

Local analytic sector subtraction at NLO and NNLO

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Introduction to NLO subtraction strategy

FSR
massless QCD

Generalities at NLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$\frac{d\sigma_{NLO} - d\sigma_{LO}}{dX} = \int d\Phi_n \textcolor{blue}{V} \delta_{X_n} + \int d\Phi_{n+1} \textcolor{blue}{R} \delta_{X_{n+1}}$$

Explicit ϵ poles

Phase space
singularities

- Subtraction algorithm: introduce **local counterterm** K and its integral I

More on subtraction
schemes → talk by
Kirill Melnikov

$$\int d\Phi_{n+1} K \delta_{X_n} = \int d\Phi_n I \delta_{X_n} \quad d\Phi_{n+1} = d\Phi_n d\Phi_{rad}$$

- Subtracted NLO cross section numerically integrable in $d = 4$

$$\frac{d\sigma_{NLO} - d\sigma_{LO}}{dX} = \int d\Phi_n \left(\textcolor{blue}{V} + \textcolor{yellow}{I} \right) \delta_{X_n} + \int d\Phi_{n+1} \left(\textcolor{blue}{R} \delta_{X_{n+1}} - \textcolor{yellow}{K} \delta_{X_n} \right)$$

finite in ϵ

finite in phase space

Strategy of the method

- ▶ Partition of radiative phase-space with **sector functions** \mathcal{W}_{ij} (as in FKS) [Frixione, Kunszt, Signer, 9512328]

$$\sum_{i,j \neq i} \mathcal{W}_{ij} = 1$$

$$R = \sum_{i,j \neq i} R\mathcal{W}_{ij}$$

Sum rules :

$$S_i \sum_{l \neq i} \mathcal{W}_{il} = 1$$

$$C_{ij}(\mathcal{W}_{ij} + \mathcal{W}_{ji}) = 1$$

Strategy of the method

- ▶ Partition of radiative phase-space with **sector functions** \mathcal{W}_{ij} (as in FKS) [Frixione, Kunszt, Signer, 9512328]
- ▶ Collect the relevant **IRC limits** for a given sector

$$R\mathcal{W}_{ij} - \hat{K}_{ij} = R\mathcal{W}_{ij} - [\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij}] R\mathcal{W}_{ij} = \text{finite}$$

(Soft + Collinear - Overlap)

Strategy of the method

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(Soft + Collinear - Overlap)

For soft gluon i :
 $(s_{ab} = 2k_a \cdot k_b)$

$$\mathbf{S}_i R \propto \sum_{k,l} \frac{s_{kl}}{s_{ik}s_{il}} B_{kl}(\{k\}_j)$$

Not yet
parametrised

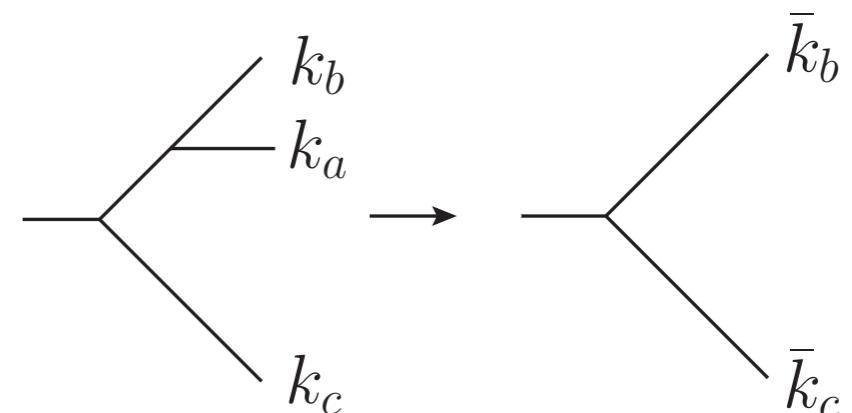
Strategy of the method

- ▶ Partition of radiative phase-space with **sector functions** \mathcal{W}_{ij} (as in FKS) [Frixione, Kunszt, Signer, 9512328]
- ▶ Collect the relevant **IRC limits** for a given sector
- ▶ **CS** [Catani, Seymour, 9605323] **dipole mapping**

* $\{k_1, \dots, k_{n+1}\} \rightarrow \{\bar{k}_1, \dots, \bar{k}_n\}^{(abc)}$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{y}{1-y} k_c$$

$$\bar{k}_c^{(abc)} = \frac{1}{1-y} k_c$$



$$y = \frac{s_{ab}}{s_{ab} + s_{ac} + s_{bc}}, \quad z = \frac{s_{ac}}{s_{ab} + s_{bc}}$$

* Phase space factorisation and parametrisation

$$d\Phi_{n+1} = d\Phi_n^{(abc)} \times d\Phi_{rad}^{(abc)} = d\Phi_n(\{\bar{k}\}^{(abc)}) \times d\Phi_{rad}(\bar{s}_{bc}^{(abc)}; y, z, \phi)$$

$$\int d\Phi_{rad}^{(abc)} \propto (\bar{s}_{bc}^{(abc)})^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon} \phi \int_0^1 dy \int_0^1 dz \left[y(1-y^2)z(1-z) \right]^{-\epsilon} (1-y)$$

Strategy of the method

- ▶ Partition of radiative phase-space with **sector functions** \mathcal{W}_{ij} (as in FKS) [Frixione, Kunszt, Signer, 9512328]
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- ▶ Promotion to **counterterm**: adapt mapping to each kernel

$$\begin{aligned} K &= \sum_{i,j \neq i} \left[\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij} - \bar{\mathbf{S}}_i \bar{\mathbf{C}}_{ij} \right] R \mathcal{W}_{ij} \\ &= \sum_i \left[\bar{\mathbf{S}}_i + \sum_{j>i} \bar{\mathbf{C}}_{ij} (1 - \bar{\mathbf{S}}_i - \bar{\mathbf{S}}_j) \right] R \end{aligned}$$

For soft gluon i :
 $(s_{ab} = 2k_a \cdot k_b)$

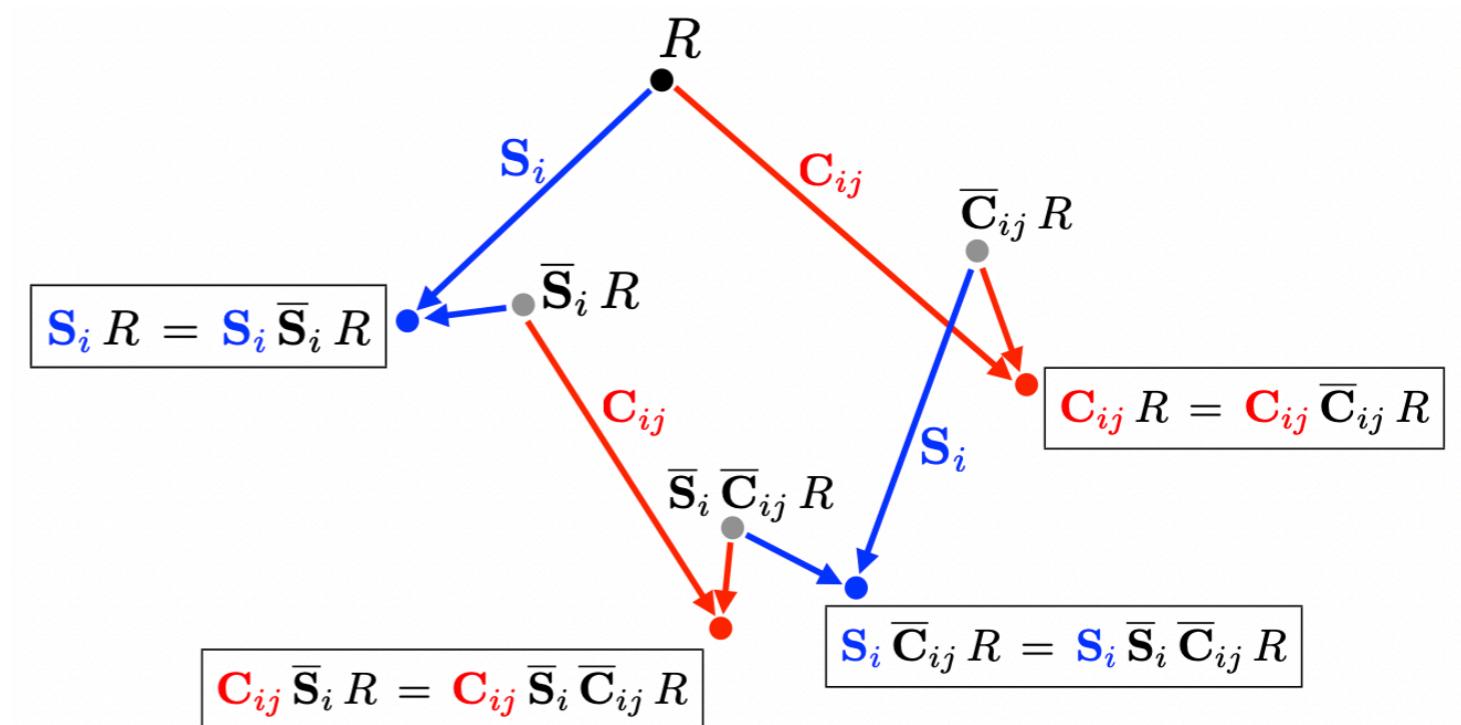
$$\bar{\mathbf{S}}_i R \propto \sum_{k,l} \frac{s_{kl}}{s_{ik}s_{il}} \bar{B}_{kl}^{(ikl)}$$

