The on-shell expansion: from Landau equations to the Feynman polytope

High Precision for Hard Processes (HP2 2022)

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OUTLINE

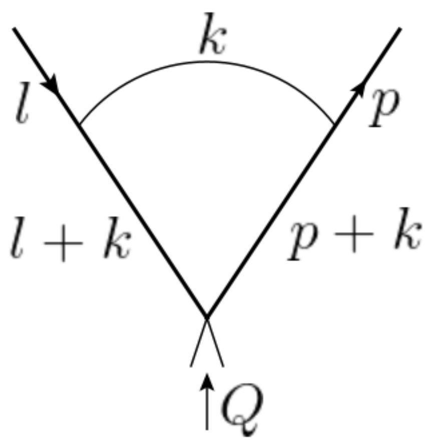
- Review of the method of regions (MoR)
  - Traditional approach in momentum space
  - The Newton polytope approach
- The Landau equations and the regions
- Criteria for the regions
- An algorithm to obtain the regions
- Conclusion
- Outlook

Review of the method of regions (MoR)

Traditional approach in momentum space

$$\mathcal{I} = \mathcal{I}^{(R_1)} + \mathcal{I}^{(R_2)} + \cdots + \mathcal{I}^{(R_n)}.$$

Traditional approach in momentum space



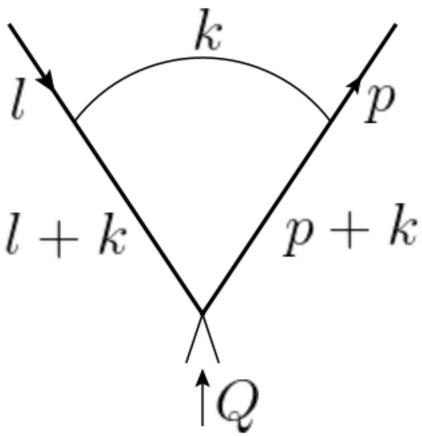
(from "Introduction to Soft-Collinear Effective Theory" by Becher, Broggio, Ferroglia)

$$I = i\pi^{-d/2}\mu^{4-d}\int d^dk\,\frac{1}{(k^2+i0)\left[(k+l)^2+i0\right]\left[(k+p)^2+i0\right]},$$

Kinematic limit: $p^\mu \sim Q(\underset{+}{\lambda}, \underset{-}{1}, \underset{\perp}{\lambda^{\frac{1}{2}}}), \quad l^\mu \sim Q(\underset{+}{1}, \underset{-}{\lambda}, \underset{\perp}{\lambda^{\frac{1}{2}}})$

$$p^2/Q^2 \sim l^2/Q^2 \sim \lambda \ll 1$$

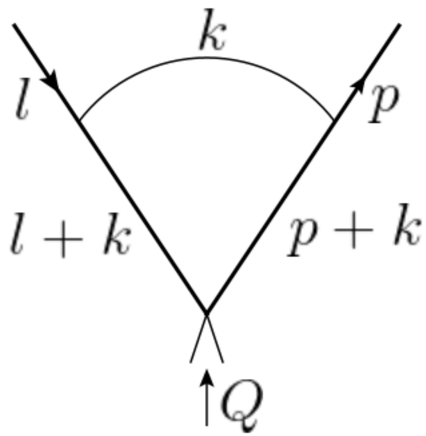
Traditional approach in momentum space



In terms of the method of regions:

The Feynman integral can be regarded as a sum over contributions from the corresponding regions.

Traditional approach in momentum space



In terms of the method of regions:

The Feynman integral can be regarded as a sum over contributions from the corresponding regions.

$$p^2/Q^2 \sim l^2/Q^2 \sim \lambda \ll 1$$

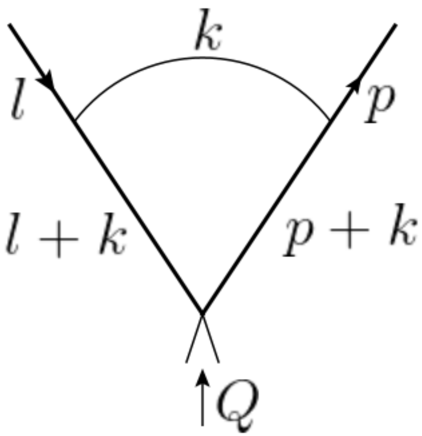
$$\text{Hard region}: \; k^\mu \sim (1,1,1)Q$$

$$\text{Collinear region to } p: k^\mu \sim (\lambda, 1, \lambda^{\frac{1}{2}})Q$$

$$\text{Collinear region to } l: k^\mu \sim (1, \lambda, \lambda^{\frac{1}{2}})Q$$

$$\text{Soft region}: k^\mu \sim (\lambda, \lambda, \lambda)Q$$

Traditional approach in momentum space



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The hard region, for example:

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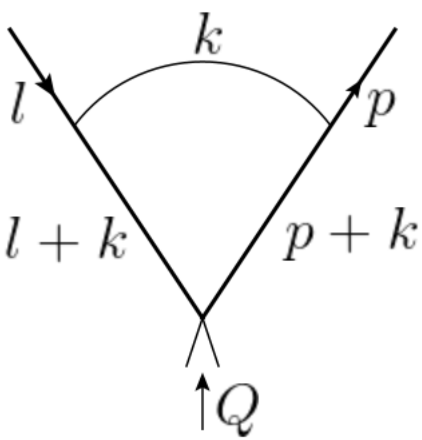
Traditional approach in momentum space



