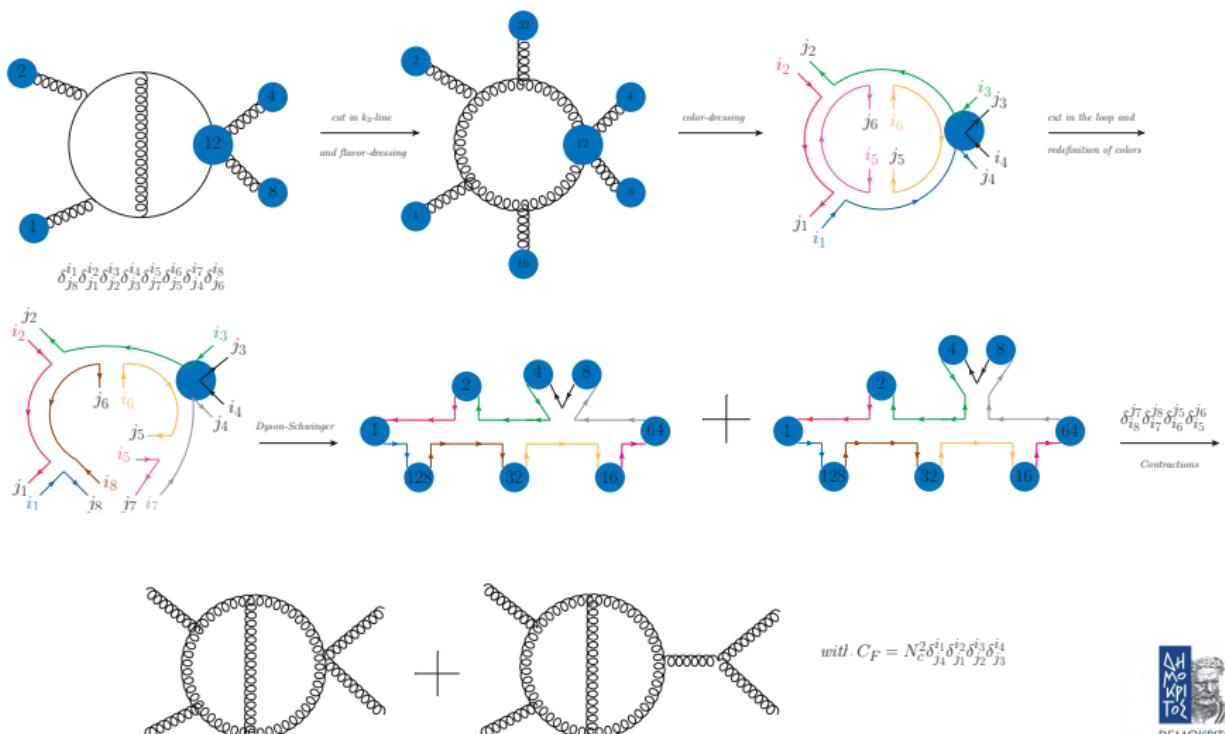
- More nonzero color-states @ 2-loop.
- Higher multiplicity factor @ 2-loop.

Box	Double-Box	Color-State ($n!$)
-1	$-N_c$	$\delta^{i_1} j_3 \delta^{i_2} j_1 \delta^{i_3} j_2 \delta^{i_4} j_4$
0	2	$\delta^{i_1} j_3 \delta^{i_2} j_1 \delta^{i_3} j_4 \delta^{i_4} j_2$
0	2	$\delta^{i_1} j_3 \delta^{i_2} j_2 \delta^{i_3} j_1 \delta^{i_4} j_4$
-1	$-N_c$	$\delta^{i_1} j_3 \delta^{i_2} j_2 \delta^{i_3} j_4 \delta^{i_4} j_1$
2	0	$\delta^{i_1} j_3 \delta^{i_2} j_4 \delta^{i_3} j_1 \delta^{i_4} j_2$
0	-4	$\delta^{i_1} j_3 \delta^{i_2} j_4 \delta^{i_3} j_2 \delta^{i_4} j_1$
N_c	$N_c^2 + 2$	$\delta^{i_1} j_4 \delta^{i_2} j_1 \delta^{i_3} j_2 \delta^{i_4} j_3$
-1	$-N_c$	$\delta^{i_1} j_4 \delta^{i_2} j_1 \delta^{i_3} j_3 \delta^{i_4} j_2$
-1	$-N_c$	$\delta^{i_1} j_4 \delta^{i_2} j_2 \delta^{i_3} j_1 \delta^{i_4} j_3$
0	2	$\delta^{i_1} j_4 \delta^{i_2} j_2 \delta^{i_3} j_3 \delta^{i_4} j_1$
0	-4	$\delta^{i_1} j_4 \delta^{i_2} j_3 \delta^{i_3} j_1 \delta^{i_4} j_2$
2	0	$\delta^{i_1} j_4 \delta^{i_2} j_3 \delta^{i_3} j_2 \delta^{i_4} j_1$