Strategy of the method

- ▶ Partition of radiative phase-space with **sector functions** \mathcal{W}_{ij} (as in FKS) [Frixione, Kunszt, Signer, 9512328]
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- ▶ Promotion to **counterterm**: adapt mapping to each kernel
- ▶ **Locality** of the cancellation ensured by *consistency relations*

$$\mathbf{S}_i R = \mathbf{S}_i (\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij} - \bar{\mathbf{S}}_i \bar{\mathbf{C}}_{ij}) R$$

$$\mathbf{C}_{ij} R = \mathbf{C}_{ij} (\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij} - \bar{\mathbf{S}}_i \bar{\mathbf{C}}_{ij}) R$$



Strategy of the method

- ▶ Partition of radiative phase-space with **sector functions** \mathcal{W}_{ij} (as in FKS) [Frixione, Kunszt, Signer, 9512328]
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- ▶ **CS** [Catani, Seymour, 9605323] **dipole mapping**
- ▶ Promotion to **counterterm**: adapt mapping to each kernel
- ▶ **Locality** of the cancellation ensured by *consistency relations*
- ▶ \mathcal{W}_{ij} sum rules + mapping adaptation = **simple analytic** counterterm integration

For soft gluon i :
 $(s_{ab} = 2k_a \cdot k_b)$

$$\begin{aligned} I^S &\propto \sum_{k,l} \bar{B}_{kl}^{(ikl)} \frac{1}{\bar{s}_{kl}^{(ikl)}} \int d\Phi_{rad}(\bar{s}_{kl}^{(ikl)}; y, z, \phi) \frac{1-z}{yz} \\ &= \sum_{k,l} \bar{B}_{kl}^{(ikl)} \frac{(4\pi)^{\epsilon-2}}{(\bar{s}_{kl}^{(ikl)})^\epsilon} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)} \end{aligned}$$

Local analytic sector subtraction at NNLO

FSR
massless QCD

Generalities at NNLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$\frac{d\sigma_{NNLO} - d\sigma_{NLO}}{dX} = \int d\Phi_n \textcolor{blue}{VV} \delta_{X_n}$$

Explicit poles up to $1/\epsilon^4$

$$+ \int d\Phi_{n+1} \textcolor{blue}{RV} \delta_{X_{n+1}}$$

Explicit poles up to $1/\epsilon^2$

Phase space singularities

$$+ \int d\Phi_{n+2} \textcolor{blue}{RR} \delta_{X_{n+2}}$$

Phase space singularities

Generalities at NNLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$\begin{aligned}
 \frac{d\sigma_{NNLO} - d\sigma_{NLO}}{dX} &= \int d\Phi_n \left(\cancel{VV} \right) \delta_{X_n} \\
 &\quad + \int d\Phi_{n+1} \left[\left(\cancel{RV} \right) \delta_{X_{n+1}} - \left(\cancel{K^{(RV)}} \right) \delta_{X_n} \right] \\
 &\qquad\qquad\qquad \underbrace{\phantom{\left[\left(\cancel{RV} \right) \delta_{X_{n+1}} - \left(\cancel{K^{(RV)}} \right) \delta_{X_n} \right]}}_{\text{finite in PS}} \\
 &\quad + \int d\Phi_{n+2} \left[\cancel{RR} \delta_{X_{n+2}} - \cancel{K^{(1)}} \delta_{X_{n+1}} - \left(\cancel{K^{(2)}} + \cancel{K^{(12)}} \right) \delta_{X_n} \right] \\
 &\qquad\qquad\qquad \underbrace{\phantom{\cancel{RR} \delta_{X_{n+2}} - \cancel{K^{(1)}} \delta_{X_{n+1}}} \delta_{X_n}}_{\text{finite in PS}}
 \end{aligned}$$

- Introduce **local counterterms** and its integrals

$$\begin{array}{ll}
 \blacksquare \int d\Phi_{n+2} K^{(1)} \delta_{X_{n+1}} = \int d\Phi_{n+1} I^{(1)} \delta_{X_{n+1}} & \blacksquare \int d\Phi_{n+2} K^{(12)} \delta_{X_n} = \int d\Phi_{n+1} I^{(12)} \delta_{X_n} \\
 \blacksquare \int d\Phi_{n+2} K^{(2)} \delta_{X_n} = \int d\Phi_n I^{(2)} \delta_{X_n} & \blacksquare \int d\Phi_{n+1} K^{(RV)} \delta_{X_n} = \int d\Phi_n I^{(RV)} \delta_{X_n}
 \end{array}$$

Generalities at NNLO

- X_i = **IRC-safe** observable computed with i-body kinematics, $\delta_{X_i} \equiv \delta(X - X_i)$

$$\begin{aligned}
 \frac{d\sigma_{NNLO} - d\sigma_{NLO}}{dX} &= \int d\Phi_n \underbrace{\left(\cancel{VV} + \cancel{I^{(2)}} + \cancel{I^{(RV)}} \right)}_{\text{finite in } d=4} \delta_{X_n} \\
 &\quad + \int d\Phi_{n+1} \underbrace{\left[\underbrace{\left(\cancel{RV} + \cancel{I^{(1)}} \right) \delta_{X_{n+1}}}_{\text{finite in } d=4, \text{ div. in PS}} - \underbrace{\left(\cancel{K^{(RV)}} - \cancel{I^{(12)}} \right) \delta_{X_n}}_{\text{finite in } d=4, \text{ div. in PS}} \right]}_{\text{finite in PS}} \\
 &\quad + \int d\Phi_{n+2} \underbrace{\left[\cancel{RR} \delta_{X_{n+2}} - \cancel{K^{(1)}} \delta_{X_{n+1}} - \left(\cancel{K^{(2)}} + \cancel{K^{(12)}} \right) \delta_{X_n} \right]}_{\text{finite in PS}}
 \end{aligned}$$

- Introduce **local counterterms** and its integrals

$$\begin{array}{ll}
 \blacksquare \int d\Phi_{n+2} K^{(1)} \delta_{X_{n+1}} = \int d\Phi_{n+1} I^{(1)} \delta_{X_{n+1}} & \blacksquare \int d\Phi_{n+2} K^{(12)} \delta_{X_n} = \int d\Phi_{n+1} I^{(12)} \delta_{X_n} \\
 \blacksquare \int d\Phi_{n+2} K^{(2)} \delta_{X_n} = \int d\Phi_n I^{(2)} \delta_{X_n} & \blacksquare \int d\Phi_{n+1} K^{(RV)} \delta_{X_n} = \int d\Phi_n I^{(RV)} \delta_{X_n}
 \end{array}$$

