Exact non-singlet n_f^2 coefficient functions for DIS at four loops

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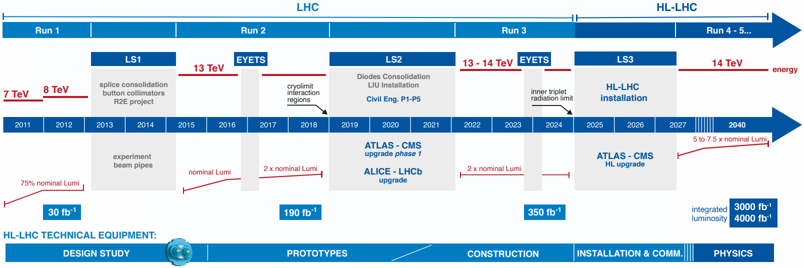
MOTIVATION

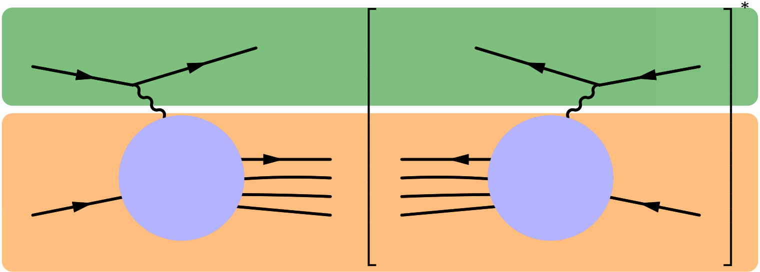
- The computation of contributions to the DIS coefficient functions represents the ideal testing ground for the possible applications for computing splitting functions.
- **Theoretical predictions** need to keep up with the ever-increasing precision of **experimental measurements**
- Need to understand the SM background in order to resolve **new physics**



- **Example:** Higgs inclusive: $8\% \to 3\%$ expected experimental uncertainty at 3000 fb^{-1}. The PDF uncertainty on the theoretical prediction cannot be neglected anymore.

DEEP INELASTIC SCATTERING

Probing the **hadron** structure by mean of high energetic **leptons** :



- Factorize the **leptonic** from the **hadronic** part in the cross-section

$$\frac{\sigma_{DIS}}{dxdy} = \frac{2\pi\alpha_{em}^2}{Q^2} L^{\mu\nu} W_{\mu\nu}$$

$$Q = -q^2, \qquad x = \frac{Q^2}{2P\cdot q}, \qquad y = \frac{P\cdot q}{P\cdot k}$$

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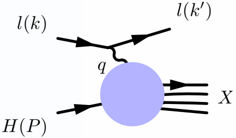
STRUCTURE FUNCTIONS

- The computation of the corresponding 4-loop Wilson coefficients is extremely challenging from the theoretical point of view

$$\frac{\sigma_{DIS}}{dxdy} = \frac{2\pi\alpha_{em}^2}{Q^2} L^{\mu\nu} W_{\mu\nu}$$

$$W_{\mu\nu} = \left(P^\mu - \frac{(P\cdot q)q_\mu}{q^2}\right)\left(P^\nu - \frac{(P\cdot q)q_\nu}{q^2}\right)\frac{\boxed{F_1(x,Q^2)}}{P\cdot q} + \left(-g_{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)\boxed{F_2(x,Q^2)} + i\epsilon_{\mu\nu\rho\sigma}\frac{P^\rho q^\sigma}{2P\cdot q}\boxed{F_3(x,Q^2)}$$



- Where the **hadronic** and **partonic** quantities are related via the **PDF**:

$$\boxed{F_a(x,Q^2)} = \sum_i \left[\boxed{f_i(\xi)} \otimes \boxed{\hat{F}_{a,i}(\xi,Q^2)}\right](x)$$

- The problem can be simplified by using the optical theorem for extracting the **Mellin moments** of the process.

