



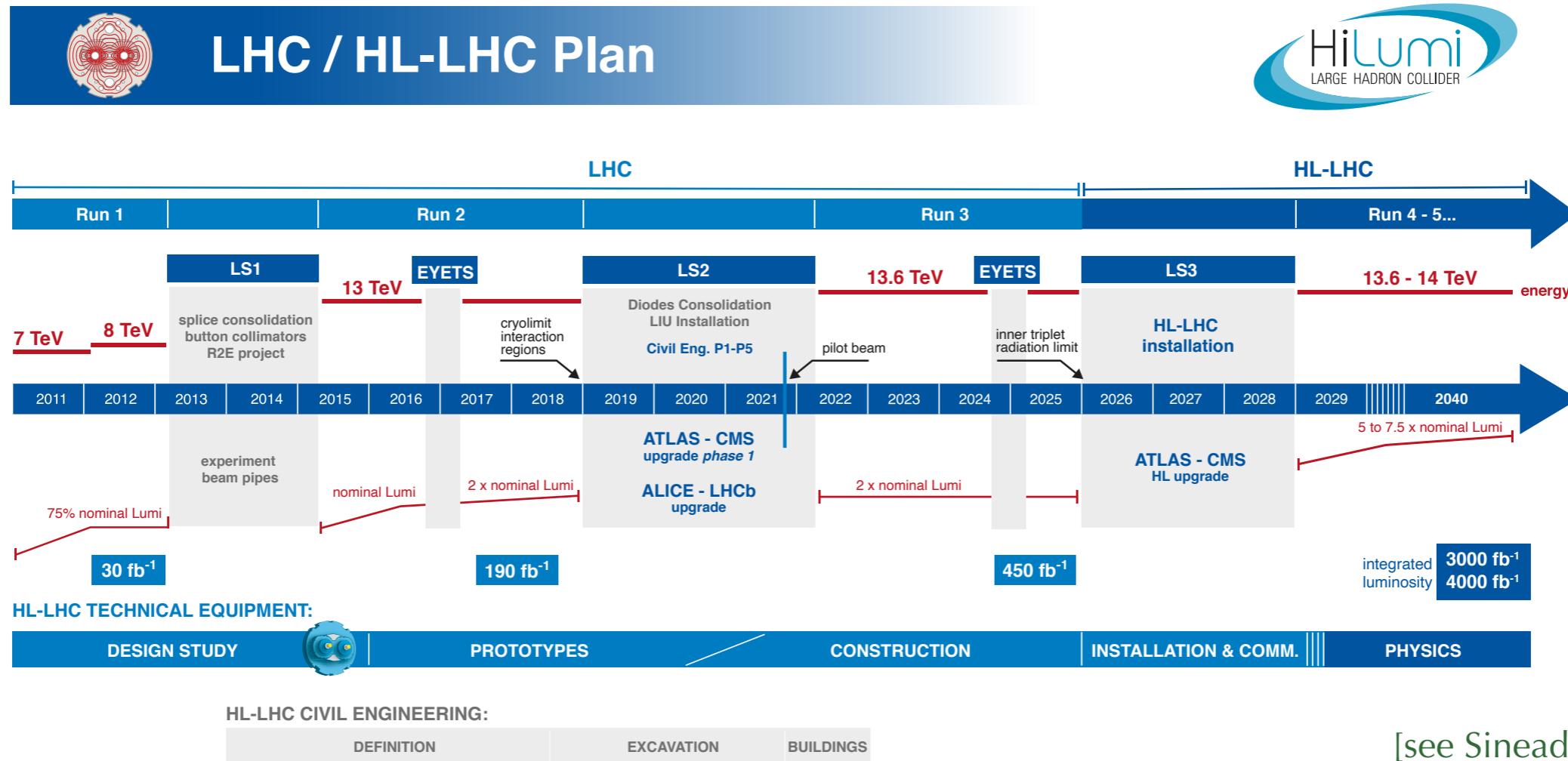
Overview of the calculation of two-loop multi-leg scattering amplitudes

Samuel Abreu
CERN & The University of Edinburgh

21st of September 2022, HP2 — Newcastle

Motivation

- Order of magnitude increase in integrated luminosity



- Percent-level precision for several observables

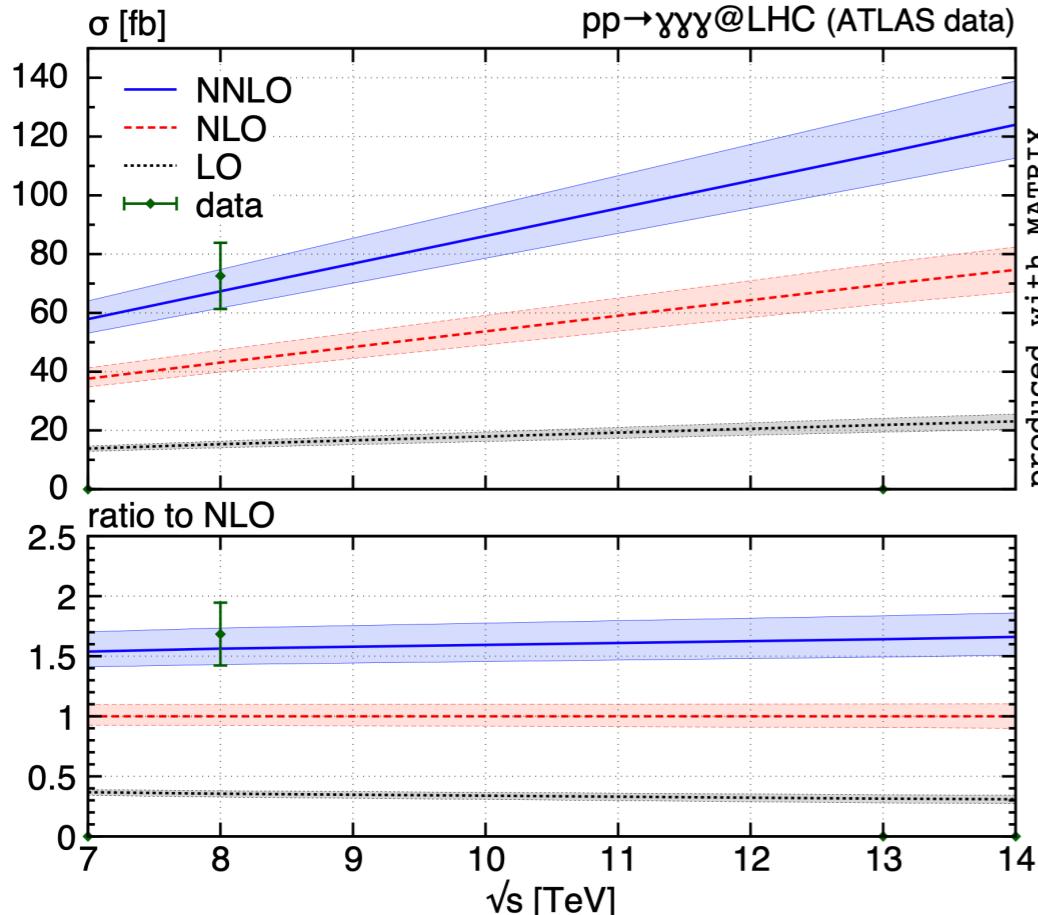
$$\sigma = \sigma_{LO} \left(1 + \alpha_s \sigma_{NLO} + \alpha_s^2 \sigma_{NNLO} \right) + \mathcal{O}(\alpha_s^3)$$