- Different powers of N_c for the same color-state @ 2-loop.



Construction: gluonic $\{\{1, 2\}, \{12\}, \{\}, \{\}, \{\}\}$ with $\delta_{j_8}^{i_1} \delta_{j_1}^{i_2} \delta_{j_2}^{i_3} \delta_{j_3}^{i_4} \delta_{j_6}^{i_5} \delta_{j_5}^{i_6}$



Results: gluonic $\{\{1, 2\}, \{12\}, \{\}, \{\}, \{\}\}$ *with* $\delta_{i_4}^{i_1} \delta_{j_1}^{i_2} \delta_{j_2}^{i_3} \delta_{j_3}^{i_4} \delta_{j_6}^{i_5} \delta_{j_5}^{i_6}$

```
INFO =====
INFO COLOR      9 out of      24
INFO number of nums   0
INFO =====
INFO COLOR      10 out of      24
INFO number of nums   208
```

→ Skeleton stored color-wised

INFO NUM		52 of		208				7											
INFO =====																			
INFO	4	80	35	9	1	1	16	35	5	64	35	7	0	0	0	0	1	2	
INFO	4	12	35	10	1	1	4	35	3	8	35	4	0	0	0	0	0	1	1
INFO	4	92	35	11	1	2	12	35	10	80	35	9	0	0	0	0	0	1	1
INFO	5	92	35	11	2	2	4	35	3	8	35	4	80	35	9	0	1	5	5
INFO	4	124	35	12	1	1	32	35	6	92	35	11	0	0	0	0	0	1	2
INFO	4	126	35	13	1	1	2	35	2	124	35	12	0	0	0	0	0	1	1
INFO	4	254	35	14	1	1	128	35	8	126	35	13	0	0	0	0	0	1	2
INFO	6	1	12	1	2	12	35	35	35	35	35	0	0	0	0	0	99	99	



Results: gluonic $\{\{1, 2\}, \{12\}, \{\}, \{\}, \{\}\}$ with $\delta_{j_4}^{i_1} \delta_{j_1}^{i_2} \delta_{j_2}^{i_3} \delta_{j_3}^{i_4} \delta_{j_6}^{i_5} \delta_{j_5}^{i_6}$