Counterterms for RR

- ▶ Partition of double-unresolved Φ_{n+2} with **sector functions** \mathcal{W}_{ijkl}

$$\sum_{i,j \neq i} \sum_{\substack{k \neq i \\ l \neq i,k}} \mathcal{W}_{ijkl} = 1$$

$$RR = \sum_{i,j \neq i} \sum_{\substack{k \neq i \\ l \neq i,k}} RR \mathcal{W}_{ijkl}$$

Sum rules :

$$\mathbf{S}_{ik} \left(\sum_{b \neq i} \sum_{d \neq i,k} \mathcal{W}_{ibkd} + \sum_{b \neq k} \sum_{d \neq k,i} \mathcal{W}_{kbid} \right) = 1$$

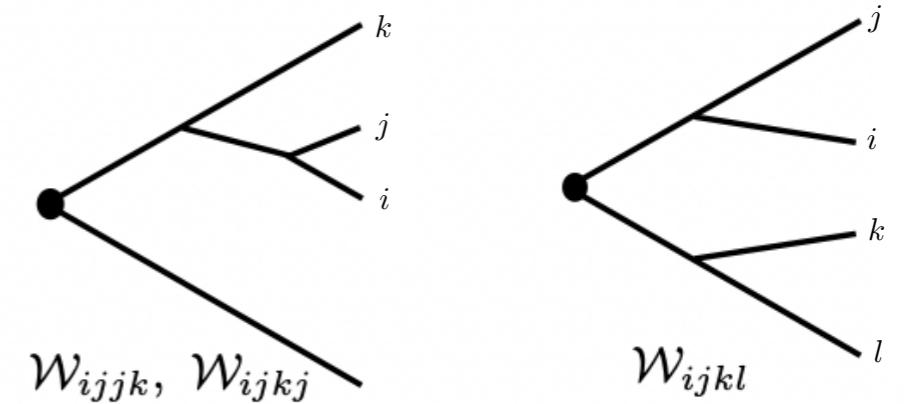
$$\mathbf{C}_{ijk} \sum_{abc \in \pi(ijk)} (\mathcal{W}_{abbc} + \mathcal{W}_{abcb}) = 1$$

...

Counterterms for RR

- ▶ Partition of double-unresolved Φ_{n+2} with **sector functions** \mathcal{W}_{ijkl}
- ▶ Collect the relevant **IRC limits** for each *topology*

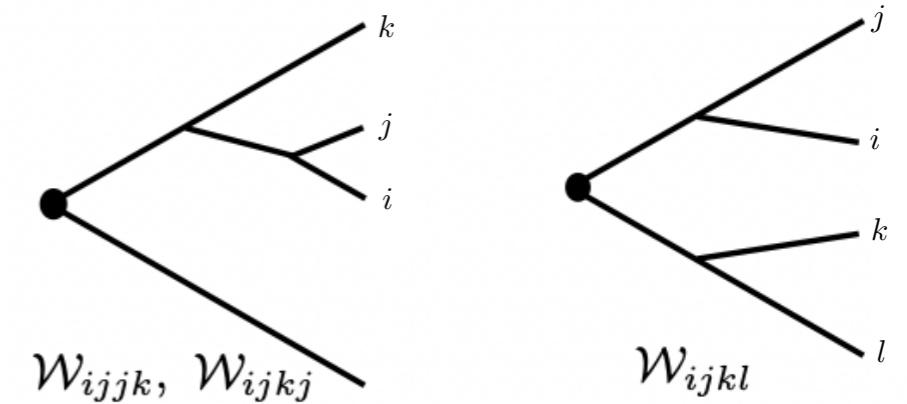
\mathcal{W}_{ijjk}	:	S_i	C_{ij}	S_{ij}	C_{ijk}	SC_{ijk}
\mathcal{W}_{ijkj}	:	S_i	C_{ij}	S_{ik}	C_{ijk}	SC_{ijk}
\mathcal{W}_{ijkl}	:	S_i	C_{ij}	S_{ik}	C_{ijkl}	SC_{ikl}



Counterterms for RR

- ▶ Partition of double-unresolved Φ_{n+2} with **sector functions** \mathcal{W}_{ijkl}
- ▶ Collect the relevant **IRC limits** for each *topology*

\mathcal{W}_{ijjk}	:	S_i	C_{ij}	
\mathcal{W}_{ijkj}	:	S_i	C_{ij}	SC_{ijk}
\mathcal{W}_{ijkl}	:	S_i	C_{ij}	SC_{ijk} SC_{kij}



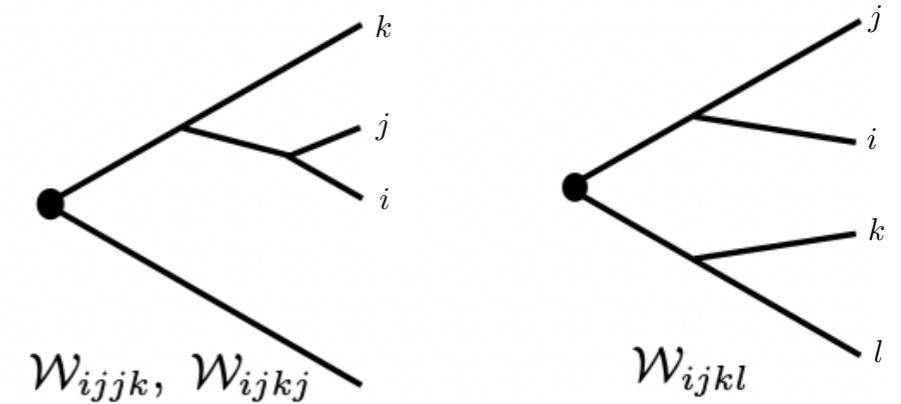
single-unresolved limits

double-unresolved limits

Counterterms for RR

- ▶ Partition of double-unresolved Φ_{n+2} with **sector functions** \mathcal{W}_{ijkl}
- ▶ Collect the relevant **IRC limits** for each *topology*

\mathcal{W}_{ijjk}	:	$S_i \quad C_{ij}$	$S_{ij} \quad C_{ijk} \quad SC_{ijk}$
\mathcal{W}_{ijkj}	:	$S_i \quad C_{ij}$	$S_{ik} \quad C_{ijk} \quad SC_{ijk} \quad SC_{kij}$
\mathcal{W}_{ijkl}	:	$S_i \quad C_{ij}$	$S_{ik} \quad C_{ijkl} \quad SC_{ikl} \quad SC_{kij}$



single-unresolved limits

$$\mathbf{L}_{ij}^{(1)} = S_i + C_{ij}(1 - S_i)$$

double-unresolved limits

$$\mathbf{L}_{ijjk}^{(2)} = S_{ij} + C_{ijk}(1 - S_{ij}) + SC_{ijk}(1 - S_{ij})(1 - C_{ijk})$$

$$\mathbf{L}_{ijkj}^{(2)} = S_{ik} + C_{ijk}(1 - S_{ik}) + (SC_{ijk} + SC_{kij})(1 - S_{ik})(1 - C_{ijk})$$

$$\mathbf{L}_{ijkl}^{(2)} = S_{ik} + C_{ijkl}(1 - S_{ik}) + (SC_{ikl} + SC_{kij})(1 - S_{ik})(1 - C_{ijkl})$$

$$RR\mathcal{W}_\tau - \left[\mathbf{L}_{ij}^{(1)} + \mathbf{L}_\tau^{(2)} - \mathbf{L}_{ij}^{(1)} \mathbf{L}_\tau^{(2)} \right] RR\mathcal{W}_\tau = \text{finite}$$

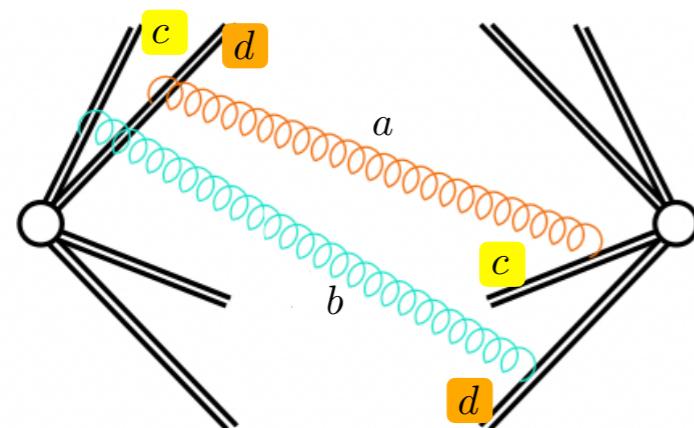
($\tau = ijjk, ijkj, ijkl$)