$\sim \mathcal{O}(10\%)$

$\sim \mathcal{O}(1\%)$

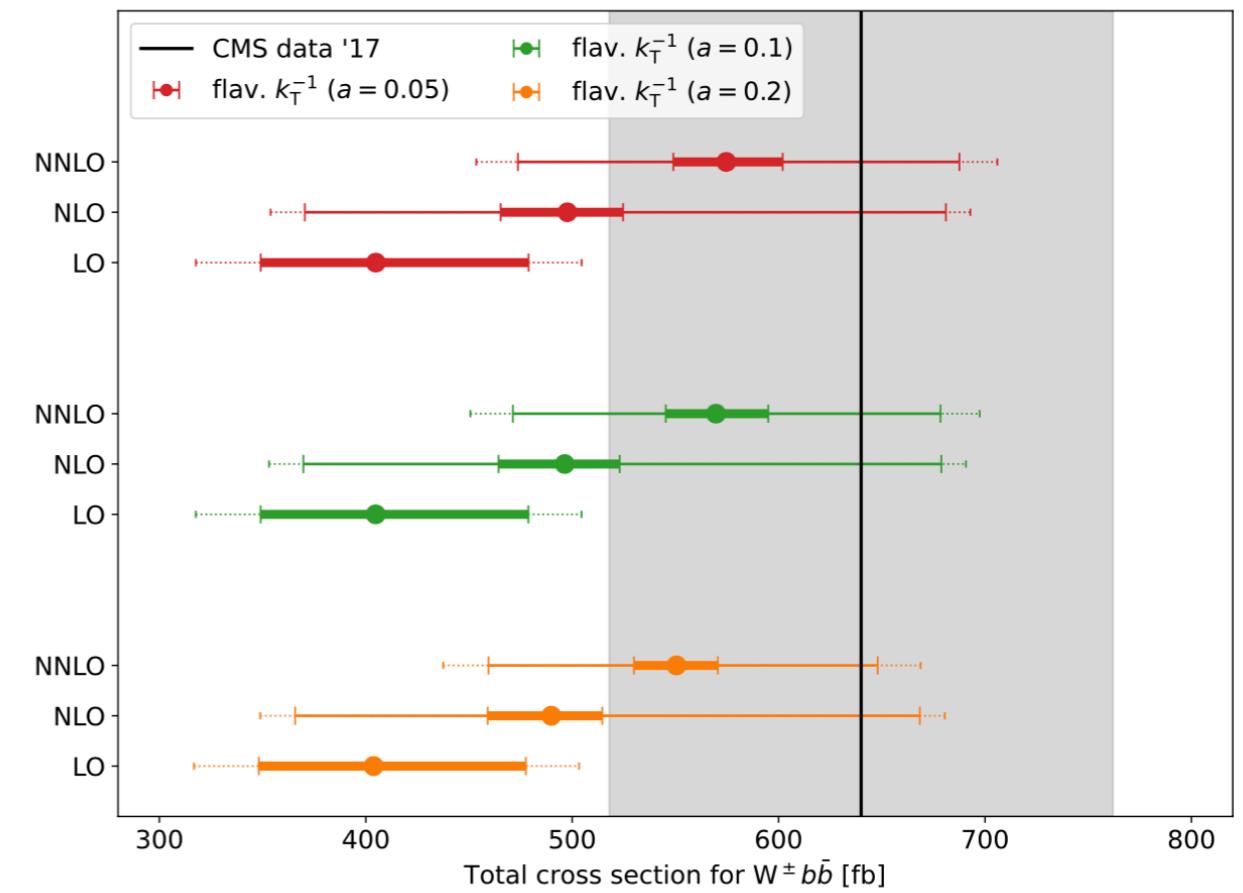
Motivation

- ◆ NNLO calculations: challenging but already needed!



[Kallweit, Sotnikov, Wiesemann, 20]

[see also Chawdhry, Czakon, Mitov, Poncelet, 19]



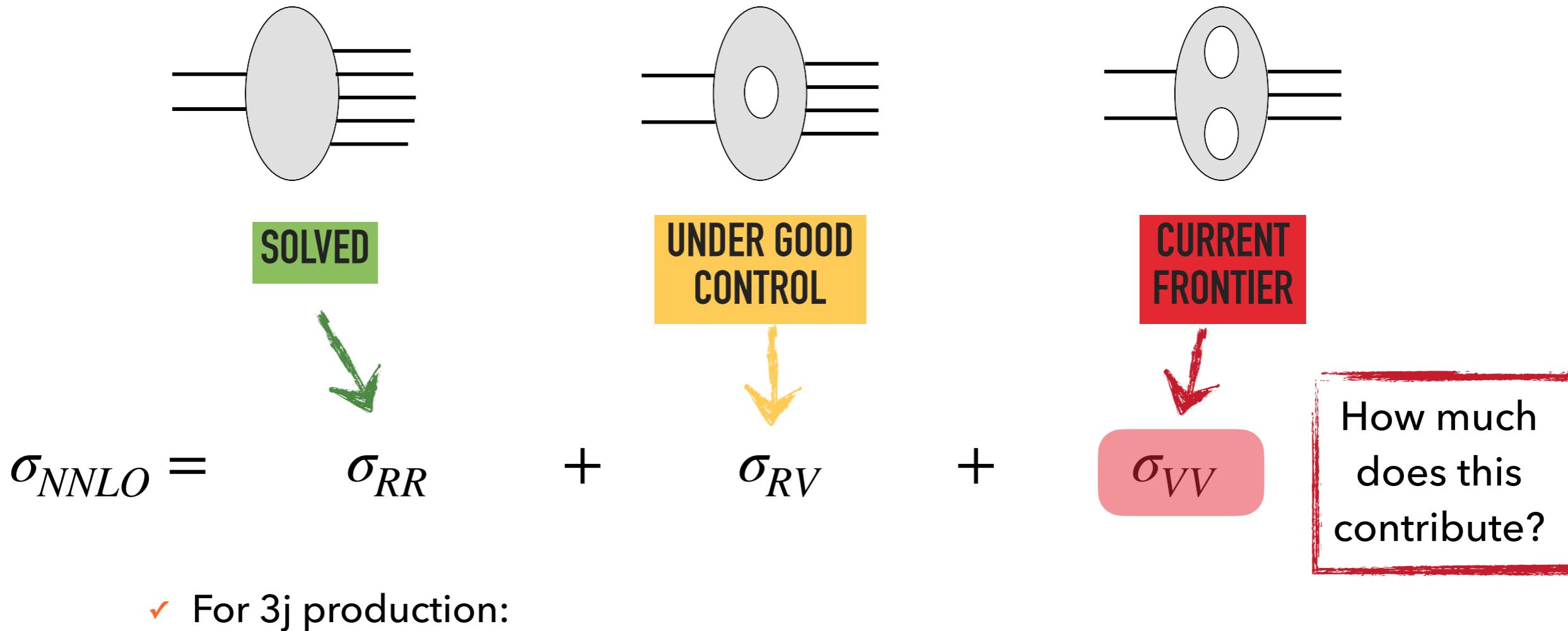
[Hartanto, Poncelet, Popescu, Zoia 22]

[see Bayu's talk]

Motivation

♦ Amplitudes for NNLO corrections to five-point processes

[see Kirill's talk]