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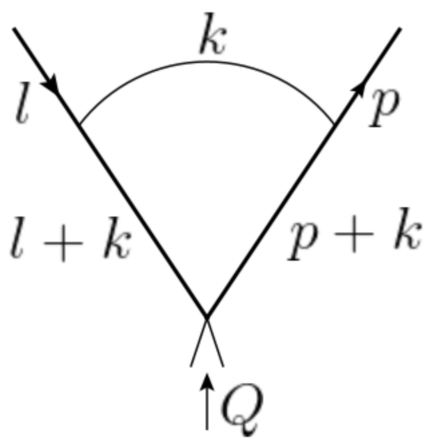
The hard region, for example:

$$I = i\pi^{-d/2}\mu^{4-d}\int d^d k \frac{1}{(k^2+i0)\,[(k+l)^2+i0]\,[(k+p)^2+i0]}\,,$$

$$I_h = i\pi^{-D/2}\mu^{4-D}\int d^D k \frac{1}{(k^2+i0)(k^2+2k_-\cdot l_+ +i0)(k^2+2k_+\cdot p_- +i0)}$$

$$+\cdots$$

Traditional approach in momentum space



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$$I_h = i\pi^{-D/2}\mu^{4-D}\int d^Dk \frac{1}{(k^2+i0)(k^2+2k_-\cdot l_+ + i0)(k^2+2k_+\cdot p_- + i0)} + \cdots$$

$$I_c = i\pi^{-D/2}\mu^{4-D}\int d^Dk \frac{1}{(k^2+i0)(2k_-\cdot l_+ + i0)((k+p)^2+i0)} + \cdots$$

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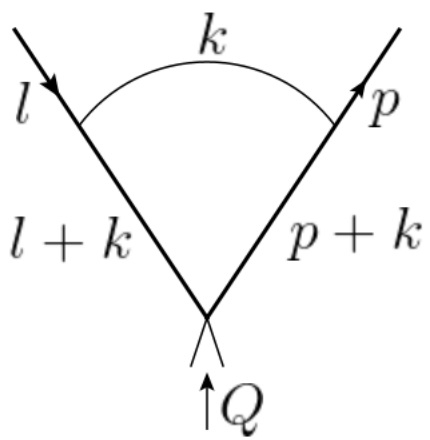
$$I_s = i\pi^{-D/2}\mu^{4-D}\int d^Dk \frac{1}{(k^2+i0)(2k_-\cdot l_+ + l^2 + i0)(2k_+\cdot p_- + p^2 + i0)} + \cdots$$

Traditional approach in momentum space



In terms of the method of regions:

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$$I = I_h + I_l + I_p + I_s = \frac{1}{Q^2}\left(\ln\frac{Q^2}{(-l^2)}\ln\frac{Q^2}{(-p^2)} + \frac{\pi^2}{3} + \mathcal{O}(\lambda)\right)$$

Actually, this equality holds to all orders of λ!

(More examples can be find in Smirnov’s book “Applied Asymptotic Expansions in Momenta and Masses”.)

Traditional approach in momentum space

To summarize:

- **The original integral can be approximated, or even restored, by the sum over contributions from each region.**

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To summarize:

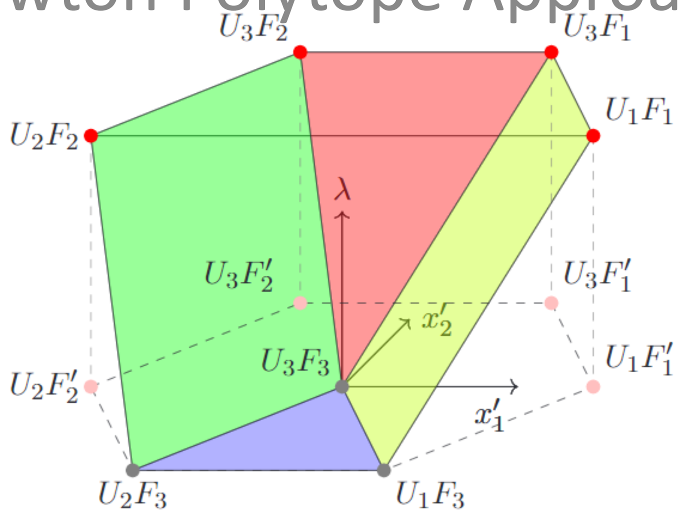
- **The original integral can be approximated, or even restored, by the sum over contributions from each region.**

- **The integration measure is the entire space for each term.**

- **The regions are chosen using heuristic methods based on examples and experience.**

Review of the method of regions (MoR)

The Newton Polytope Approach



The Newton polytope approach

Lee-Pomeransky representation of the Feynman integral

$$\mathcal{I}(G) = \frac{\Gamma(D/2)}{\Gamma((L+1)D/2-\nu)\prod_{e\in G}\Gamma(\nu_e)}\int_0^\infty \left(\prod_{e\in G} dx_e x_e^{\nu_e-1}\right)\left(\mathcal{P}\left(\boldsymbol{x},\boldsymbol{s}\right)\right)^{-D/2},$$

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) \equiv \mathcal{U}(\boldsymbol{x}) + \mathcal{F}(\boldsymbol{x},\boldsymbol{s}),$$

$$\mathcal{U}(\boldsymbol{x}) = \sum_{T^1}\prod_{e\notin T^1} x_e, \qquad \mathcal{F}(\boldsymbol{x},\boldsymbol{s}) = -\sum_{T^2} s_{T^2}\prod_{e\notin T^2} x_e + \mathcal{U}(\boldsymbol{x})\sum_e m_e^2 x_e\ .$$

The Newton polytope approach

Lee-Pomeransky representation of the Feynman integral

Our work focuses on the on-shell expansion
for wide-angle scattering:

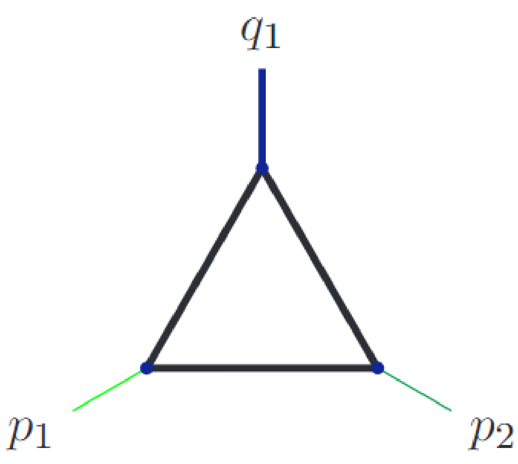
$$\text{on-shell expansion:} \quad p_i^2 \sim \lambda Q^2, \qquad q_j^2 \sim Q^2, \qquad m^2 = 0$$
$$\text{wide-angle scattering:} \quad p_k \cdot p_l \sim Q^2 \quad (\forall k \neq l).$$

where $\lambda \ll 1$ is a small scaling vector.