Counterterms for RR

- ▶ Partition of double-unresolved Φ_{n+2} with **sector functions** \mathcal{W}_{ijkl}
- ▶ Collect the relevant **IRC limits** for each *topology*
- ▶ **Nested** Catani-Seymour mappings
 - * mapping from $(n + 2) \rightarrow n$ kinematics
 - * simple phase-space factorisation
 - * parametrisation simplifies kernels expressions

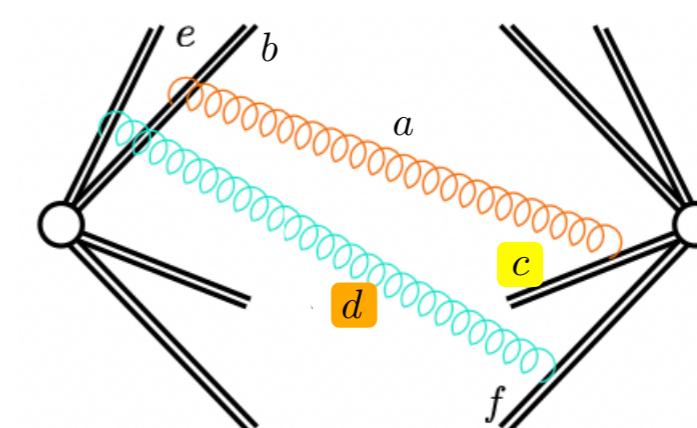
$$\{k\} \rightarrow \{\bar{k}\}^{(abcd)}$$

$$d\Phi_{n+2} = d\Phi_n^{(abcd)} \times d\Phi_{\text{rad},2}^{(abcd)}$$



$$\{k\} \rightarrow \{\bar{k}\}^{(abc,def)}$$

$$d\Phi_{n+2} = d\Phi_n^{(abcd)} \times d\Phi_{\text{rad}}^{(abc)} d\Phi_{\text{rad}}^{(def)}$$



Counterterms for RR

- ▶ Partition of double-unresolved Φ_{n+2} with **sector functions** \mathcal{W}_{ijkl}
- ▶ Collect the relevant **IRC limits** for each *topology*
- ▶ **Nested** Catani-Seymour mappings
- ▶ Promotion to **counterterms**: adapt mapping to each kernel

$$\blacksquare \quad K^{(1)} = \sum_{i,j \neq i} \sum_{\substack{k \neq i \\ l \neq i,k}} \bar{\mathbf{L}}_{ij}^{(1)} RR \mathcal{W}_{ijkl}$$

single-unresolved limits

$$\blacksquare \quad K^{(2)} = \sum_{i,j \neq i} \sum_{\substack{k \neq i \\ l \neq i,k}} \bar{\mathbf{L}}_{ijkl}^{(2)} RR \mathcal{W}_{ijkl}$$

uniform
double-unresolved limits

$$\blacksquare \quad K^{(12)} = - \sum_{i,j \neq i} \sum_{\substack{k \neq i \\ l \neq i,k}} \bar{\mathbf{L}}_{ij}^{(1)} \bar{\mathbf{L}}_{ijkl}^{(2)} RR \mathcal{W}_{ijkl}$$

strongly-ordered
double-unresolved limits

$$RR \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - (K^{(2)} + K^{(12)}) \delta_{X_n} = \text{finite}$$

Counterterms for RR

- ▶ Partition of double-unresolved Φ_{n+2} with **sector functions** \mathcal{W}_{ijkl}
- ▶ Collect the relevant **IRC limits** for each *topology*
- ▶ **Nested** Catani-Seymour mappings
- ▶ Promotion to **counterterms**: adapt mapping to each kernel



$$K^{(1)} = \sum_{i,j \neq i} \sum_{k \neq i} \bar{\mathbf{L}}_{ij}^{(1)} RR \mathcal{W}_{ijkl}$$

single-unresolved limits



$$K^{(2)} = \sum_{i,j \neq i} \sum_{\substack{k \neq i \\ l \neq i,k}} \bar{\mathbf{L}}_{ijkl}^{(2)} RR \mathcal{W}_{ijkl}$$

uniform
double-unresolved limits

$$\begin{aligned}
 &= \left\{ \sum_{i,k>i} \bar{\mathbf{S}}_{ik} + \sum_{i,j>i} \sum_{k>j} \bar{\mathbf{C}}_{ijk} \left(1 - \bar{\mathbf{S}}_{ij} - \bar{\mathbf{S}}_{ik} - \bar{\mathbf{S}}_{jk} \right) \right. \\
 &\quad + \sum_{i,j>i} \sum_{\substack{k \neq j \\ k>i}} \sum_{l>k} \bar{\mathbf{C}}_{ijkl} \left[1 - \bar{\mathbf{S}}_{ik} - \bar{\mathbf{S}}_{il} - \bar{\mathbf{S}}_{jk} - \bar{\mathbf{S}}_{jl} \right. \\
 &\quad \quad \quad \left. - \bar{\mathbf{SC}}_{ikl} (1 - \bar{\mathbf{S}}_{ik} - \bar{\mathbf{S}}_{il}) - \bar{\mathbf{SC}}_{jkl} (1 - \bar{\mathbf{S}}_{jk} - \bar{\mathbf{S}}_{jl}) \right. \\
 &\quad \quad \quad \left. - \bar{\mathbf{SC}}_{kij} (1 - \bar{\mathbf{S}}_{ik} - \bar{\mathbf{S}}_{jk}) - \bar{\mathbf{SC}}_{lij} (1 - \bar{\mathbf{S}}_{il} - \bar{\mathbf{S}}_{jl}) \right] \\
 &\quad \left. + \sum_{i,j>i} \sum_{\substack{k \neq i \\ k>j}} \bar{\mathbf{SC}}_{ijk} (1 - \bar{\mathbf{S}}_{ij} - \bar{\mathbf{S}}_{ik}) (1 - \bar{\mathbf{C}}_{ijk}) \right\} RR
 \end{aligned}$$

Collection of
universal kernels!