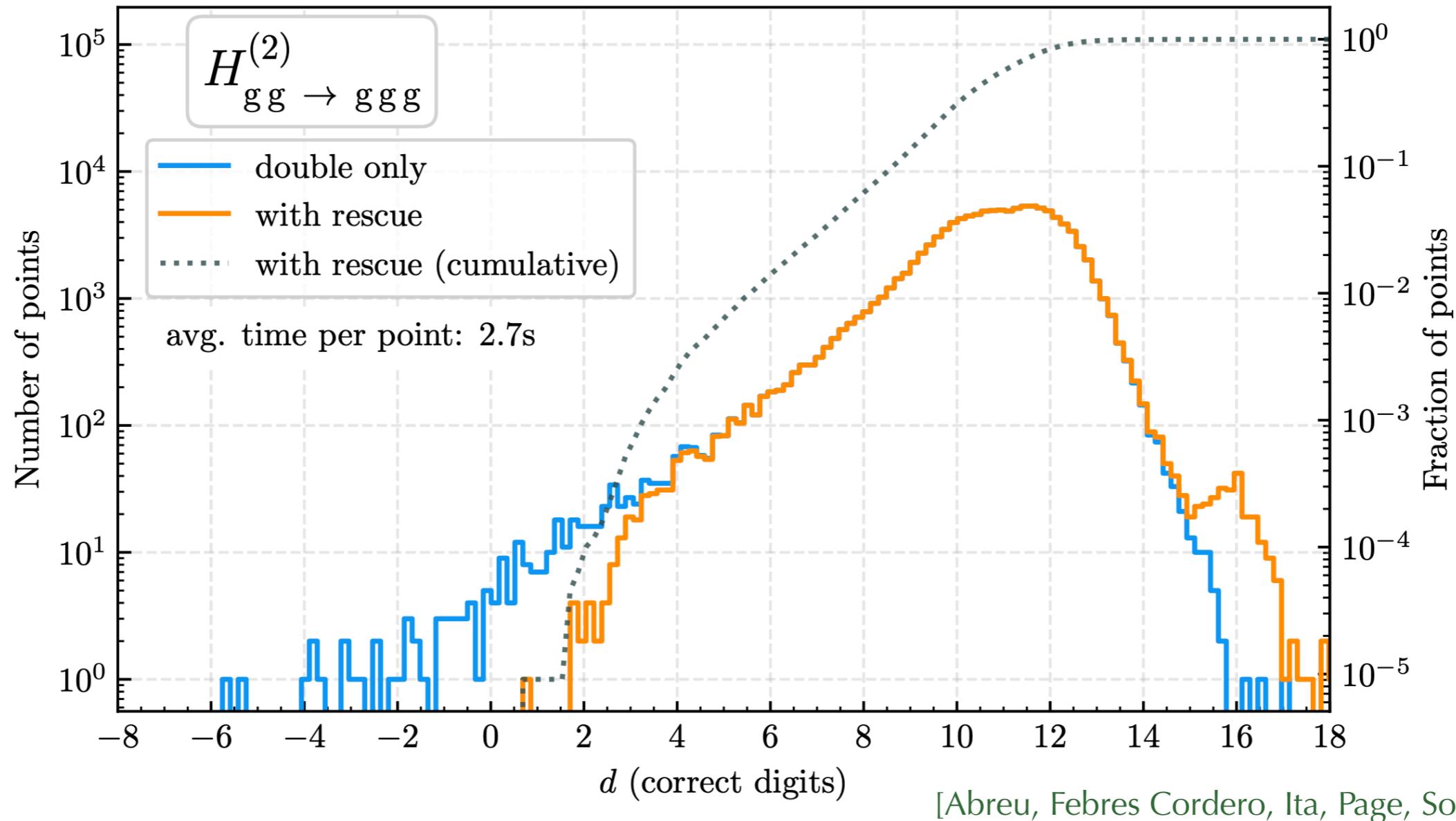


- ✓ For 3j production:

This implies that the double virtual contribution is about $\approx 10\%$ of the total NNLO cross-section in contrast to our previous findings of $\approx 2\%$. With this, the naive estimate for corrections from sub-leading colour terms would correspond to 1% corrections of the NNLO QCD prediction.

[see Rene's talk]

- ◆ Goal: fast and stable evaluation of two-loop amplitudes



✓ σ_{VV} for 3-jet production at NNLO

✓ σ_{RVV} for 2-jet production at N3LO

[see Giulio's talk]

Status of five-point two-loop amplitude calculations

	Comment	Complete analytic results	Public numerical code	Cross sections	
$pp \rightarrow jjj$	l.c.	[9]	[9]	[1, 2]	[see several talks: Ryan, Bayu, Giuseppe, Bakul, Ben, Jakub, ...]
$pp \rightarrow \gamma\gamma j$	l.c.*	[10, 11]	[10]	[12]	
$pp \rightarrow \gamma\gamma\gamma$	l.c.*	[13, 14]	[13]	[15, 16]	
$pp \rightarrow \gamma\gamma j$		[3]			
$gg \rightarrow \gamma\gamma g$	NLO loop induced	[4]	[4]	[17]	
$pp \rightarrow W b\bar{b}$	l.c.*, on-shell W	[18]			
$pp \rightarrow W(l\nu) b\bar{b}$	l.c.	[19, 20]		[20] + 2209.03280	
$pp \rightarrow W(l\nu) jj$	l.c.	[19]			[Hartanto, Poncelet, Popescu, Zoia 22]
$pp \rightarrow Z(l\bar{l}) jj$	l.c.*	[19]			
$pp \rightarrow W(l\nu) \gamma j$	l.c.*	[21]			
$pp \rightarrow H b\bar{b}$	l.c., b -quark Yukawa	[22]			

Table 1: Known two-loop QCD corrections for five-point scattering processes at hadron colliders. “l.c.” refers to the calculations in the leading-color approximation; “l.c.*” means that in addition non-planar l.c. contributions are omitted. All public codes employ `PentagonFunctions++` [23, 24] for numerical evaluation of special functions.

Status of five-point two-loop amplitude calculations

[1] M. Czakon, A. Mitov and R. Poncelet, *Next-to-Next-to-Leading Order Study of Three-Jet Production at the LHC*, *Phys. Rev. Lett.* **127** (2021) 152001 [[2106.05331](#)]. (pages 2 and 3)

[2] X. Chen, T. Gehrmann, N. Glover, A. Huss and M. Marcoli, *Automation of antenna subtraction in colour space: gluonic processes*, [2203.13531](#). (pages 2 and 3)

[3] B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, *Two-Loop Helicity Amplitudes for Diphoton Plus Jet Production in Full Color*, *Phys. Rev. Lett.* **127** (2021) 262001 [[2105.04585](#)]. (pages 2 and 3)

[4] S. Badger, C. Brønnum-Hansen, D. Chicherin, T. Gehrmann, H.B. Hartanto, J. Henn et al., *Virtual QCD corrections to gluon-initiated diphoton plus jet production at hadron colliders*, *JHEP* **11** (2021) 083 [[2106.08664](#)]. (pages 2, 3, and 5)

[9] S. Abreu, F.F. Cordero, H. Ita, B. Page and V. Sotnikov, *Leading-color two-loop QCD corrections for three-jet production at hadron colliders*, *JHEP* **07** (2021) 095 [[2102.13609](#)]. (pages 3 and 7)

[10] B. Agarwal, F. Buccioni, A. von Manteuffel and L. Tancredi, *Two-loop leading colour QCD corrections to $q\bar{q} \rightarrow \gamma\gamma g$ and $qg \rightarrow \gamma\gamma q$* , *JHEP* **04** (2021) 201 [[2102.01820](#)]. (page 3)