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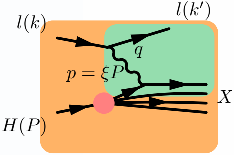

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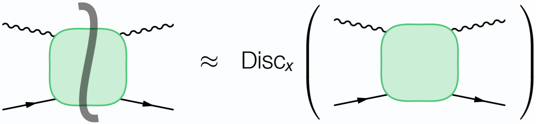
MELLIN MOMENTS

Objective:

- Compute the hadronic cross-section $\hat{W}_{\mu\nu}$ using the forward scatting $\hat{T}_{\mu\nu}$



How:

- Compute the **Mellin moments** of the structure functions:

$$\boxed{F_a(x,Q^2)} = \sum_i \left[\boxed{f_i(\xi)} \otimes \boxed{\hat{F}_{a,i}(\xi,Q^2)} \right](x)$$

with the **Mellin transform** defined by

$$M[f(x)](N) = \int_0^1 \mathrm{d}x\, x^{N-1} f(x)$$

- The Mellin moments of **cross-section** correspond to the expansion coefficients around $\omega = \frac{1}{x} = 0$ of the **Forward Scattering Amplitude** :

$$M[\hat{W}_{\mu\nu}](N) = \frac{1}{N!} \left[\frac{\mathrm{d}^N T_{\mu\nu}}{\omega^N} \right|_{\omega=0}$$

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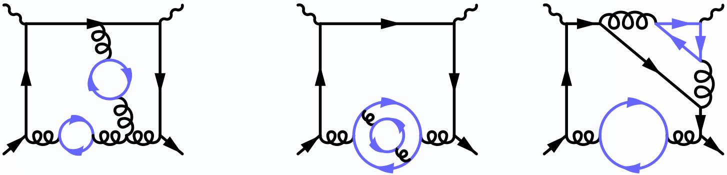
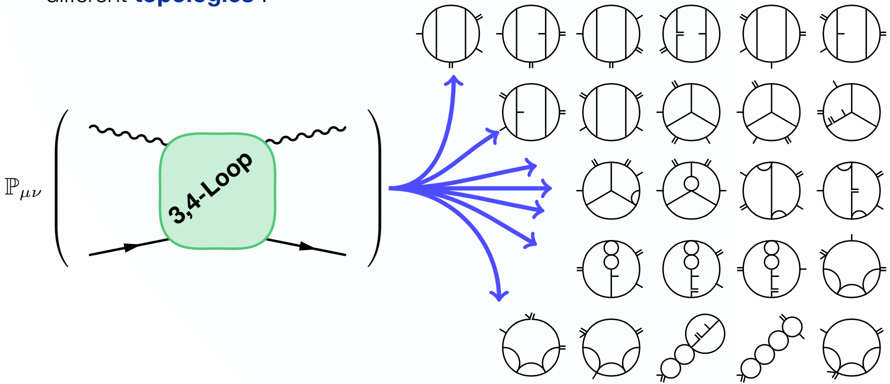
SYSTEM OF DIFFERENTIAL EQUATIONS

- We focus our attention on the non-singlet coefficient functions contribution of order **$[n_f^3, n_f^2]$** for $q + \gamma \rightarrow q + \gamma$:



SYSTEM OF DIFFERENTIAL EQUATIONS:

We process all the diagrams for the relevant process and cast them into 24 different **topologies** :



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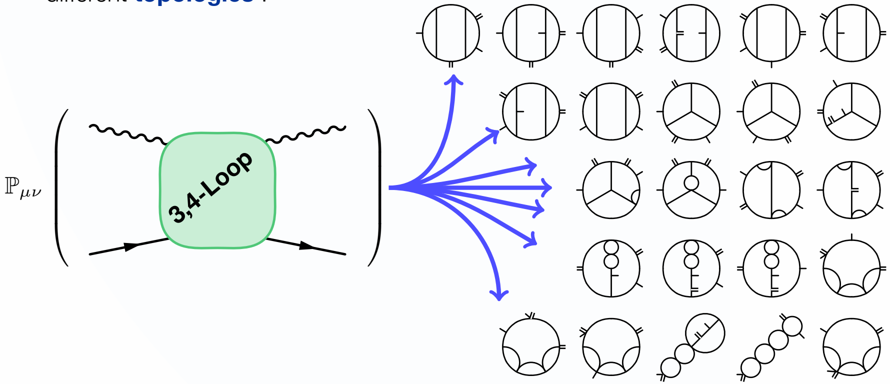

Still left with $\approx 10^5$ different integrals to be computed!

- We can resize the problem by using **IBP** relations among these integrals.
- Many publicly available programs to perform **reductions to master integrals** (e.g FIRE, Reduze, Kira) each with its strengths and weaknesses.