The Newton polytope approach

Lee-Pomeransky representation of the Feynman integral

Example:



$$\mathcal{U} = x_1 + x_2 + x_3,$$

$$\mathcal{F} = (-p_1^2)x_1x_3 + (-p_2^2)x_2x_3 + (-q_1^2)x_1x_2.$$

$$\mathcal{P}(\boldsymbol{x}, \boldsymbol{s}) \equiv \mathcal{U}(\boldsymbol{x}) + \mathcal{F}(\boldsymbol{x}, \boldsymbol{s}),$$

$$\mathcal{I}(G) = \mathcal{C} \int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} x_3^{\nu_3 - 1}$$
$$\cdot \left(x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2\right)^{-D/2},$$

The Newton polytope approach

Lee-Pomeransky representation of the Feynman integral

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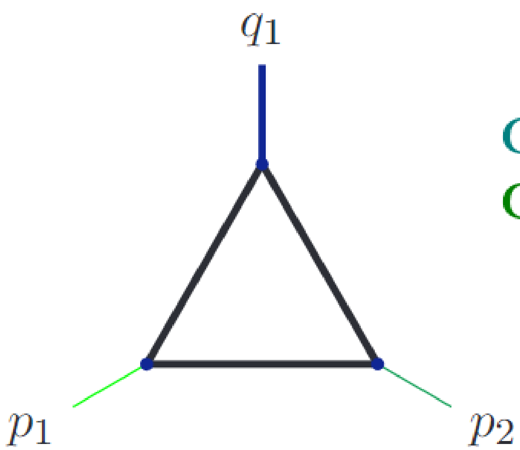

$$\textbf{Hard region}: \; x_1, x_2, x_3 \sim \lambda^0$$

$$\textbf{Collinear region to } p_1: x_1, x_3 \sim \lambda^{-1}, \; x_2 \sim \lambda^0$$

$$\textbf{Collinear region to } p_2: x_1 \sim \lambda^0, x_2, x_3 \sim \lambda^{-1}$$

$$\textbf{Soft region}: x_1, x_2 \sim \lambda^{-1}, \; x_3 \sim \lambda^{-2}$$

$$\mathcal{I}(G) = \; \mathcal{C} \int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} x_3^{\nu_3 - 1} \cdot \left(x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2\right)^{-D/2},$$

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The Newton polytope approach

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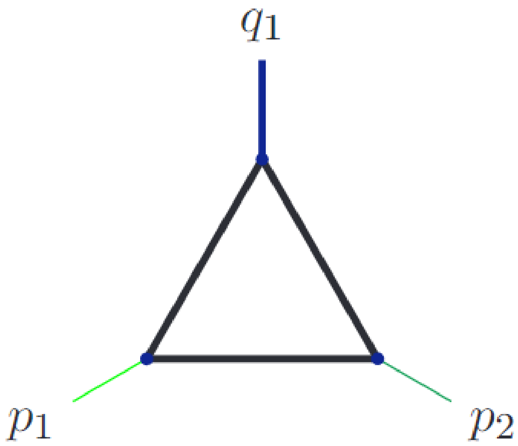


Hard region : $x_1, x_2, x_3 \sim \lambda^0$

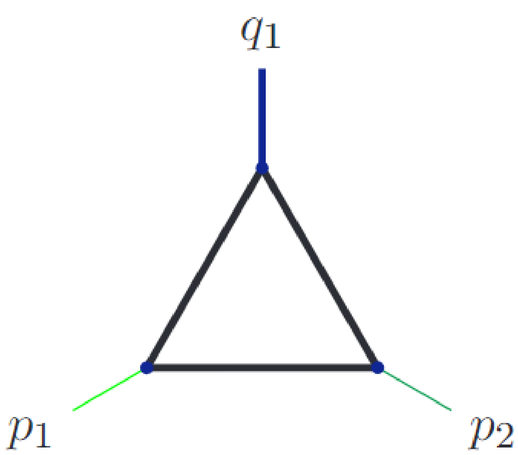
$$\mathcal{I}_h = \mathcal{C}\int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\nu_3-1} \cdot \left(x_1+x_2+x_3-q_1^2 x_1 x_2\right)^{-D/2},$$

$$+\cdots$$

The Newton polytope approach

Lee-Pomeransky representation of the Feynman integral

Example:



Collinear region to p_1 : $x_1, x_3 \sim \lambda^{-1},\ x_2 \sim \lambda^0$

$$\mathcal{I}_{p_1} = \mathcal{C}\int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\nu_3-1} \cdot \left(x_1 + x_3 - p_1^2 x_1 x_3 - q_1^2 x_1 x_2\right)^{-D/2},$$

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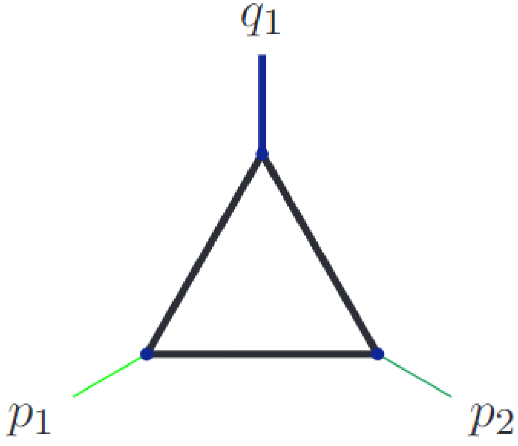

Collinear region to $p_2 : x_1 \sim \lambda^0, x_2, x_3 \sim \lambda^{-1}$

$$\mathcal{I}_{p_2} = \mathcal{C} \int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} x_3^{\nu_3 - 1} \cdot \left(x_2 + x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2\right)^{-D/2},$$

$$+ \cdots$$

The Newton polytope approach

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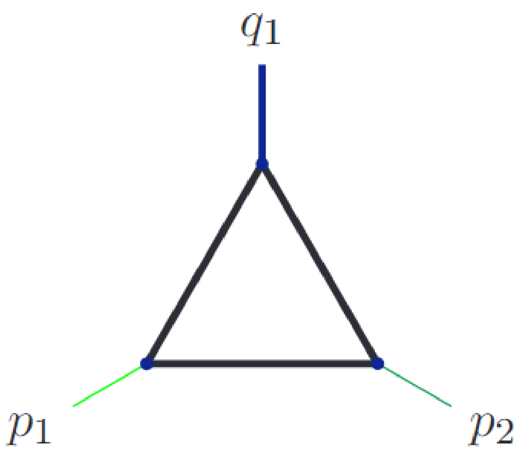

Soft region : $x_1, x_2 \sim \lambda^{-1},\ x_3 \sim \lambda^{-2}$

$$\mathcal{I}_s = \mathcal{C} \int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\nu_3-1} \cdot \left(x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2\right)^{-D/2},$$

$$+ \cdots$$

The Newton polytope approach

A systematic way to determine these regions:

To construct the Newton polytope associated to the Lee-Pomeransky polynomial:

$$\mathcal{P}(\boldsymbol{x}, \boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

More precisely, the Newton polytope is the convex hull of the exponents of the Lee-Pomeransky polynomial, which is (N+1)-dimensional.