[11] H.A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, *Two-loop leading-colour QCD helicity amplitudes for two-photon plus jet production at the LHC*, *JHEP* **07** (2021) 164 [[2103.04319](#)]. (page 3)

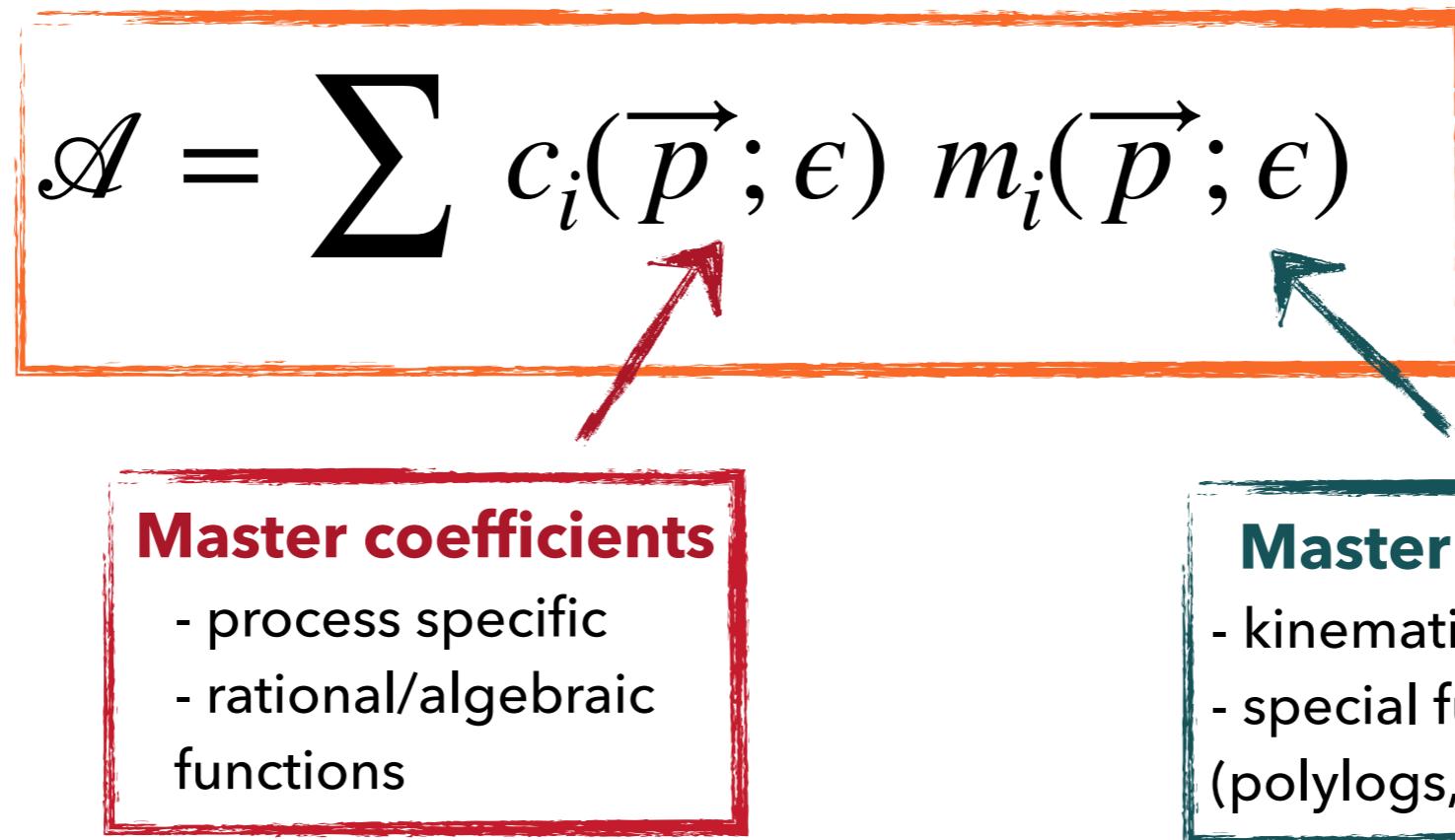
- [12] H.A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, *NNLO QCD corrections to diphoton production with an additional jet at the LHC*, *JHEP* **09** (2021) 093 [[2105.06940](#)]. (page 3)
- [13] S. Abreu, B. Page, E. Pascual and V. Sotnikov, *Leading-Color Two-Loop QCD Corrections for Three-Photon Production at Hadron Colliders*, *JHEP* **01** (2021) 078 [[2010.15834](#)]. (page 3)
- [14] H.A. Chawdhry, M. Czakon, A. Mitov and R. Poncelet, *Two-loop leading-color helicity amplitudes for three-photon production at the LHC*, *JHEP* **06** (2021) 150 [[2012.13553](#)]. (page 3)
- [15] H.A. Chawdhry, M.L. Czakon, A. Mitov and R. Poncelet, *NNLO QCD corrections to three-photon production at the LHC*, *JHEP* **02** (2020) 057 [[1911.00479](#)]. (page 3)
- [16] S. Kallweit, V. Sotnikov and M. Wiesemann, *Triphoton production at hadron colliders in NNLO QCD*, *Phys. Lett. B* **812** (2021) 136013 [[2010.04681](#)]. (page 3)
- [17] S. Badger, T. Gehrmann, M. Marcoli and R. Moodie, *Next-to-leading order QCD corrections to diphoton-plus-jet production through gluon fusion at the LHC*, *Phys. Lett. B* **824** (2022) 136802 [[2109.12003](#)]. (page 3)
- [18] S. Badger, H.B. Hartanto and S. Zoia, *Two-Loop QCD Corrections to Wbb^- Production at Hadron Colliders*, *Phys. Rev. Lett.* **127** (2021) 012001 [[2102.02516](#)]. (pages 3 and 7)
- [19] S. Abreu, F. Febres Cordero, H. Ita, M. Klinkert, B. Page and V. Sotnikov, *Leading-color two-loop amplitudes for four partons and a W boson in QCD*, *JHEP* **04** (2022) 042 [[2110.07541](#)]. (pages 3 and 5)
- [20] H.B. Hartanto, R. Poncelet, A. Popescu and S. Zoia, *NNLO QCD corrections to $Wb\bar{b}$ production at the LHC*, [2205.01687](#). (pages 3 and 4)
- [21] S. Badger, H.B. Hartanto, J. Kryś and S. Zoia, *Two-loop leading colour helicity amplitudes for $W\gamma + j$ production at the LHC*, *JHEP* **05** (2022) 035 [[2201.04075](#)]. (page 3)
- [22] S. Badger, H.B. Hartanto, J. Kryś and S. Zoia, *Two-loop leading-colour QCD helicity amplitudes for Higgs boson production in association with a bottom-quark pair at the LHC*, *JHEP* **11** (2021) 012 [[2107.14733](#)]. (page 3)

[from Sotnikov, [2207.12295](#)]

FIVE-POINT TWO-LOOP AMPLITUDES

SETUP OF CALCULATION

Master integral decomposition



- ◆ How?
 - ✓ construct **amplitude integrand** (QGRAF+projectors, Generalised Unitarity, ...)
 - ✓ IBP reduce (FiniteFlow, Fire, Kira, Reduze, ...)
 - ◆ **Extremely complicated coefficients**, even with “good basis”
 - ◆ Decomposition valid to all orders in ϵ ...
 - ... but we only care about the first orders
- Can we use this to simplify our life?