Counterterms for RR

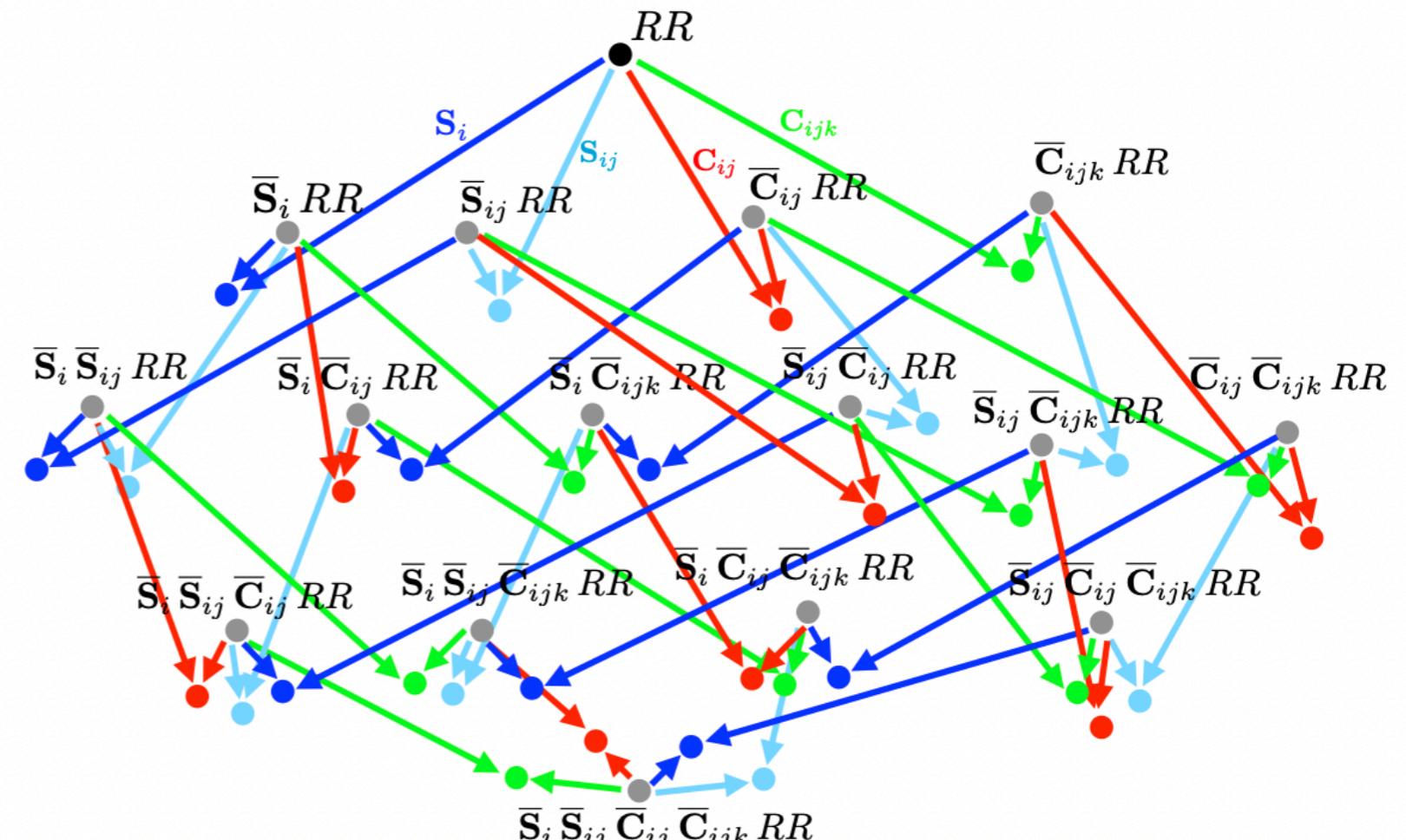
- ▶ Partition of double-unresolved Φ_{n+2} with **sector functions** \mathcal{W}_{ijkl}
- ▶ Collect the relevant **IRC limits** for each *topology*
- ▶ **Nested** Catani-Seymour mappings
- ▶ Promotion to **counterterms**: adapt mapping to each kernel

$$K^{(1)}, K^{(2)}, K^{(12)}$$

- ▶ **Locality** of the cancellation ensured by *consistency relations*

*verified
sector by sector*

$$\bar{S}_i, \bar{C}_{ij}, \bar{S}_{ij}, \bar{C}_{ijk}$$



SC limits not displayed.

Counterterms for RR

► \mathcal{W}_{ijkl} sum rules + mapping adaptation = **feasible analytic** counterterms integration

NNLO
complexity

$$\blacksquare \quad I^{(1)} = \int d\Phi_{\text{rad}} K^{(1)}$$

$$\blacksquare \quad I^{(12)} = \int d\Phi_{\text{rad}} K^{(12)}$$

$$\blacksquare \quad I^{(2)} = \int d\Phi_{\text{rad},2} K^{(2)}$$

$$K^{(2)} \supset \bar{\mathbf{S}}_{ij} RR, \bar{\mathbf{C}}_{ijk} RR$$

[Catani, Grazzini, 9903516, 9810389]

Counterterms for RR

- \mathcal{W}_{ijkl} sum rules + mapping adaptation = **feasible analytic** counterterms integration

NNLO
complexity

$$\begin{aligned} \blacksquare \quad I^{(1)} &= \int d\Phi_{\text{rad}} K^{(1)} & \blacksquare \quad I^{(12)} &= \int d\Phi_{\text{rad}} K^{(12)} & \blacksquare \quad I^{(2)} &= \int d\Phi_{\text{rad},2} K^{(2)} \\ &&&& K^{(2)} &\supset \bar{\mathbf{S}}_{ij}RR, \bar{\mathbf{C}}_{ijk}RR \end{aligned}$$

[Catani, Grazzini, 9903516, 9810389]

Counterterm for RV

- Apply NLO strategy to **define** and **analytically** integrate in single-unresolved phase space

$$\blacksquare \quad K^{(\text{RV})} \supset \sum_{i,j \neq i} \left[\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij}(1 - \bar{\mathbf{S}}_i) \right] RV \mathcal{W}_{ij} \quad \blacksquare \quad I^{(\text{RV})} = \int d\Phi_{\text{rad}} K^{(\text{RV})}$$

- **Analytically** checked

- * $RV + I^{(1)} \rightarrow$ free of ϵ poles
- * $K^{(\text{RV})} - I^{(12)} \rightarrow$ free of ϵ poles
- * $I^{(1)} + I^{(12)} \rightarrow$ finite in phase space
- * $RV - K^{(\text{RV})} \rightarrow$ finite in phase space

NNLO subtraction formula

massless FSR

$$\frac{d\sigma_{NNLO} - d\sigma_{NLO}}{dX} = \int d\Phi_n \left(\textcolor{blue}{VV} + I^{(2)} + I^{(\text{RV})} \right) \delta_{X_n} \quad \checkmark$$

$$+ \int d\Phi_{n+1} \left[\left(\textcolor{blue}{RV} + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\text{RV})} - I^{(12)} \right) \delta_{X_n} \right] \quad \checkmark$$

$$+ \int d\Phi_{n+2} \left[\textcolor{blue}{RR} \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} + K^{(12)} \right) \delta_{X_n} \right] \quad \checkmark$$

Analytically verified for
an arbitrary number of final-state partons

$$VV + I^{(2)} + I^{(\text{RV})} \rightarrow \text{free of } \epsilon \text{ poles}$$

NNLO subtraction formula

massless FSR

$$\begin{aligned}
 \frac{d\sigma_{NNLO} - d\sigma_{NLO}}{dX} &= \int d\Phi_n \left(\boxed{VV + I^{(2)} + I^{(\text{RV})}} \right) \delta_{X_n} \quad \checkmark \\
 &+ \int d\Phi_{n+1} \left[\left(RV + I^{(1)} \right) \delta_{X_{n+1}} - \left(K^{(\text{RV})} - I^{(12)} \right) \delta_{X_n} \right] \quad \checkmark \\
 &+ \int d\Phi_{n+2} \left[RR \delta_{X_{n+2}} - K^{(1)} \delta_{X_{n+1}} - \left(K^{(2)} + K^{(12)} \right) \delta_{X_n} \right] \quad \checkmark
 \end{aligned}$$

Analytic
and compact!

$$\begin{aligned}
 VV + I^{(2)} + I^{(\text{RV})} = & \left(\frac{\alpha_s}{2\pi} \right)^2 \left\{ \left[I^{(0)} + \sum_j I_j^{(1)} \mathbf{L}_{jr} + \sum_j I_j^{(2)} \mathbf{L}_{jr}^2 + \frac{1}{2} \sum_{j,l \neq j} \gamma_j^{\text{hc}} \gamma_l^{\text{hc}} \mathbf{L}_{jr} \mathbf{L}_{lr} \right] \mathbf{B} \right. \\
 &+ \sum_j \left[I_{jr}^{(0)} + I_{jr}^{(1)} \mathbf{L}_{jr} \right] \mathbf{B}_{jr} - 2(1-\zeta_2) \sum_{j,c \neq j,r} \gamma_j^{\text{hc}} (2 - \mathbf{L}_{cr}) \mathbf{B}_{cr} \\
 &+ \sum_{c,d \neq c} \mathbf{L}_{cd} \left[I_{cd}^{(0)} + I_{cd}^{(1)} \mathbf{L}_{cd} + \frac{\beta_0}{12} \mathbf{L}_{cd}^2 + (4 - \mathbf{L}_{cd}) \sum_j \gamma_j^{\text{hc}} \mathbf{L}_{jr} \right] \mathbf{B}_{cd} \\
 &+ \sum_{c,d \neq c} \left[-2 + \zeta_2 + 2\zeta_3 - \frac{5}{4}\zeta_4 + 2(1-\zeta_3) \mathbf{L}_{cd} \right] \mathbf{B}_{cdcd} \\
 &+ (1-\zeta_2) \sum_{\substack{c,d \neq c \\ e \neq d}} \mathbf{L}_{cd} \mathbf{L}_{ed} \mathbf{B}_{cded} + \sum_{\substack{c,d \neq c \\ e,f \neq e}} \mathbf{L}_{cd} \mathbf{L}_{ef} \left[1 - \frac{1}{2} \mathbf{L}_{cd} \left(1 - \frac{1}{8} \mathbf{L}_{ef} \right) \right] \mathbf{B}_{cdef} \\
 &+ \pi \sum_{\substack{c,d \neq c \\ e \neq c,d}} \left[\ln \frac{s_{ce}}{s_{de}} \mathbf{L}_{cd}^2 + \frac{1}{3} \ln^3 \frac{s_{ce}}{s_{de}} + 2 \text{Li}_3 \left(-\frac{s_{ce}}{s_{de}} \right) \right] \mathbf{B}_{cde} \Big\} \\
 &+ \left(\frac{\alpha_s}{2\pi} \right) \left\{ \left[\Sigma_\phi - \sum_j \gamma_j^{\text{hc}} \mathbf{L}_{jr} \right] \mathbf{V}^{\text{fin}} + \sum_{c,d \neq c} \mathbf{L}_{cd} \left(2 - \frac{1}{2} \mathbf{L}_{cd} \right) \mathbf{V}_{cd}^{\text{fin}} \right\} + \mathbf{VV}^{\text{fin}}
 \end{aligned}$$