(N: the number of propagators of G)

The Newton polytope approach

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- x_1 → **(1,0,0,0)**
- x_2 → **(0,1,0,0)**
- x_3 → **(0,0,1,0)**
- $p_1^2 x_1 x_3$ → **(1,0,1,1)**
- $p_2^2 x_2 x_3$ → **(0,1,1,1)**
- $q_1^2 x_1 x_2$ → **(1,1,0,0)**

The Newton polytope approach

The Newton polytope is the convex hull of the exponents of the Lee-Pomeransky polynomial.

Suppose a graph has N propagators, then the Newton polytope is (N+1)-dimensional.

The regions are identified as the lower facets of the Newton polytope.

The Newton polytope approach

Coming back to our example:

$$\mathcal{P}(x,s) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

x_1 → **(1,0,0,0)**

x_2 → **(0,1,0,0)**

x_3 → **(0,0,1,0)**

$p_1^2 x_1 x_3$ → **(1,0,1,1)**

$p_2^2 x_2 x_3$ → **(0,1,1,1)**

$q_1^2 x_1 x_2$ → **(1,1,0,0)**

Each region is considered as a specific facet containing certain points of the polytope.

The Newton polytope approach

Coming back to our example:

$$\mathcal{P}(\boldsymbol{x}, \boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

(1,0,0,0) (0,1,0,0) (0,0,1,0) (1,0,1,1) (0,1,1,1) (1,1,0,0)

These points are in the hard facet:

The Newton polytope approach

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- x_1 → (1,0,0,0)
- x_2 → (0,1,0,0)
- x_3 → (0,0,1,0)
- $p_1^2 x_1 x_3$ → (1,0,1,1)
- $p_2^2 x_2 x_3$ → (0,1,1,1)
- $q_1^2 x_1 x_2$ → (1,1,0,0)

These points are in the hard facet:

Hard region : $x_1, x_2, x_3 \sim 1$

$$\mathcal{I}_h = \mathcal{C} \int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} x_3^{\nu_3 - 1} \cdot (x_1 + x_2 + x_3 - q_1^2 x_1 x_2)^{-D/2},$$

$$+ \cdots$$

The Newton polytope approach

The points **r** on a lower facet are those with the minimum value of **r·v$_R$**, where **v$_R$** is the vector normal to the facet.

Hard region vector **v$_h$**=(0,0,0,1)

$$\mathcal{P}(\boldsymbol{x},\boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

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- x_2 → **(0,1,0,0)**
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These points are in the hard facet:

The Newton polytope approach

Coming back to our example:

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x_1 → **(1,0,0,0)** x_3 → **(0,0,1,0)** $p_1^2 x_1 x_3$ → **(1,0,1,1)** $q_1^2 x_1 x_2$ → **(1,1,0,0)**

These points are in the collinear-p1 facet:

$$\textbf{Collinear region to } p_1 : x_1, x_3 \sim 1,\ x_2 \sim \lambda$$

$$\mathcal{I}_{p_1} = \mathcal{C} \int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} x_3^{\nu_3 - 1} \cdot \left(x_1 + x_3 - p_1^2 x_1 x_3 - q_1^2 x_1 x_2\right)^{-D/2},$$

$$+ \cdots$$

The Newton polytope approach

Coming back to our example:

$$\mathcal{P}(\boldsymbol{x}, \boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

$x_2 \rightarrow$ (0,1,0,0)

$x_3 \rightarrow$ (0,0,1,0)

$p_2^2 x_2 x_3 \rightarrow$ (0,1,1,1)

$q_1^2 x_1 x_2 \rightarrow$ (1,1,0,0)

These points are on the collinear-p2 facet:

$$\textbf{Collinear region to } \boldsymbol{p_2} : \boldsymbol{x_1 \sim \lambda, x_2, x_3 \sim 1}$$

$$\mathcal{I}_{p_2} = \mathcal{C} \int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} x_3^{\nu_3 - 1} \cdot \left(x_2 + x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2 \right)^{-D/2},$$

$$+ \cdots$$

The Newton polytope approach

Coming back to our example:

$$\mathcal{P}(\boldsymbol{x}, \boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$

x_3 → **(0,0,1,0)**; $p_1^2 x_1 x_3$ → **(1,0,1,1)**; $p_2^2 x_2 x_3$ → **(0,1,1,1)**; $q_1^2 x_1 x_2$ → **(1,1,0,0)**

These points are on the soft facet:

$$\textbf{Soft region}: \boldsymbol{x_1, x_2 \sim \lambda,\ \ x_3 \sim 1}$$

$$\mathcal{I}_s = \mathcal{C} \int_0^\infty dx_1 dx_2 dx_3 x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} x_3^{\nu_3 - 1} \cdot \left(x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2\right)^{-D/2},$$

$$+ \cdots$$

The Newton polytope approach

The Newton polytope is the convex hull of the exponents of the Lee-Pomeransky polynomial.

The regions are identified as the lower facets of the Newton polytope.

There have been computer codes based on this approach, such as Asy2, ASPIRE and pySecDec.

We aim to find an analytic way to determine the regions, by relating them to the Landau equations.

The Landau equations

ON THE ANALYTIC PROPERTIES OF VERTEX PARTS IN QUANTUM FIELD THEORY

L. D. LANDAU

Institute of Physical Problems, Academy of Sciences, U.S.S.R.