Pentagon Functions

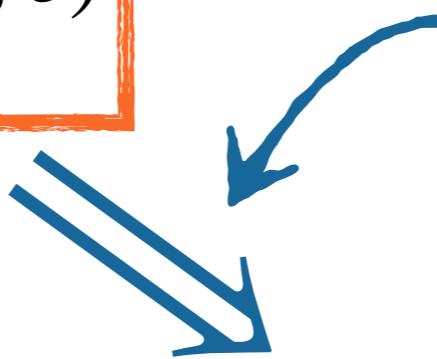
- ♦ Relations after expansion in ϵ :

$$\sim r_0 + r_1 \epsilon \ln(s) + r_2 \epsilon^2 \ln^2(s) + \dots$$

- ♦ Make relations explicit: basis of transcendental functions at each order

$$\mathcal{A} = \sum c_i(\vec{p}; \epsilon) m_i(\vec{p}; \epsilon)$$

$$m(\vec{p}) = \sum \epsilon^j h_j(\vec{p})$$



$$\mathcal{A} = \sum \epsilon^j \sum d_{j,k}(\vec{p}) h_k(\vec{p})$$

[Gehrmann, Henn, Lo Presti, 18]

[Chicherin, Sotnikov, 20]

[Chicherin, Sotnikov, Zoia, 21]

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, to appear]

$$\mathcal{A} = \sum \epsilon^j \sum d_{j,k}(\vec{p}) h_k(\vec{p})$$

1. Master integrals:

- ✓ Decompose in terms of **pentagon functions**
- ✓ Efficient and stable **numerical evaluation** to required order in ϵ

2. Coefficients

- ✓ Directly compute **coefficients in ϵ expansion**
- ✓ Simplify for efficient and stable **numerical evaluation**

MASTER INTEGRALS

PENTAGON FUNCTIONS AND NUMERICAL EVALUATION

- ◆ How to compute them? Many ways...

- ✓ Identify independent transcendental components: pentagon functions
- ✓ Convenient representation for fast/stable numerical evaluation

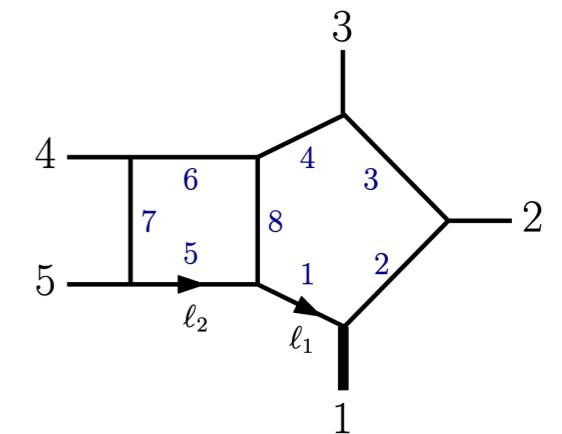
- ◆ Solution: differential equations in canonical form

[Remiddi, 97]
 [Gehrmann, Remiddi, 99]
 [Henn, 13]

$$d\vec{M} = \epsilon A \vec{M}$$

$$A(\vec{p}) = \sum A_i \ d \ln W_i(\vec{p})$$

- ✓ Find special (pure) basis
- ✓ Determine the $d \ln W_i$ \implies “symbol alphabet”: precious analytic information, in usable form
- ✓ Determine the matrices A_i



$$d\vec{M} = \epsilon A \vec{M}$$

$$A(\vec{p}) = \sum A_i d \ln W_i(\vec{p})$$

[Abreu, Page, Zeng, 19]

♦ **Complexity:** similar to amplitude calculation... find shortcut!

♦ **Finding pure basis**

- ✓ Several semi-automated methods, but not yet systematic
- ✓ Maximal cuts/leading singularities, dimension shifting, ...
- ✓ Still important bottleneck, often based on trial an error

♦ **Finding the symbol alphabet**

- ✓ Special points in phase space: Landau conditions
- ✓ Construct educated guesses and test
- ✓ Simplify for compact representation

♦ **Finding the constant matrices A_i**

- ✓ Rational numbers: use numerical evaluations
- ✓ Trivial once pure basis and alphabet known

All investigations
done with finite
field numerical
evaluations

[FiniteFlow, Fire, Kira, ...]

[Schabinger, von Manteuffel, 14]

[Peraro, 16]

$$d\vec{M} = \epsilon A \vec{M}$$

$$A(\vec{p}) = \sum A_i d \ln W_i(\vec{p})$$

- ◆ Solve order by order

$$m(\vec{p}) = \sum \epsilon^j h_j(\vec{p})$$

- ◆ Multiple Polylogarithms

- ✓ Cumbersome representation, region specific
- ✓ Relations not explicit
- ✓ Use **Ginac** for evaluation, but slow...

[Papadopoulos et al, 15, 19, 20, 22]

- ◆ Chen iterated integrals

- ✓ Relations are explicit
- ✓ Write **dedicated code for evaluation**

[Chicherin, Sotnikov, 20]

[Chicherin, Sotnikov, Zoia, 21]

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, to appear]

- ◆ Numeric alternatives: currently too slow for precision required [see Matijn's talk]

- ✓ **pySecDec**

[Heinrich et al. 15, 17, 18, 21]

- ✓ **Diffexp**

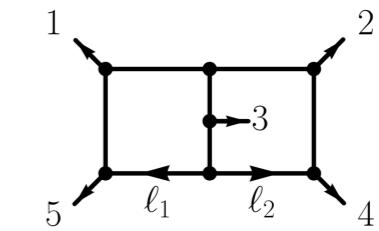
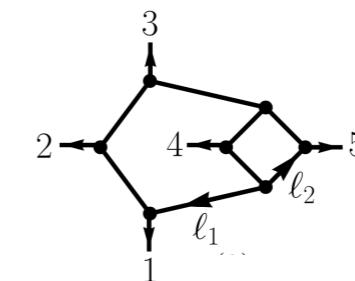
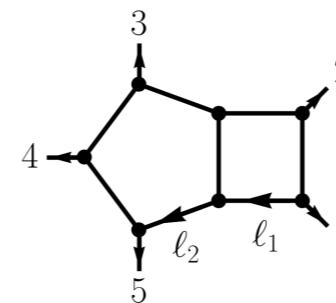
[Moriello 19]

[Hidding 20]

- ✓ **AMFlow**

[Liu, Ma, (Wang), (17), 21, 22]

✓ Five-point massless



[Papadopoulos, Tommasini, Wever, 15]

[Gehrmann, Henn, Lo Presti, 18]

[Abreu, Page, Zeng, 18]

[Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser, 18]