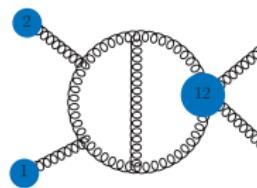
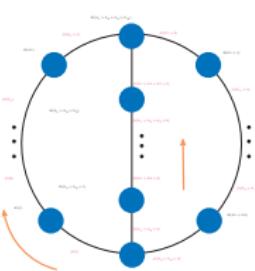
```
INFO =====
INFO COLOR      9 out of      24
INFO number of nums      0
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```

→ Skeleton stored color-wised

INFO NUM	52	of	208	7																
INFO	4	80	35	9	1	1	16	35	5	64	35	7	0	0	0	0	0	1	2	
INFO	4	12	35	10	1	1	4	35	3	8	35	4	0	0	0	0	0	1	1	
INFO	4	92	35	11	1	2	12	35	10	80	35	9	0	0	0	0	0	1	1	
INFO	5	92	35	11	2	2	4	35	3	8	35	4	80	35	9	0	1	5		
INFO	4	124	35	12	1	1	32	35	6	92	35	11	0	0	0	0	0	1	2	
INFO	4	126	35	13	1	1	2	35	2	124	35	12	0	0	0	0	0	1	1	
INFO	4	254	35	14	1	1	128	35	8	126	35	13	0	0	0	0	0	1	2	
INFO	6	1	12	1	2	12	35	35	35	35	35	35	0	0	0	0	99	9		

$$ID = \begin{cases} (2)^L 1 (3)^L 2 (5)^L 3 (7)^L A (11)^L B, & \text{Theta} \\ (2)^L 1 (3)^L 2, & \text{Infinity} \\ (2)^L 1 (3)^L 2 (5)^L C (7)^L A (11)^L B, & \text{Dumbbell} \end{cases}$$

$$loopnum = \begin{cases} 1, & \text{Theta} \\ 2, & \text{Infinity} \\ 3, & \text{Dumbbell} \end{cases}$$



Numerators and numerics in 4 dimensions

Process	loop-flavors	Color	Size	Time	Numerators
$gg \rightarrow gg$	{g}	leading	2.4 MB	30 m	1248
$gg \rightarrow gg$	{g}	full	37.1 MB	4h 40m	26048
$gg \rightarrow ggg$	{g}	leading	63.8 MB	2d 10h	17040

Comments on the skeletons:

- 1 n increase \longrightarrow complexity increase
- 2 leading color to full color \longrightarrow complexity increase
- 3 Timings a bit large \longrightarrow Skeleton constructed only once per process!
- 4 Much numerators (some are identical) \longrightarrow Room for improving efficiency!



$p_1 = (250, 0, 0, 250)$, $p_2 = (250, 0, 0, -250)$, $p_3 = (250, 49, -176, -171)$, $k_1 = (0.2, 0.3, 0.5, 0.7)$ and $k_3 = (0.9, 0.11, 0.13, 0.15)$



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- 4 Much numerators (some are identical) → Room for improving efficiency!

Numerical results for some numerators (*helicity = -- → --*)²:

- 1 $N_{\{\{1,2\},\{12\},\{\},\{\},\{\}}}=17052219.315419123+64639250.888367772i.$
- 2 $N_{\{\{1,2\},\{4,8\},\{\},\{\},\{\}}}= -12231870819598.090+5124375444085.5430i.$
- 3 $N_{\{\{1,2\},\{4\},\{8\},\{\},\{\}}}= -1268111397619.5310+195312105699.88257i.$
- 4 $N_{\{\{2,1\},\{8\},\{\},\{4\},\{\}}}= -49731029299.352333+15599344.440385548i.$

- Perfect agreement in cross-checks with FeynArts + FeynCalc!
- Quarks and ghosts on the loop not completely implemented yet → Work in progress!

² $p_1 = (250, 0, 0, 250)$, $p_2 = (250, 0, 0, -250)$, $p_3 = (250, 49, -176, -171)$, $k_1 = (0.2, 0.3, 0.5, 0.7)$ and $k_3 = (0.9, 0.11, 0.13, 0.15)$

Conclusion

Summary

- Implementation of a two-loop algorithm for the 4-dimensional computation of the integrand numerators using a hybrid Dyson-Schwinger recursion → validations for gluon-loops!
- Currently working in the implementation of the rest of QCD particles in the loop and in the partially optimization and parallelism in different cores of the code.



Conclusion

Summary

- Implementation of a two-loop algorithm for the 4-dimensional computation of the integrand numerators using a hybrid Dyson-Schwinger recursion → validations for gluon-loops!
- Currently working in the implementation of the rest of QCD particles in the loop and in the partially optimization and parallelism in different cores of the code.