Submitted to JETP editor February 19, 1959; resubmitted April 7, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) 37, 62-70 (July, 1959)

A general method is developed, on the basis of the diagram technique, for finding the singularities of quantities involved in quantum field theory.

The Landau equations

In the Feynman parameterized integral

$$\mathcal{I}(\boldsymbol{s}) = \frac{\Gamma(\nu)}{\prod_{e\in G}\Gamma(\nu_e)} \int_0^\infty \left(\prod_{e\in G} d\alpha_e \alpha_e^{\nu_e - 1}\right) \delta\left(\sum_{e\in G}\alpha_e - 1\right) \int [d\boldsymbol{k}] \frac{1}{[\mathcal{D}(k,p,q;\alpha)]^\nu}.$$

with $$\mathcal{D}(k,p,q;\alpha) = \sum_{e\in G} \alpha_e \left(-l_e^2(k,p,q) + m_e^2 - i\epsilon\right),$$

The Landau equations

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with $$\mathcal{D}(k,p,q;\alpha) = \sum_{e\in G}\alpha_e\left(-l_e^2(k,p,q) + m_e^2 - i\epsilon\right),$$

The Landau equations read:

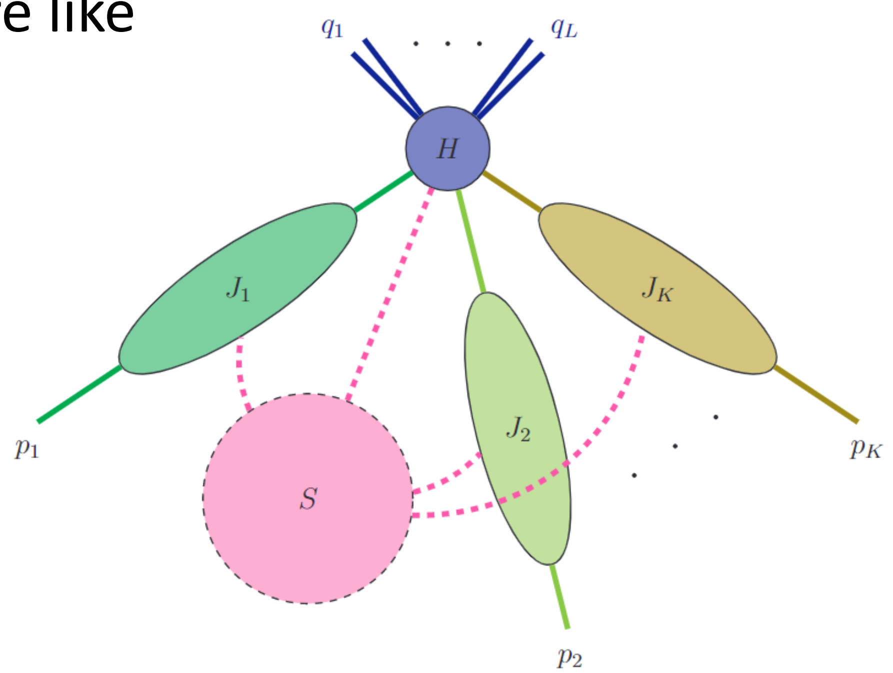
$$\alpha_e l_e^2(k,p,q) = 0 \qquad \forall e \in G$$

$$\frac{\partial}{\partial k_a}\mathcal{D}\left(k,p,q;\alpha\right) = 0 \qquad \forall a \in \{1,\ldots,L\}.$$

The Landau equations

There are other representations of the Landau equations.

The solution(s) of the Landau equations are called the "pinch surfaces", which are like



H: hard subgraph

J_1, J_2,..., J_K: jet subgraphs

S: soft subgraph

The Landau equations

Given an amplitude G

some external momenta on-shell $p_i^2 = 0 \quad (i = 1, \ldots, K)$

the submanifold of the momentum space

$$
\begin{aligned}
&\text{Hard: } k_H^\mu = (k_H^t, k_H^x, k_H^y, k_H^z) \sim Q(1,1,1,1),\\
&\text{Jet: } k_{J_i}^\mu = \left(k_{J_i}\cdot\overline{\beta}_i,\ k_{J_i}\cdot\beta_i,\ k_{J_i}\cdot\beta_{i\perp}\right) \sim Q\,(1,0,0)\,,\\
&\text{Soft: } k_S^\mu = (k_S^t, k_S^x, k_S^y, k_S^z) \sim Q(0,0,0,0).
\end{aligned}
$$

is a solution of the Landau equations

$$
\begin{aligned}
&\alpha_e l_e^2(k,p,q) = 0 \qquad \forall e \in G\\
&\frac{\partial}{\partial k_a}\mathcal{D}\,(k,p,q;\alpha) = 0 \qquad \forall a \in \{1,\ldots,L\}\,.
\end{aligned}
$$

The Landau equations

Observation: if we take
the on-shell expansion condition $p_i^2 \sim \lambda Q^2 \quad (i = 1, \ldots, K),$

the vicinity of the pinch surface in momentum space

$$\text{Hard:}\ \ k_H^\mu = (k_H^t, k_H^x, k_H^y, k_H^z) \sim Q(1,1,1,1),$$
$$\text{Jet:}\ \ k_{J_i}^\mu = \left(k_{J_i}\cdot\overline{\beta}_i,\ k_{J_i}\cdot\beta_i,\ k_{J_i}\cdot\beta_{i\perp}\right) \sim Q\left(1,\lambda,\lambda^{1/2}\right),$$
$$\text{Soft:}\ \ k_S^\mu = (k_S^t, k_S^x, k_S^y, k_S^z) \sim Q(\lambda,\lambda,\lambda,\lambda).$$

and modify the Landau equations to

$$\alpha_e l_e^2(k,p,q) \lesssim \lambda \qquad \forall e \in G$$
$$\frac{\partial}{\partial k_a}\mathcal{D}\left(k,p,q;\alpha\right) \lesssim \lambda \qquad \forall a \in \{1,\ldots,L\}\,.$$

The Landau equations

The scaling of the Lee-Pomeransky parameters reads

$$x^{[H]} \lesssim \lambda^0, \quad x^{[J]} \lesssim \lambda^{-1}, \quad x^{[S]} \lesssim \lambda^{-2}$$