SYSTEM OF DIFFERENTIAL EQUATIONS

- Within each **topology** we perform a reduction to **master integrals** :

$$I^{(n)}(\omega, \epsilon) = \sum_i c_i(\omega, \epsilon) \cdot M_i^{(n)}(\omega, \epsilon), \qquad \omega = \frac{1}{x}$$

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Expand in ω

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Assuming:

- The DE matrix has at most a simple pole in ω:

$$A = \frac{A_{-1}}{w} + \sum_{k=0}^{\infty} A_k\omega^k$$

Note: Can always be done by applying a linear transformation T:

$$\vec{M}\to T\cdot\vec{M}, \qquad A \to \frac{\partial T}{\partial\omega}T^{-1} + T\cdot A\cdot T^{-1}$$

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$$\boxed{\underbrace{((k+1)\mathbb{1} - A_{-1})}_{:=B_k}\cdot\vec{m}_{k+1} = \sum_{j=0}^{k} A_j\vec{m}_{k-j}}$$

$\det(B_k)\neq 0$ $\qquad$ $\det(B_k) = 0$

Recursive Expression:

$$\vec{m}_{k+1} = B_k^{-1}\cdot\left(\sum_{j=0}^{k} A_j\vec{m}_{k-j}\right)$$

Gaussian Elimination:

Required by a **finite number** of k

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Mellin moments generation

$$\frac{\partial}{\partial\omega}\vec{M}(\omega,\epsilon) = A(\omega,\epsilon)\cdot\vec{M}(\omega,\epsilon).$$

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EXPANSION

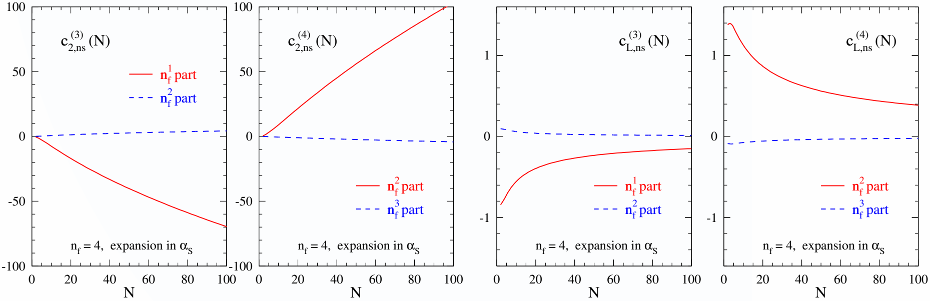
- We need to fix the **boundary** condition for $p \to 0$ (FORCER)
- In our case the **transformation** matrix T consists of a simple rescaling of the master integrals

$$T = \text{diag}\left(\omega^{\vec{a}}\right), \qquad \vec{a} \in \mathbb{N}_0^{\#\text{ of masters}}$$

- We can perform a simultaneous expansion in the **dimensional regulator** ϵ in order to speed up the computation
- Results can be expanded to high order in **Mellin moments**
- Faster than an expansion at the integrand level (Tensor decomposition)
- The **reductions** to master integrals remain the main bottleneck of the computation

MELLIN MOMENTS AT 4-LOOP

- Starting to explore the DIS expression for a simple subgroup **$[n_f^3, n_f^2]$** for $q + \gamma \to q + \gamma$:



- **Expansion** in ω is possible to **high orders** within a day:

$$\mathcal{O}\left(\omega^{1500}\right)$$

- Allows for the reconstruction of the **structure functions** in **x**-space at **all orders**

Harmonic series: $S_{\vec{m}}(N) \quad \rightarrow \quad$ Harmonic Polylogarithms: $H_{\vec{n}}(x)$

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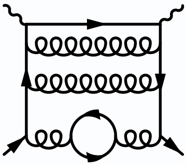
ADVENTURING FURTHER INTO 4-LOOP

The **new** goal is to push the same technique to new horizons:

- Consider the diagrams contributing to $C_F^3 n_f$



- Effectively 3-loop topologies with **bubble insertions**!
- The added degrees of freedom start to become a real problem for the reduction to **master integrals:**
  - Moving from 11 to 12 propagators
  - Higher powers in the numerator
- Implement a tailored reduction routine for this specific problem

SUMMARY

- Use IBP identities for a **reduction to master integrals** and build a system of differential equations
- **Transform** the system to allow for an efficient recursive expression for the extraction of the series coefficients
- Tested the method by computing high numbers of Mellin moments for the DIS **Wilson coefficients** $\hat{F}_L$, $\hat{F}_2$ and $\hat{F}_3$ at **3-loop**
- Generated 1500 Mellin moments for the non-singlet n_f^2 contribution at **4-loop** to obtain for the first time the corresponding **Wilson coefficients**

Future:

- Consider the additional diagrams necessary for the computation of $\hat{F}_3$ (11 extra topologies)
- Apply the same method for extracting Mellin moments at **4-loop** for the $C_F^3 n_f$ to extract the **splitting functions**

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Thank you!

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