Next milestones for HELAC-2LOOP

- Development of a two-loop OPP-like method for a 4-dimensional amplitude reduction at the integrand level

$$A_{2\text{-loop}} = \frac{\bar{N}_I(\bar{k}_1, \bar{k}_2, p_1, \dots, p_{n-1}, \gamma^\mu, \epsilon^\mu)}{\prod_{\{i_1, i_2, i_3\} \in I} \bar{D}_{i_1}(\bar{k}_1) \bar{D}_{i_2}(\bar{k}_2) \bar{D}_{i_3}(\bar{k}_1, \bar{k}_2)} = \sum_i c_i(\mathbf{s}) I_i + \sum_j \tilde{c}_j(\mathbf{s}) S_j$$

- Computation of rational terms³ originating from $\bar{k}_{1,2} \rightarrow k_{1,2} + \tilde{k}_{1,2}$

$$A_{2\text{-loop}} = \sum_i c_i(\mathbf{s}) F_i(\mathbf{s}, \varepsilon) + R_1^{2\text{-loop}}(\mathbf{s}, \varepsilon) + R_2^{2\text{-loop}}(\mathbf{s}, \varepsilon) + \mathcal{O}(\varepsilon)$$

Alternatively implementation of d -dimensional integrand reduction [H. Ita, Phys.Rev.D 94 (2016) 11, 116015]!

- Calculation of Feynman Integrals and creation of a library (like pentagon functions) for efficient calculations!



³progress made on that direction → J. Lang, S. Pozzorini, H. Zhang and M. Zoller, JHEP 10 (2020) 016

Thank you!

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Color connection representation

- In the color connection representation, the gluons are represented by a pair of color/anti-color indices (i, j) and the quarks (anti-quarks) by a single color $(i, 0)$ (anti-color $(0, j)$) index, with $i, j \in (1, \dots, N_C)$. All the other particles that do not carry color have $(0, 0)$.
- The amplitude takes the following form

$$\mathcal{M}_{j_1, j_2, \dots, j_k}^{i_1, i_2, \dots, i_k} = \sum_{\sigma} \delta_{i_{\sigma_1}, j_1} \delta_{i_{\sigma_2}, j_2} \dots \delta_{i_{\sigma_k}, j_k} A_{\sigma}$$

with $k = n_g + n_q$ and the sum is running over all the permutations (equal to $k!$). The color-stripped amplitudes, A_{σ} , are calculated using properly defined Feynman rules [A. Cafarella, C. G. Papadopoulos and M. Worek, *Comput. Phys. Commun.* **180** (2009), 1941-1955].

- The total color factor is a product of δ 's, and thus the color summed squared amplitude takes the form

$$\sum_{\{i\}, \{j\}} |\mathcal{M}_{j_1, j_2, \dots, j_k}^{i_1, i_2, \dots, i_k}|^2 = \sum_{\sigma, \sigma'} A_{\sigma}^* C_{\sigma', \sigma} A_{\sigma}$$

where the color matrix $C_{\sigma', \sigma}$ is given by

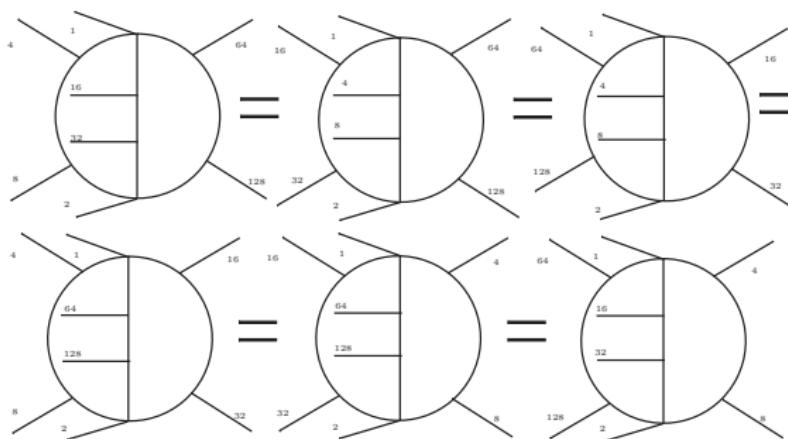
$$C_{\sigma', \sigma} = \sum_{\{i\}, \{j\}} \delta_{i_{\sigma'_1}, j_1} \delta_{i_{\sigma'_2}, j_2} \dots \delta_{i_{\sigma'_k}, j_k} \delta_{i_{\sigma_1}, j_1} \delta_{i_{\sigma_2}, j_2} \dots \delta_{i_{\sigma_k}, j_k} = N_C^{m(\sigma', \sigma)}$$

with $m(\sigma', \sigma)$ counting the number of common cycles of the 2 permutations.



Graph symmetries

The graphs are symmetric on (combined or individual) mirror transformations on the vertical and the horizontal axis (swap of the three loop lines). For example



All the symmetries of the graphs can be expressed in symmetries of the lists using one or both of the following 2 actions:

- **Swap:** corresponds to the swap of two sublists. E.g. $\{\{k_1\}, \{k_2\}\} \rightarrow \{\{k_2\}, \{k_1\}\}$.
- **Reversion:** corresponds to the reversion of the elements of a sublist. E.g. $\{1, 2, 4\} \rightarrow \{4, 2, 1\}$.



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