Therefore, it is natural to propose the following region vector for G:

$$\boldsymbol{v}_R = (u_{R,1}, u_{R,2}, \ldots, u_{R,N}; 1), \qquad u_{R,e} \in \{0, -1, -2\},$$

$$u_{R,e} = 0 \quad \Leftrightarrow \quad e \in H$$

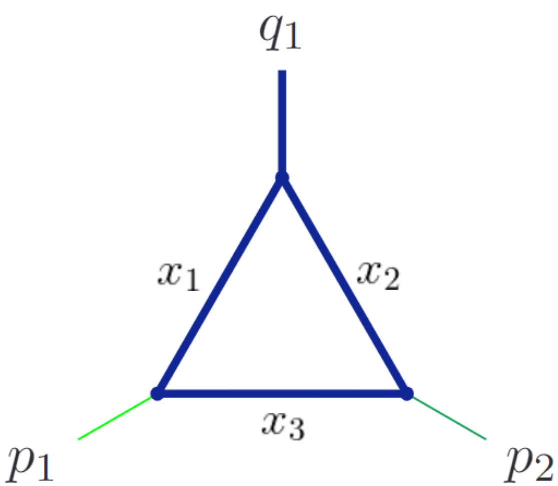
$$u_{R,e} = -1 \quad \Leftrightarrow \quad e \in J \equiv \cup_{i=1}^{K} J_i$$

$$u_{R,e} = -2 \quad \Leftrightarrow \quad e \in S$$

The Landau equations

Coming back to our example again:

$$\mathcal{P}(\boldsymbol{x}, \boldsymbol{s}) = x_1 + x_2 + x_3 - p_1^2 x_1 x_3 - p_2^2 x_2 x_3 - q_1^2 x_1 x_2$$



Four regions in total:

$$\begin{aligned} v_H &= (0, 0, 0; 1), \\ v_{C_1} &= (-1, 0, -1; 1), \\ v_{C_2} &= (0, -1, -1; 1), \\ v_S &= (-1, -1, -2; 1). \end{aligned}$$

compatible to the proposition.

The Landau equations

Question: is the proposition true in general?

At higher loops:

1. Does each solution of the Landau equations correspond to a particular region?

2. Does each region correspond to a particular solution of the Landau equations?

The criteria for the regions

The criteria for the regions

The all order result: a solution of the Landau equations corresponds to one of the regions, if and only if the following requirements on the subgraphs H, J, S are satisfied:

- *Requirement of H: the integral over any hard loop momentum is not scaleless after we set all the jet and soft momenta entering H to be exactly on-shell.*[5]

- *Requirement of J: the total momentum flowing into (and out of) each 1VI component of $\widetilde{J}_i$ must be equal to the jet momentum p_i^μ.*[6]

- *Requirement of S: every connected component of S must be attached to at least two different jet subgraphs J_i and J_j.*

The criteria for the regions

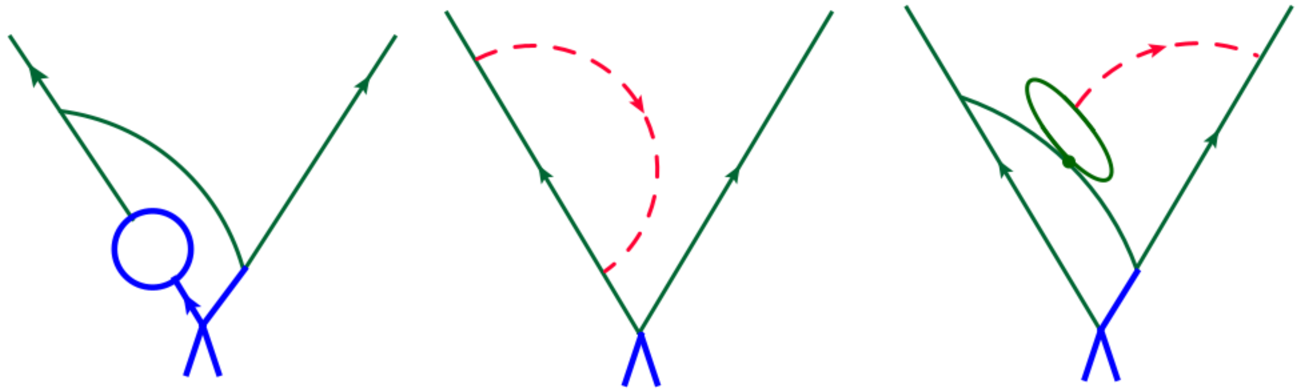
An equivalent graph-theoretical set of requirements:

1. *For any* $i = 1, \ldots, K$, *the subgraph* $H \cup J \setminus J_i$ *is mojetic.*
2. *Every connected component of* S *must be attached to at least two different jets* J_i *and* J_j.

Mojetic (invented from “motic”): a graph is called mojetic if it is one-vertex irreducible (1VI) after contracting all its external vertices to one vertex.

The criteria for the regions

The following examples are NOT regions



because the requirements on the subgraphs are not satisfied.

The criteria for the regions

In order that R is a region of G, several requirements on the subgraphs of G must be met.

These requirements rule out the scaleless integrals that possibly appear in the on-shell expansion.

These requirements can be translated into a graph-theoretical language.

From this graph-theoretical language, we can design an algorithm to find the regions of G directly.

An algorithm to obtain the regions

An algorithm to obtain the regions

- *Step 1*: For each $i = 1, \dots, K$, construct the one-external subgraph γ_i in the p_i channel, such that the subgraph $H_i \equiv G \setminus \gamma_i$ is mojetic.
- *Step 2*: Consider all possible sets $\{\gamma_1, \dots, \gamma_K\}$. For each such set focus on each edge of G. If it has been assigned to two or more γ_i's, it belongs to the soft subgraph S; if it has been assigned to exactly one γ_i, it belongs to the jet subgraph J_i; if it has not been assigned to any γ_i's, it belongs to H. We also denote $J \equiv \cup_{i=1}^{K} J_i$.
- *Step 3*: We now check the obtained result from three aspects: (i) each jet subgraph J_i is connected; (ii) each hard subgraph H is connected; (iii) each of the K subgraphs $H \cup J \setminus J_i$ $(i = 1, \dots, K)$ is mojetic. The region would be ruled out if any of these conditions are not satisfied.