Latest developments:
NLO extension to ISR

NLO extension to ISR

- ▶ Sector functions extended to ISR satisfy the same FSR sum rules: **key for integration**
- ▶ Catani-Seymour [Catani, Seymour, 9605323] **initial-state** dipole mappings

NLO extension to ISR

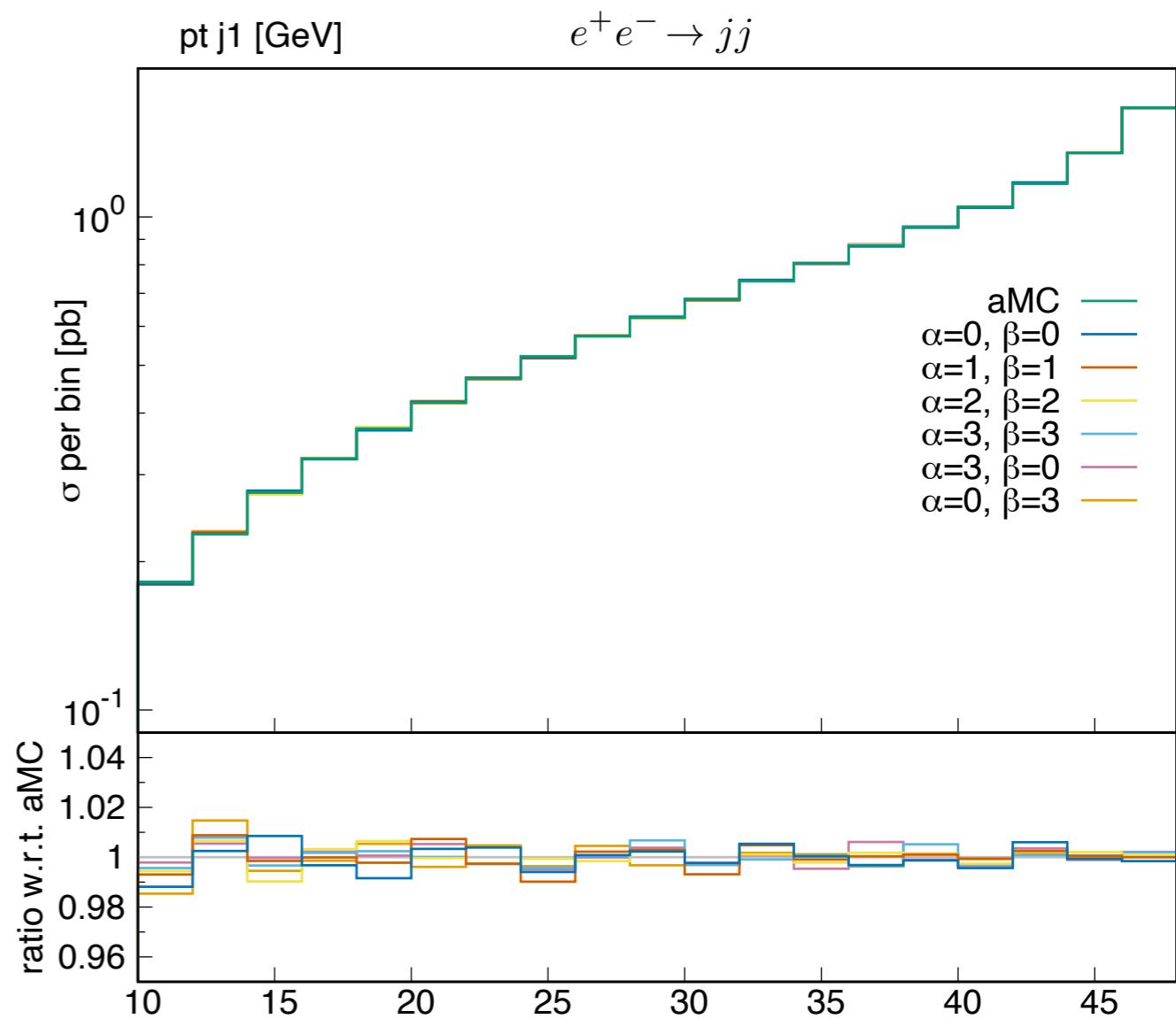
- ▶ Sector functions extended to ISR satisfy the same FSR sum rules: **key for integration**
- ▶ Catani-Seymour [Catani, Seymour, 9605323] **initial-state** dipole mappings
- ▶ Systematic optimisation with **damping factors**: multiplicative powers of kinematic invariants smoothly turning off the local counterterms away from the singular regions

* For final-state splitting:

$$(s_{ab} = 2k_a \cdot k_b)$$

$$\bar{S}_i \bar{C}_{ij} R \propto \delta_{f_i g} \frac{s_{jr}}{s_{ij} s_{ir}} (1-z)^\alpha (1-y)^\beta \bar{B}^{(ijr)}$$

* Final results independent of the damping exponents



NLO extension to ISR

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- ▶ **Trivial analytic** integration reproducing all NLO virtual + collinear factorisation poles

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- ▶ **Trivial analytic** integration reproducing all NLO virtual + collinear factorisation poles
- ▶ Numerical implementation in **MadNkLO** [Hirschi, Deutschmann, Lionetti, et al.] :
automated MG5-inspired python framework
 - * Cancellation of IRC singularities and ϵ poles checked up to $pp \rightarrow 3j$
 - * Validation on physical cross sections for both leptonic and hadronic collisions

Process	aMC LO	MADNkLO LO	aMC NLO corr.	MADNkLO NLO corr.	[pb]
$e^+e^- \rightarrow jj$	0.53209(6)	0.53208(6)	0.019991(7)	0.019991(10)	
$e^+e^- \rightarrow jjj$	0.4739(3)	0.4740(3)	-0.1461(1)	-0.1463(6)	
$pp \rightarrow Z$	46361(3)	46362(3)	6810.9(8)	6810.8(4)	
$pp \rightarrow Zj$	11270(7)	11258(5)	3770(6)	3776(17)	
$pp \rightarrow W^+W^-j$	42.42(1)	42.39(2)	10.68(5)	10.53(13)	

Status

- General analytic subtraction formula for massless FSR and ISR at NLO
- Numerical implementation and validation of NLO subtraction formula
- General analytic subtraction formula for massless FSR at NNLO