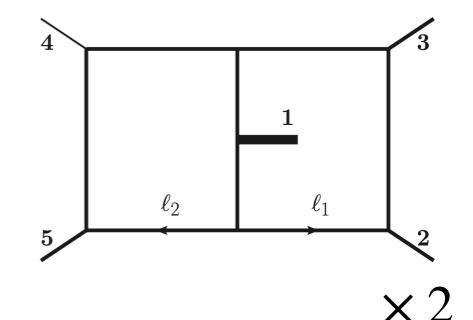
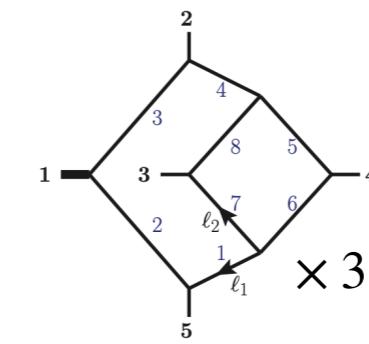
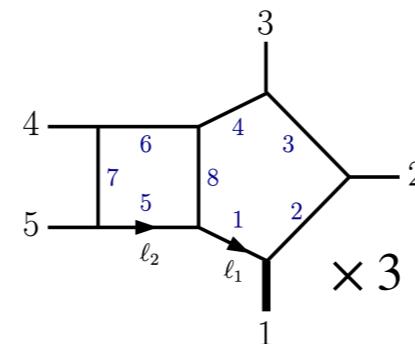
[Abreu, Dixon, Herrmann, Page, Zeng, 18]

[Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, 18]

[Gehrmann, Henn, Lo Presti, 18]

[Chicherin, Sotnikov, 20]

✓ Five-point one mass



[Abreu, Ita, Moriello, Page, Tschernow, Zeng, 20]

[Abreu, Ita, Page, Tschernow, 21]

[Papadopoulos, Tommasini, Wever, 15]

[Papadopoulos, Wever, 19]

[Canko, Papadopoulos, Syrrakos, 20]

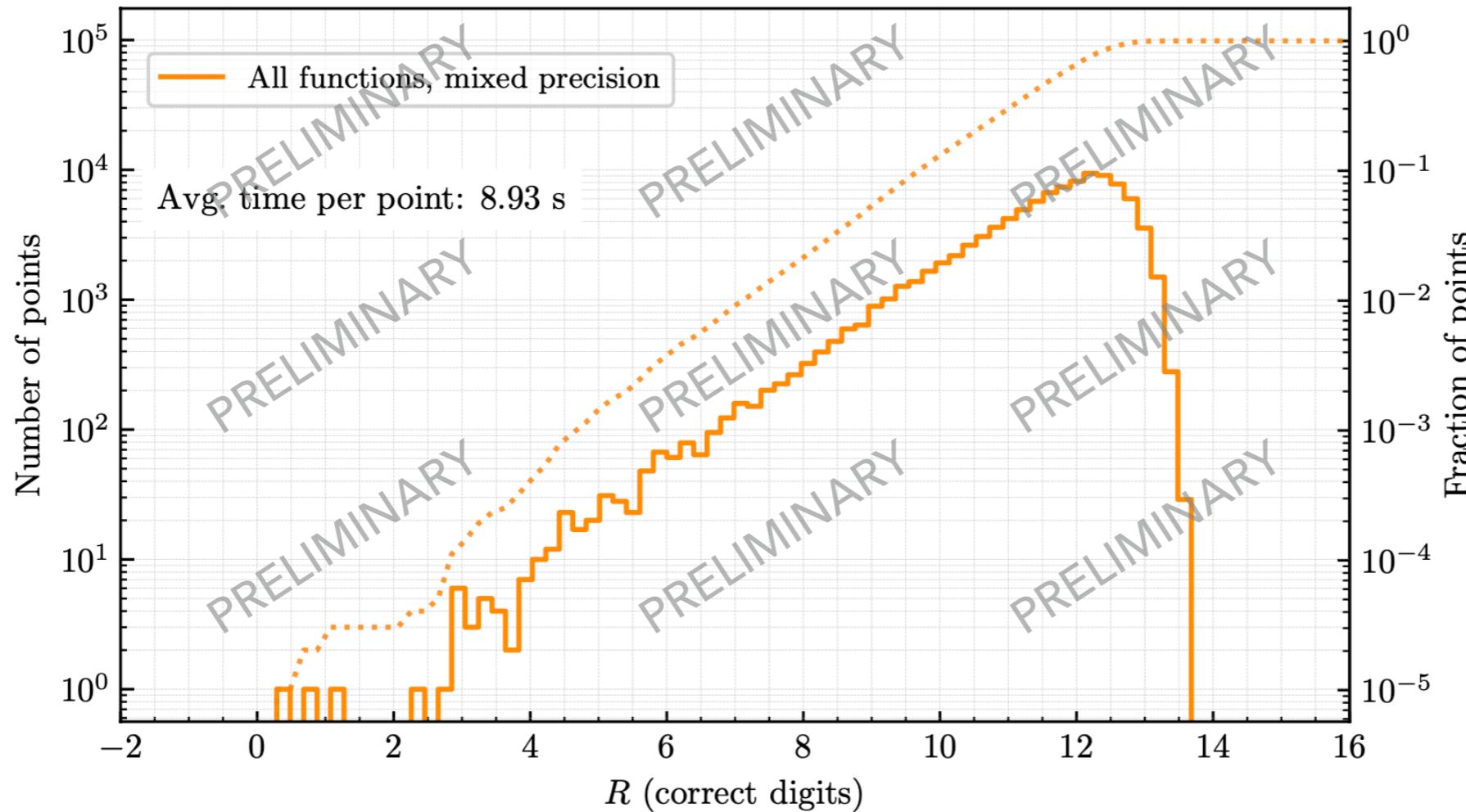
[Kardos, Papadopoulos, Smirnov, Syrrakos, Wever, 19]

[Chicherin, Sotnikov, Zoia, 21]

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, to appear]

- ◆ All five-point one-mass pentagon functions

- ✓ $\mathcal{O}(200)$ symbol letters
- ✓ $\mathcal{O}(1250)$ pentagon functions, $\mathcal{O}(1000)$ at weight 4



✓ Note: Conservative time estimate

All the functions you need for five-point
one-mass processes at two loops

COEFFICIENTS

COMPUTATION AND SIMPLIFICATION

$$\mathcal{A} = \sum \epsilon^j \sum d_{j,k}(\vec{p}) h_k(\vec{p})$$

- ◆ Now that the h_k are known, determine $d_{j,k}$
- ◆ Strategy: Ansatz constrained by (finite field) numerical evaluations

$$d(\vec{p}) = \frac{\mathcal{N}(\vec{p})}{\mathcal{D}(\vec{p})}$$

[Schabinger, von Manteuffel, 14]

[Peraro, 16]

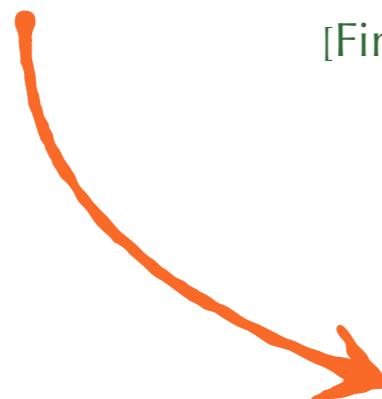
- ✓ Numerator/Denominator are polynomials of high degree
- ✓ Generate the “numerical data”

“Feynman diagrammatic”

[Badger et al] [Czakon et al]
 [von Manteuffel et al]

- ✓ Generate integrand: QGRAF, ...
- ✓ Project on form factors
- ✓ IBP reduce (with syzygy, ...)