An algorithm to obtain the regions

Roughly speaking,

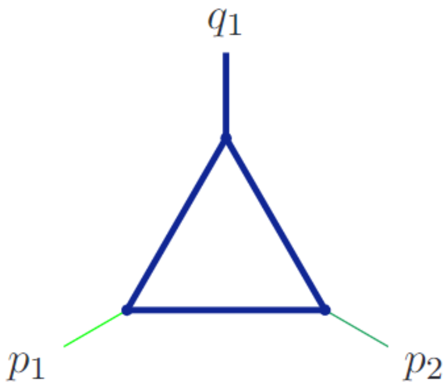
Step 1: For every on-shell momentum p_i, construct a jet J_i that includes p_i, such that all the propagators in $G \setminus J_i$ are hard;

Step 2: Examine each propagator of G; if it has been assigned to more than one jets in the step above, then it should be soft; if it is assigned to exactly one jet J_i, it should be part of J_i; otherwise (if assigned to no jet) it is hard;

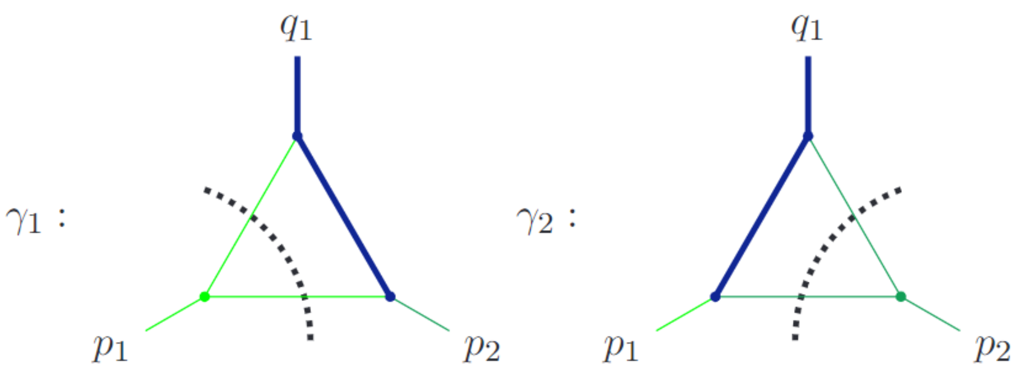
Step 3: Exclude some “bad results” according to some graph-theoretical rules.

An algorithm to obtain the regions

Examples:

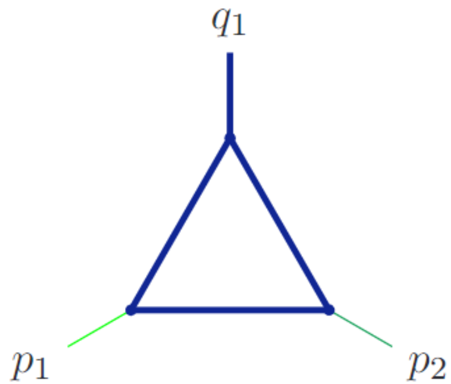


Step 1:



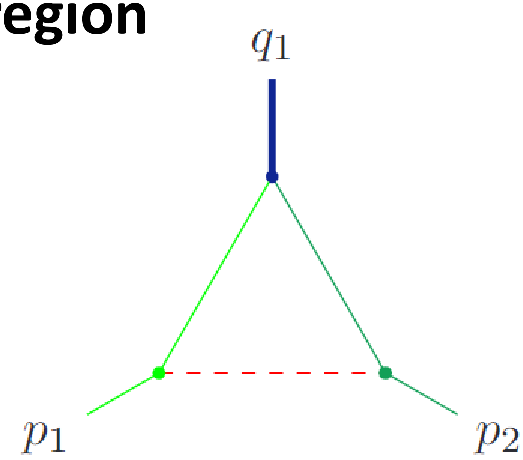
An algorithm to obtain the regions

Examples:



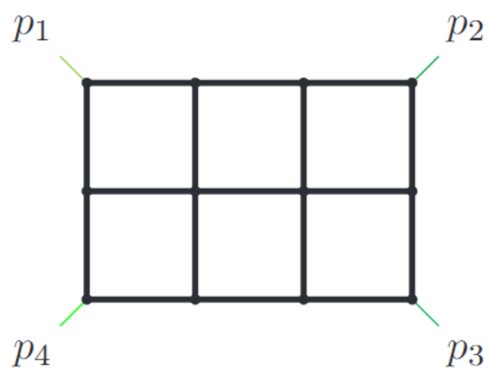
Step 2: we obtain a soft region

$\gamma_1 \sqcup \gamma_2$:



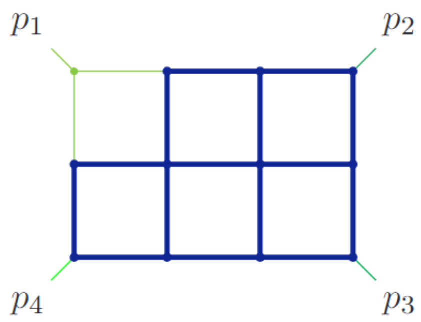
An algorithm to obtain the regions

$G_{3\times 2} =$



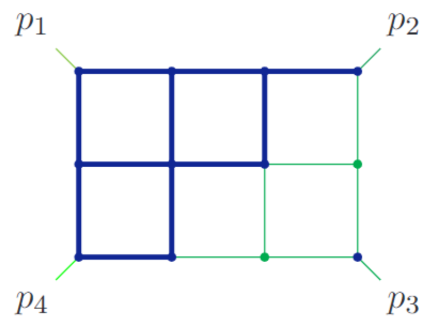
Step 1:

γ_1 :



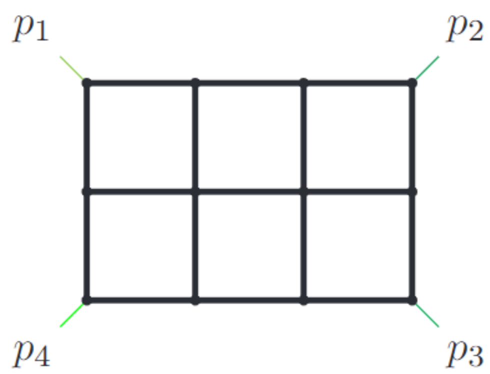
,

γ_3 :