Outlook

- ▶ Framework optimisation for relevant phenomenology
(phase-space integration routine, low-level code, ...)
- ▶ Numerical implementation of NNLO massless FSR
- ▶ NLO treatment of massive coloured particles; future extension to NNLO
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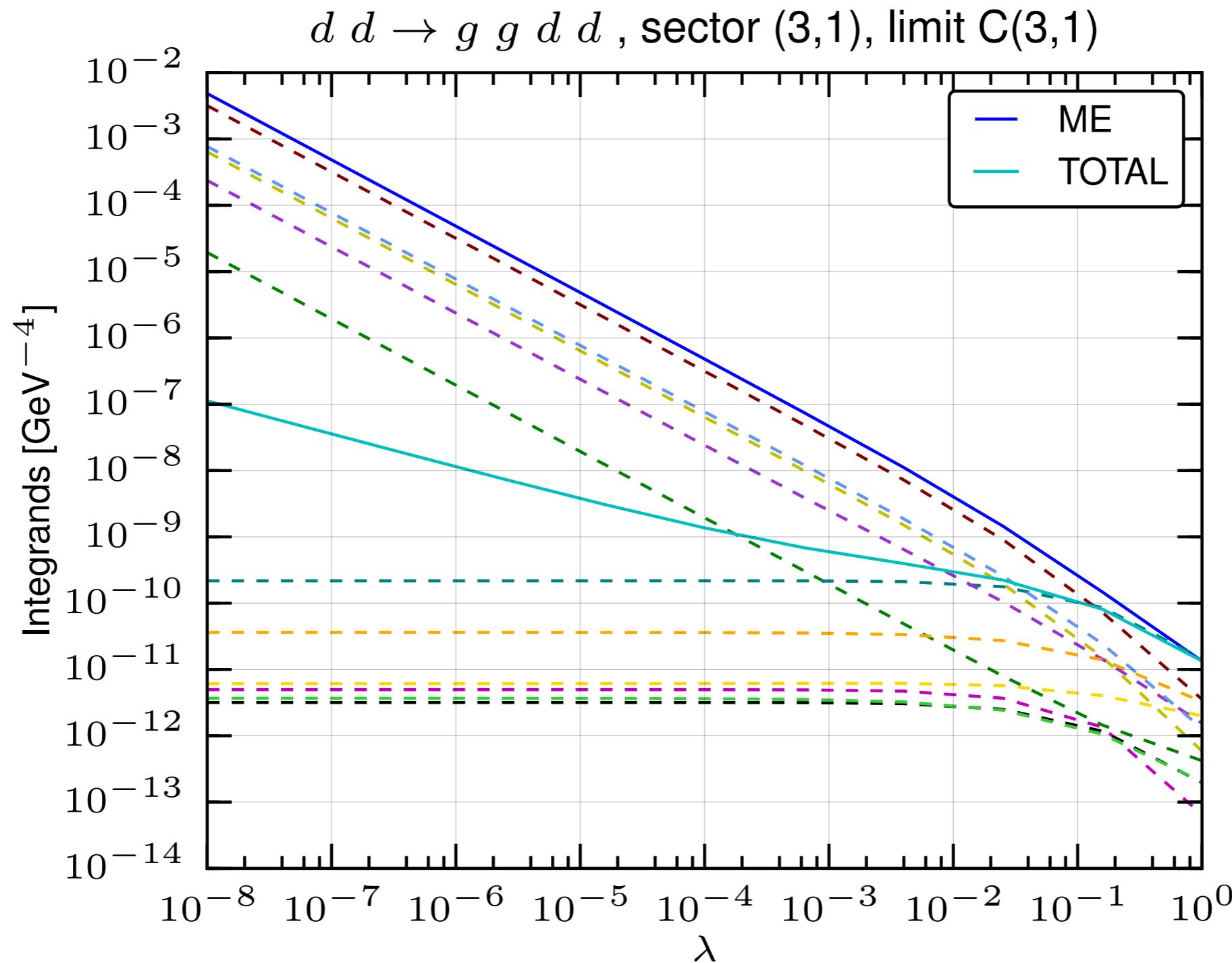
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Thanks
for your attention!

Backup slides

Collinear limit : $\lambda \sim \theta_{ij}^2$

ME $\sim \lambda^{-1}$
Subtracted ME $\sim \lambda^{-1/2}$



NLO sector function:

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{a,b \neq i} \sigma_{ab}} \quad \sigma_{ij} = \frac{1}{\mathcal{E}_i \omega_{ij}} \quad \begin{aligned} \mathcal{E}_i \rightarrow 0 & \quad \text{when particle } i \text{ becomes soft} \\ w_{ij} \rightarrow 0 & \quad \text{when particle } i \text{ and } j \text{ become collinear} \end{aligned}$$

NNLO sector function:

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} \quad \sigma_{ijkl} = \frac{1}{(\mathcal{E}_i w_{ij})^\alpha} \frac{1}{(\mathcal{E}_k + \delta_{kj} \mathcal{E}_i) w_{kl}} \quad \alpha > 1$$

$$\sigma = \sum_{\substack{i, j \neq i \\ k \neq i, l \neq i, k}} \sigma_{ijkl}$$

$$VV + I^{(2)} + I^{(\text{RV})} = \left(\frac{\alpha_s}{2\pi}\right)^2 \left\{ \left[\boxed{I^{(0)}} + \sum_j \boxed{I_j^{(1)}} \mathbf{L}_{jr} + \sum_j \boxed{I_j^{(2)}} \mathbf{L}_{jr}^2 + \frac{1}{2} \sum_{j,l \neq j} \gamma_j^{\text{hc}} \gamma_l^{\text{hc}} \mathbf{L}_{jr'} \mathbf{L}_{lr'} \right] \mathbf{B} \right. \\ \left. + \sum_j \boxed{I_{jr}^{(0)}} + \boxed{I_{jr}^{(1)}} \mathbf{L}_{jr} \right] \\ + \sum_{c,d \neq c} \mathbf{L}_{cd} \boxed{I_{cd}^{(0)}} + \boxed{I_{cd}^{(1)}} \mathbf{L}_{cd} \\ + \sum_{c,d \neq c} \left[-2 + \zeta_2 + 2\zeta_3 \right] \\ + (1 - \zeta_2) \sum_{\substack{c,d \neq c \\ e \neq d}} \mathbf{L}_{cd} \mathbf{L}_{ed} \mathbf{L}_{de} \\ + \pi \sum_{\substack{c,d \neq c \\ e \neq c,d}} \left[\ln \frac{s_{ce}}{s_{de}} \mathbf{L}_{cd}^2 + \right. \\ \left. + \left(\frac{\alpha_s}{2\pi}\right) \left\{ \left[\Sigma_\phi - \sum_j \gamma_j^{\text{hc}} \mathbf{L}_{jr} \right] \mathbf{V} \right. \right. \\ \left. \left. + I^{(0)} = N_q^2 C_F^2 \left[\frac{101}{8} - \frac{141}{8} \zeta_2 + \frac{245}{16} \zeta_4 \right] + N_g N_q C_F \left[C_A \left(\frac{13}{3} - \frac{125}{6} \zeta_2 + \frac{245}{8} \zeta_4 \right) + \beta_0 \left(\frac{77}{12} - \frac{53}{12} \zeta_2 \right) \right] \right. \\ \left. + N_g^2 \left[C_A^2 \left(\frac{20}{9} - \frac{13}{3} \zeta_2 + \frac{245}{16} \zeta_4 \right) + \beta_0^2 \left(\frac{73}{72} - \frac{1}{8} \zeta_2 \right) + C_A \beta_0 \left(-\frac{1}{9} - \frac{11}{3} \zeta_2 \right) \right] \right. \\ \left. + N_q C_F \left[C_F \left(\frac{53}{32} - \frac{57}{8} \zeta_2 + \frac{1}{2} \zeta_3 + \frac{21}{4} \zeta_4 \right) + C_A \left(\frac{677}{432} + \frac{5}{3} \zeta_2 - \frac{25}{2} \zeta_3 + \frac{47}{8} \zeta_4 \right) \right. \right. \\ \left. \left. + \beta_0 \left(\frac{5669}{864} - \frac{85}{24} \zeta_2 - \frac{11}{12} \zeta_3 \right) \right] \right. \\ \left. + N_g \left[C_F C_A \left(-\frac{737}{48} + 11\zeta_3 \right) + C_F \beta_0 \left(\frac{67}{16} - 3\zeta_3 \right) + \beta_0^2 \left(\frac{73}{72} - \frac{3}{8} \zeta_2 \right) \right. \right. \\ \left. \left. + C_A^2 \left(-\frac{4289}{216} + \frac{15}{2} \zeta_2 - 14\zeta_3 + \frac{89}{8} \zeta_4 \right) + C_A \beta_0 \left(\frac{647}{54} - \frac{53}{8} \zeta_2 - \frac{11}{12} \zeta_3 \right) \right] \right. \\ \left. I_j^{(1)} = \delta_{f_a \{q,\bar{q}\}} C_F \left[N_q C_F \left(\frac{5}{2} - \frac{7}{4} \zeta_2 \right) + N_g C_A \left(\frac{1}{3} - \frac{7}{4} \zeta_2 \right) + \frac{2}{3} N_g \beta_0 \right. \right. \\ \left. \left. + C_F \left(-\frac{3}{8} - 4\zeta_2 + 2\zeta_3 \right) + C_A \left(\frac{25}{12} - 3\zeta_2 + 3\zeta_3 \right) + \beta_0 \left(\frac{1}{24} + \zeta_2 \right) \right] \right. \\ \left. + \delta_{f_a g} \left[N_q C_F C_A (10 - 7\zeta_2) - N_q C_F \beta_0 \left(\frac{5}{2} - \frac{7}{4} \zeta_2 \right) + N_g C_A^2 \left(\frac{4}{3} - 7\zeta_2 \right) + N_g C_A \beta_0 \left(\frac{7}{3} + \frac{7}{4} \zeta_2 \right) \right. \right. \\ \left. \left. - \frac{2}{3} (N_g + 1) \beta_0^2 + \frac{11}{4} C_F C_A - \frac{3}{4} C_F \beta_0 + C_A^2 \left(\frac{28}{3} - \frac{23}{2} \zeta_2 + 5\zeta_3 \right) - C_A \beta_0 \left(\frac{2}{3} - \frac{5}{2} \zeta_2 \right) \right] \right. \\ \left. I_j^{(2)} = \frac{1}{8} (15 C_A - 7 \beta_0 - 15) C_{f_j} - \frac{1}{4} (5 C_A - 2 \beta_0) \gamma_j + 2 \zeta_2 C_{f_j}^2 \right. \\ \left. I_{jr}^{(0)} = (-1 + 3\zeta_2 - 2\zeta_3) C_A - \frac{1}{2} (13 + 10\zeta_2 + 2\zeta_3) C_{f_j} + (5 + 2\zeta_3) \gamma_j \right. \\ \left. I_{jr}^{(1)} = (1 - \zeta_2) C_A + \frac{1}{2} (4 + 7\zeta_2) C_{f_j} - (2 + \zeta_2) \gamma_j \right. \\ \left. I_{cd}^{(0)} = \left(\frac{20}{9} - 2\zeta_2 - \frac{7}{2} \zeta_3 \right) C_A + \frac{31}{9} \beta_0 + 2 \Sigma_\phi + 8 (1 - \zeta_2) C_{f_d} \right. \\ \left. I_{cd}^{(1)} = -\left(\frac{1}{3} - \frac{1}{2} \zeta_2 \right) C_A - \frac{11}{12} \beta_0 - \frac{1}{2} \Sigma_\phi \right]$$