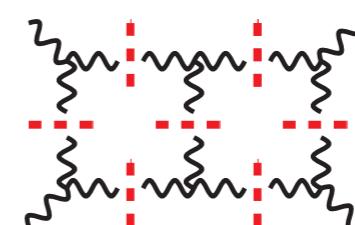
[FiniteFlow — Peraro, 19]



[Ita et al]

“Two-loop generalised unitarity”

- ✓ Parametrise generic integrand as surface terms/master integrands
- ✓ Generate integrand: product of trees



Caravel

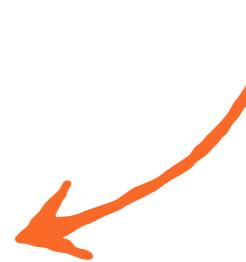
[Abreu, Febres Cordero, Ita, Jaquier, Page, Zeng, 17]

[Abreu, Febres Cordero, Ita, Page, Zeng, 17]

- ♦ Almost done:

$$\mathcal{A}_0 / . \left\{ m_i(\vec{p}) \rightarrow \sum \epsilon^j h_j(\vec{p}) \right\} \implies$$

$$\mathcal{A}_0 = \sum \epsilon^j \sum d_{j,k}(\vec{p}_0) h_k(\vec{p})$$



$$d(\vec{p}) = \frac{\mathcal{N}(\vec{p})}{\mathcal{D}(\vec{p})}$$

[Badger et al]

[von Manteuffel et al]

[Ita et al]

- ♦ Denominator: essentially **nothing to do if you have the integrals!**

$$\mathcal{D}(\vec{p}) = \prod W_i^k$$

- ♦ Much **simpler problem** (but still hard): only need $\mathcal{N}(\vec{p})$!

- ✓ Morally easy: determine a polynomial from exact numerical data
- ✓ Algorithms scale very badly with degree/number of variables
- ✓ Many ways to improve performance, very important in practice:

[see Bakul's, Ben's talk]

- ✓ Univariate slices
- ✓ Univariate/multivariate partial fractions
- ✓ Choice of variables
- ✓ p -adic numbers

$$\mathcal{A} = \sum \epsilon^j \sum d_{j,k}(\vec{p}) h_k(\vec{p})$$

- ♦ Important for **pheno-ready results**
- ♦ **Assemble full amplitude:** all colour structures/permuations/channels
 - ✓ Large combinatorial factors...
- ♦ Most $d_{j,k}$ are related \Rightarrow write in **basis of rational functions**
- ♦ Clean-up basis for **fast evaluation**
 - ✓ Multivariate partial fractions

[Pak 11]
[Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov 19]
[Boehm, Wittmann, Wu, Xu, Zhang '20]
[Heller, von Manteuffel, 21]
- ♦ Implement everything in numerical C++ code
 - ✓ Precision rescue system: hiding in the symbol alphabet (**cheap!**)

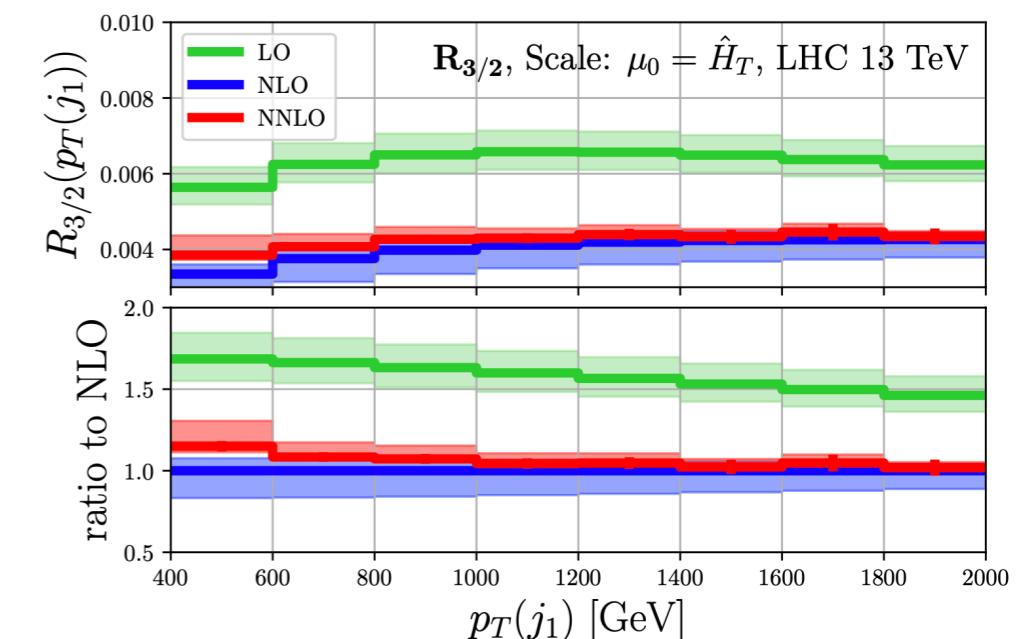
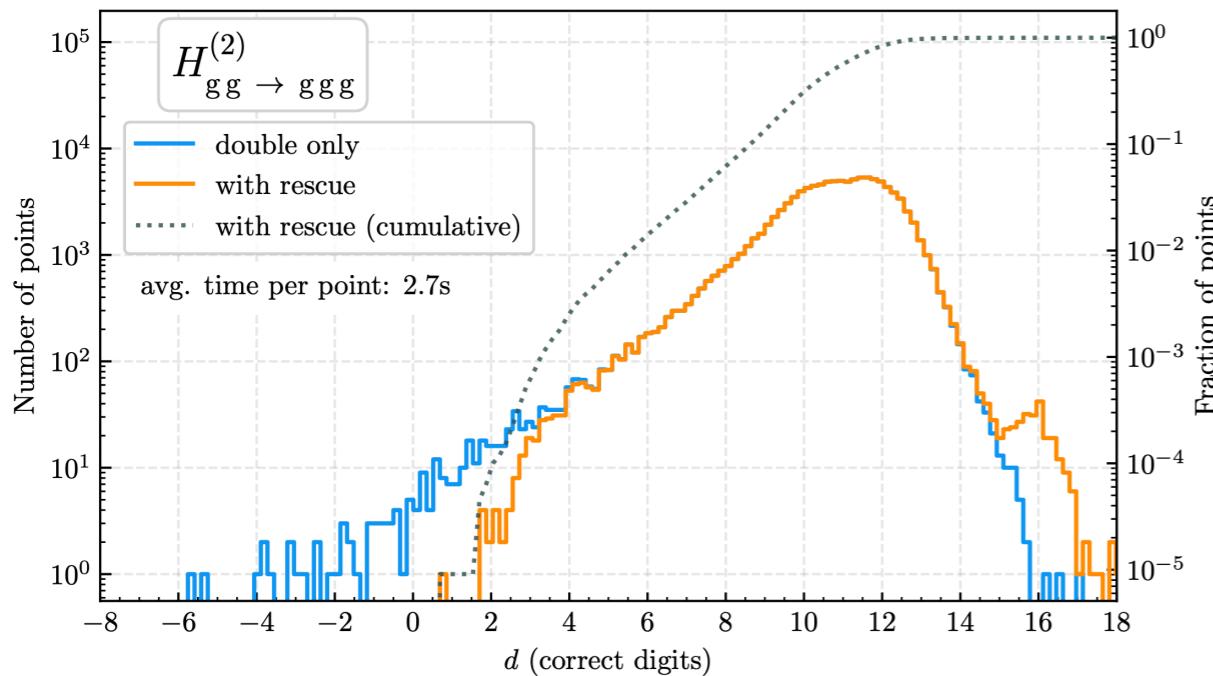
[Abreu, Page, Pascual, Sotnikov, 20]