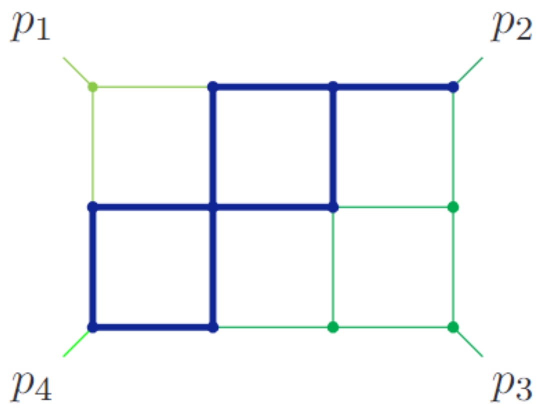


An algorithm to obtain the regions

$G_{3\times 2} =$

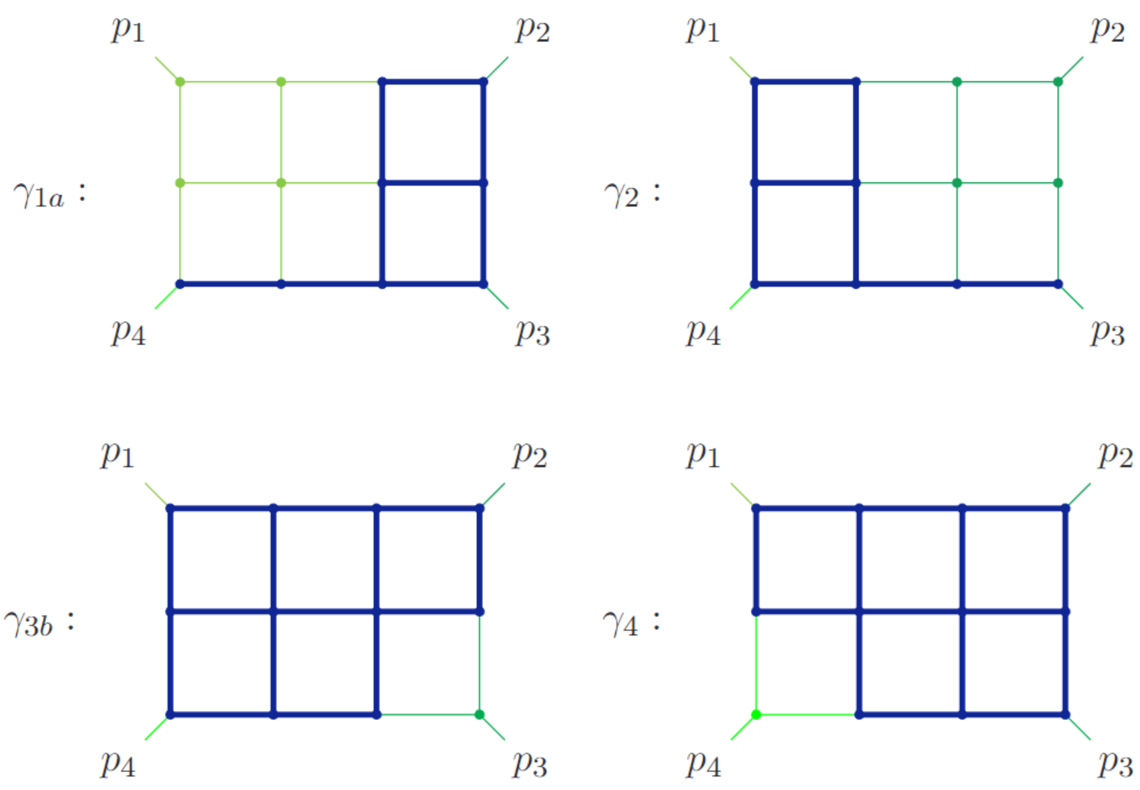


Step 2:



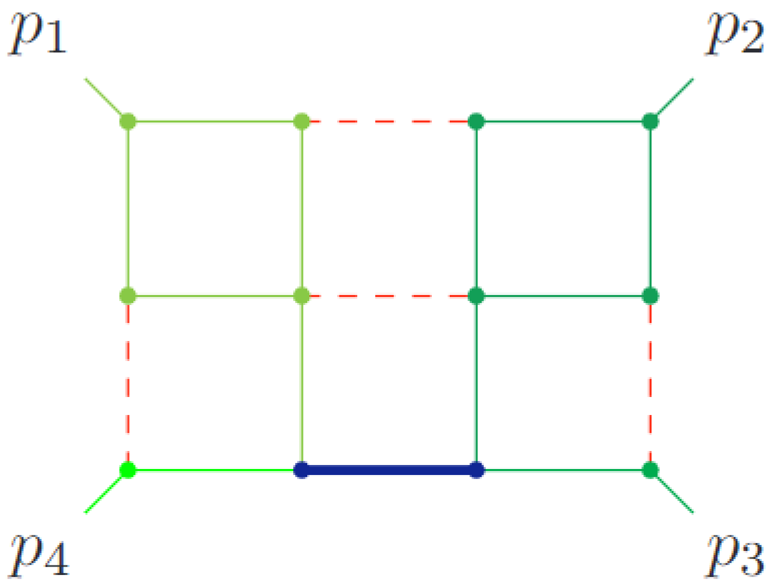
An algorithm to obtain the regions

Step 1:



An algorithm to obtain the regions

Step 2:



Conclusions and Outlook

Conclusions

In this talk, we have

1, introduced two approaches to the method of regions (momentum space & parameter space)

2, related the Landau equations to the region vectors;

3, found a set of criteria for the subgraphs of a region;

4, constructed and explained the algorithm to obtain the set of regions.

Conclusions

In this work, we also

5, studied the action of consecutive expansions, and derived a criterion for the commutativity of two expansions;

6, related the expansion by regions to the infrared forest formula.

Outlook

Here are some interesting topics:

1. For the on-shell expansion, does each region correspond to a particular solution of the Landau equations?

2. What will the conclusions be in some other expansions/processes?

3. There may be a geometric interpretation of the (infrared) forest formula.

4. Can the method of regions be justified with the help of this approach?

5. ….

Thank you!