Double-unresolved phase space

- ▶ Catani-Seymour variables $y, z, y', z', x' \in [0, 1]$ for mapping $\{k\} \rightarrow \{\bar{k}\}^{(abcd)}$:

$$\begin{aligned} s_{ab} &= y' y s_{abcd}, & s_{cd} &= (1 - y') (1 - y) (1 - z) s_{abcd}, \\ s_{ac} &= z' (1 - y') y s_{abcd}, & s_{bc} &= (1 - y') (1 - z') y s_{abcd}, \\ s_{ad} &= (1 - y) \left[y' (1 - z') (1 - z) + z' z - 2 (1 - 2x') \sqrt{y' z' (1 - z') z (1 - z)} \right] s_{abcd}, \\ s_{bd} &= (1 - y) \left[y' z' (1 - z) + (1 - z') z + 2 (1 - 2x') \sqrt{y' z' (1 - z') z (1 - z)} \right] s_{abcd}, \end{aligned}$$

- ▶ Phase-space factorisation:

$$d\Phi_{n+2} = d\Phi_n^{(abcd)} d\Phi_{\text{rad},2}^{(abcd)},$$

$$\begin{aligned} \int d\Phi_{\text{rad},2}^{(abcd)} &= \int d\Phi_{\text{rad},2} (s_{abcd}; y, z, \phi, y', z', x') \\ &= N^2(\epsilon) (s_{abcd})^{2-2\epsilon} \int_0^1 dx' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi (\sin \phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \\ &\quad \times \left[4 x' (1 - x') y' (1 - y')^2 z' (1 - z') y^2 (1 - y)^2 z (1 - z) \right]^{-\epsilon} \\ &\quad \times [x' (1 - x')]^{-1/2} (1 - y') y (1 - y). \end{aligned}$$

Analytic integration of double-unresolved counterterms

- ▶ Exploit as much as possible **symmetries of $d\Phi_{\text{rad},2}^{(abcd)}$** :

$$\text{perm}(k_a, k_b, k_c, k_d), \quad s_{ab} \leftrightarrow s_{cd}, \quad s_{ac} \leftrightarrow s_{bd}, \quad s_{ad} \leftrightarrow s_{bc}.$$

- ▶ Possible denominator structures reduce to

$$\begin{aligned} s_{ab} &= y' y s_{abcd}, \\ s_{ac} &= z' (1 - y') y s_{abcd}, \\ s_{bc} &= (1 - y') (1 - z') y s_{abcd}, \\ s_{cd} &= (1 - y') (1 - y) (1 - z) s_{abcd}, \\ s_{bd} &= (1 - y) \left[y' z' (1 - z) + (1 - z') z + 2 (1 - 2w') \sqrt{y' z' (1 - z') z (1 - z)} \right] s_{abcd}, \\ s_{ac} + s_{bc} &= (1 - y') y s_{abcd}, \\ s_{ad} + s_{bd} &= (y' + z - y' z) (1 - y) s_{abcd}, \\ s_{ab} + s_{bc} &= (1 - z' + z' y') y s_{abcd}. \end{aligned}$$

- ▶ Integration measure

$$\begin{aligned} \int d\Phi_{\text{rad},2}^{(abcd)} &= N(\epsilon) (s_{abcd})^{2-2\epsilon} \int_0^1 dw' \int_0^1 dy' \int_0^1 dz' \int_0^1 dy \int_0^1 dz [w' (1 - w')]^{-1/2-\epsilon} \\ &\quad \times \left[y' (1 - y')^2 z' (1 - z') y^2 (1 - y)^2 z (1 - z) \right]^{-\epsilon} (1 - y') y (1 - y). \end{aligned}$$

Analytic integration of double-unresolved counterterms

- ▶ Integration measure

$$\int d\Phi_{\text{rad},2}^{(abcd)} = N(\epsilon) (s_{abcd})^{2-2\epsilon} \int_0^1 dy \int_0^1 dw' \int_0^1 dz \int_0^1 dy' \int_0^1 dz' [w' (1-w')]^{-1/2-\epsilon} \\ \times \left[y' (1-y')^2 z' (1-z') y^2 (1-y)^2 z (1-z) \right]^{-\epsilon} (1-y') y (1-y).$$

- ▶ Integrate y : fully factorised dependence → Beta functions.
 - ▶ Integrate w' (azimuth): at worst one gets rational $\times {}_2F_1 [1, 1+\epsilon, 1-\epsilon, \frac{y'z'(1-z)}{z(1-z')}]$.
 - ▶ Integrate z : at worst one gets rational $\times {}_2F_1 [1, n+1-\epsilon, 1-\epsilon, -\frac{y'z'}{1-z'}]$.
 - ▶ ${}_2F_1 \rightarrow$ integral representation in t ; integrate in z' and get at worst
- $$\int_0^1 dy' dt t^a (1-t)^b y'^c (1-y')^d {}_2F_1 [n, m-\epsilon, p-2\epsilon, 1-ty'] , \quad n, m, p \in \mathbb{N}$$
- ▶ Expand in ϵ and integrate in $dt dy'$.
 - ▶ Checked against numerical integration (with no symmetries or relabellings encoded).