CONCLUSION AND OUTLOOK

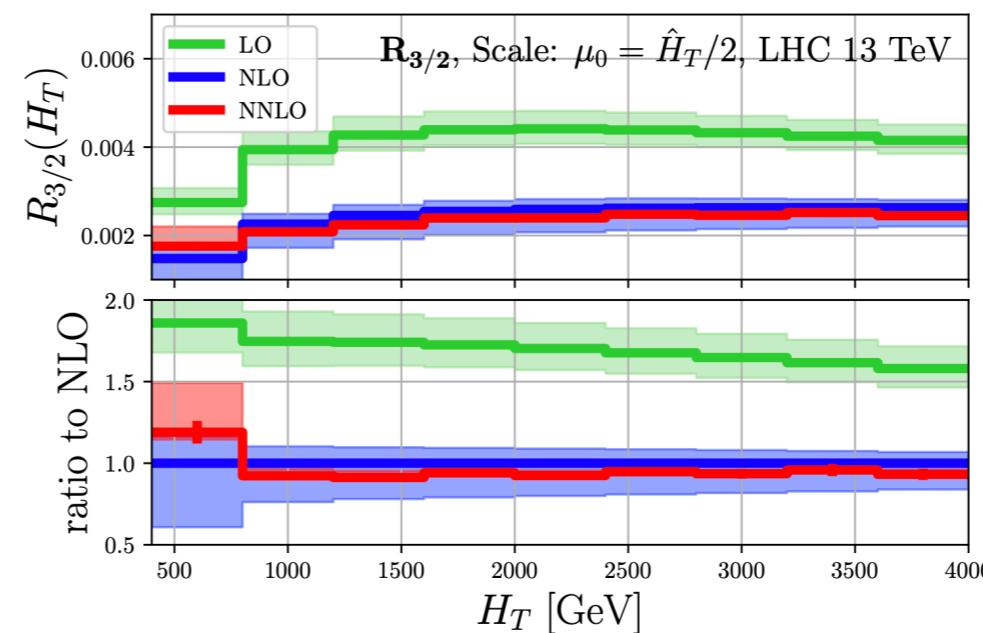
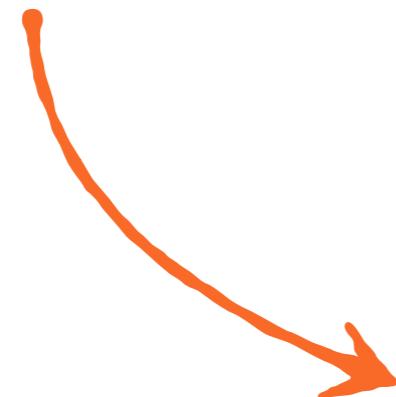
Towards phenomenology

♦ Subtraction methods are ready: we need pheno-ready two-loop amplitudes

[Abreu, Febres Cordero, Ita, Page, Sotnikov, 21]



[Czakon, Mitov, Poncelet 21]



[see Rene's talk]

- ♦ A lot of progress in recent years in two-loop five-point calculations
- ♦ Two-loop amplitudes start to be available in pheno-ready form
- ♦ Some progress needed for non-planar corrections...
- ♦ ... but ingredients available for massless and one-mass

✓ e.g.: sub-leading colour for 3j production, H/W/Z+2j amplitudes, ...

- Incremental improvement for other cases with massless propagators
 - ✓ Some progress for six-point massless integrals [Henn, Peraro, Xu, Zhang 21]
- Bigger improvements when more complicated special functions are involved ($t\bar{t} + j$, $t\bar{t} + H$, ...)
 - ✓ Massive propagators or a lot of external legs
 - ✓ Elliptic integrals: what are the new “pentagon functions”
 - ✓ Is it time to go fully numerical?

THANK YOU!

RADCOR 2023

16th International Symposium on Radiative Corrections:
Applications of Quantum Field Theory to Phenomenology

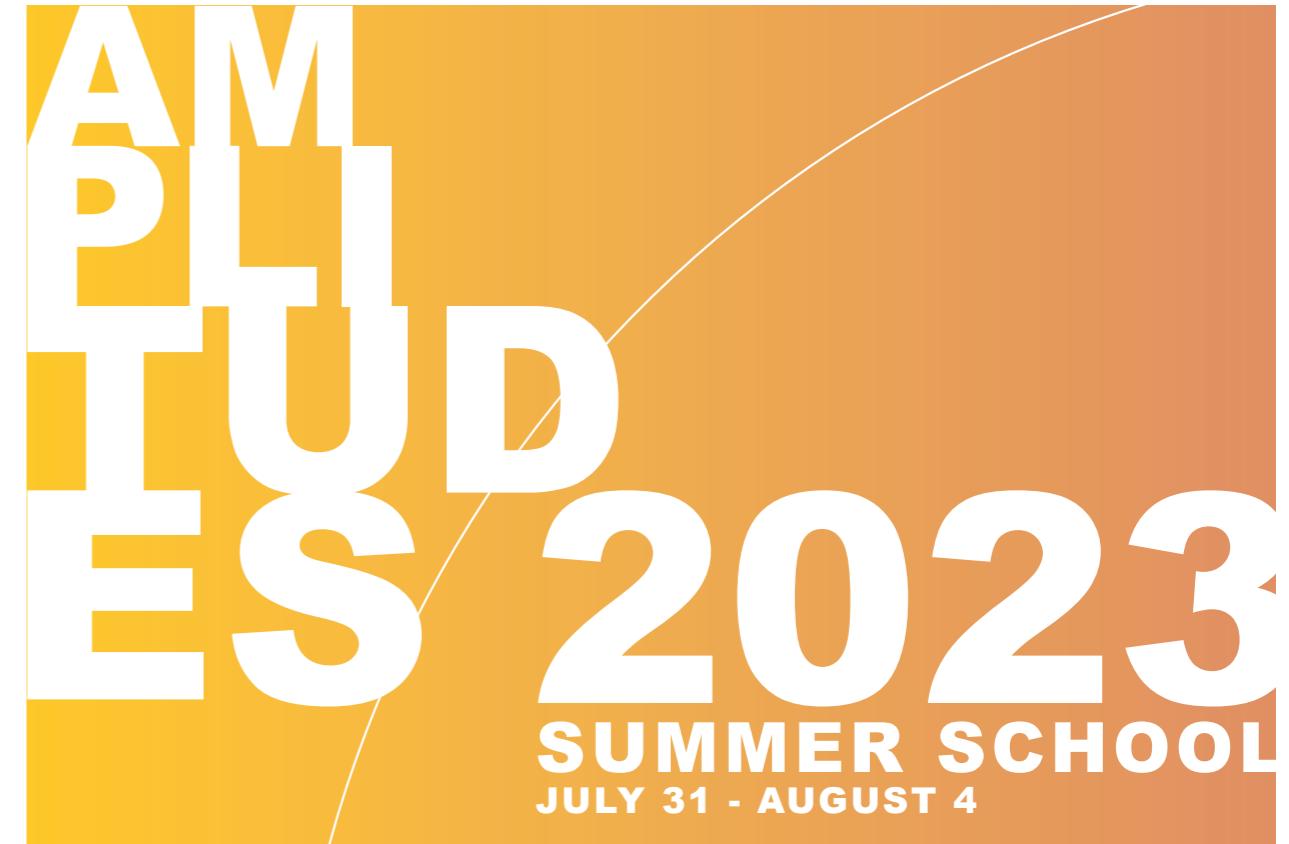
Sunday 28th May - Friday 2nd June, Crieff, Scotland



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Organisers: Andrew McLeod, Ben Page,
Lorenzo Tancredi, Samuel Abreu

Registration opens soon!