STRONGLY-ORDERED INFRARED LIMITS FROM FACTORISATION

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Outline

- Introduction

- Infrared Counterterms
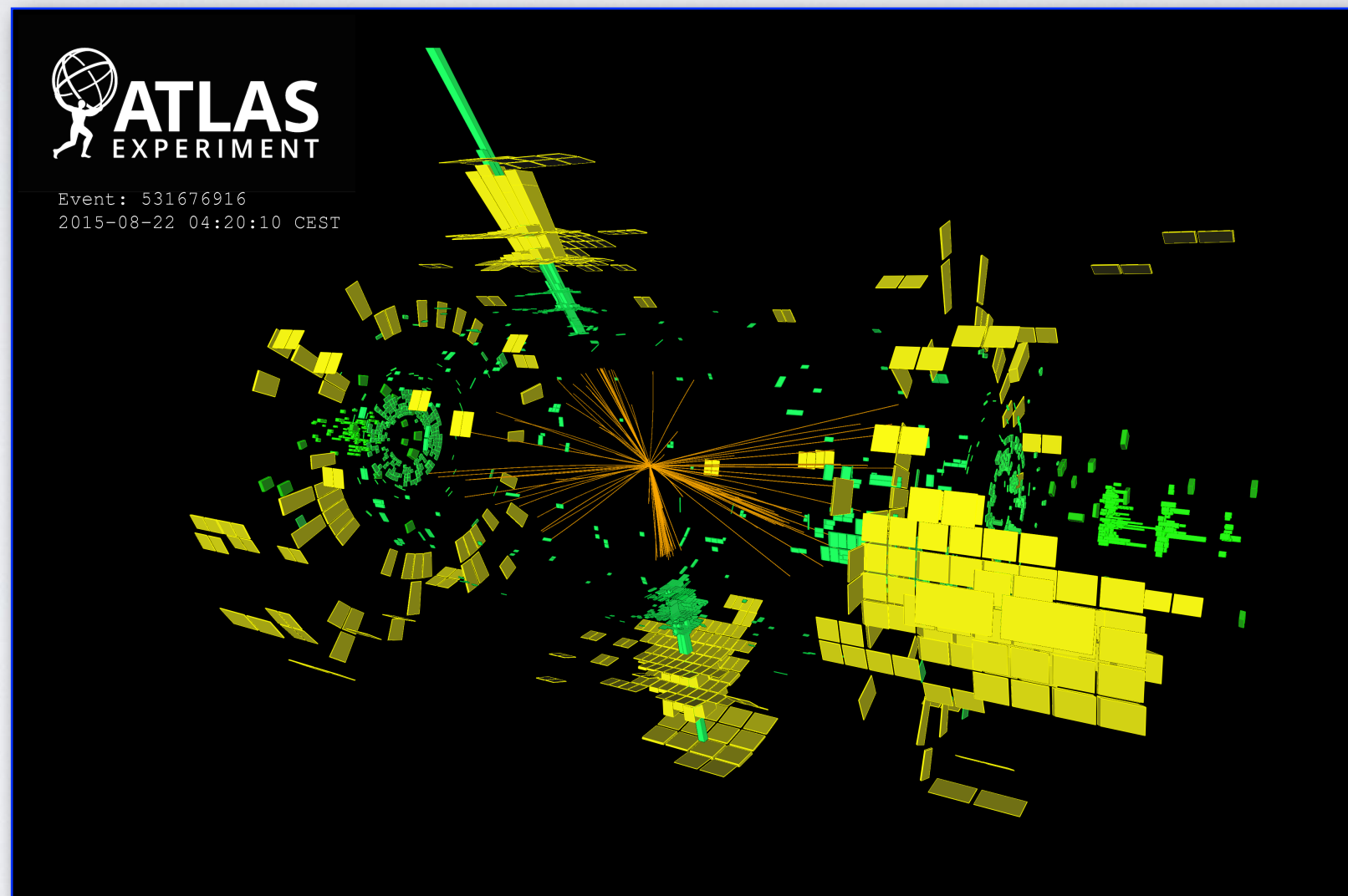
- Strong Ordering

- Outlook

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In collaboration with

Calum Milloy

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INTRODUCTION

The computation of a generic IRC-safe observable at NLO requires the combination

$$\frac{d\sigma_{\text{NLO}}}{dX} = \lim_{d \to 4} \left\{ \int d\Phi_n \, V_n \, \delta_n(X) + \int d\Phi_{n+1} \, R_{n+1} \, \delta_{n+1}(X) \right\},$$

The necessary numerical integrations require finite ingredients in d=4. Define counterterms

$$K_{n+1}^{(1)} = \mathbf{L}^{(1)} \, R_{n+1}\,.$$

$$I_n^{(1)} \equiv \int d\Phi_{\text{r},1}^{n+1} \, K_{n+1}^{(1)}\,,$$

Add and subtract the same quantity to the observable: each contribution is now finite.

$$\frac{d\sigma_{\text{NLO}}}{dX} = \int d\Phi_n \left(V_n + I_n^{(1)} \right) \delta_n(X) + \int d\Phi_{n+1} \left(R_{n+1} \, \delta_{n+1}(X) - K_{n+1}^{(1)} \, \delta_n(X) \right),$$

Search for the simplest fully local integrand K_{n+1} with the correct singular limits.

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Search for the simplest fully local integrand K_{n+1} with the correct singular limits.

Defining $\mathbf{L}^{(1)}$ with sectors

Minimize complexity: split phase space in sectors with sector function $\mathcal{W}_{ij}$ in order to have at most one soft (i) and one collinear (ij) singularity in each sector (FKS).

- Sector functions must form a partition of unity.

- In order not to appear in analytic integrations, sector functions must obey sum rules. Denoting with $\mathbf{S}_i$ the soft limit for parton i and $\mathbf{C}_{ij}$ the collinear limit for the ij pair,

$$\mathbf{S}_i \sum_{k \neq i} \mathcal{W}_{ik} = 1, \qquad \mathbf{C}_{ij} \sum_{ab \in \mathrm{perm}(ij)} \mathcal{W}_{ab} = 1, \qquad \longleftarrow \quad \text{sum rules}$$

- Sector functions are defined in terms of Lorentz invariants before choosing an explicit parametrisation of phase space. A possible choice is

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{k, l \neq k} \sigma_{kl}}, \qquad \text{with} \qquad \sigma_{ij} = \frac{1}{e_i \, w_{ij}}, \qquad e_i = \frac{s_{qi}}{s}, \qquad w_{ij} = \frac{s \, s_{ij}}{s_{qi} \, s_{qj}}.$$

- With the help of sector functions, one can now define a candidate counterterm

$$\mathbf{L}^{(1)} R_{n+1} = \sum_i \sum_{j \neq i} \left(\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij} \right) R_{n+1} \mathcal{W}_{ij}.$$

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Phase-space mappings at NLO

In order to factorise a Born matrix element B_n with n on-shell particles conserving momentum, we need a mapping from the (n+1)-particle to the Born phase spaces. We use (CS)

$$\bar{k}_i^{(abc)} = k_i, \quad \text{if } i \neq a, b, c,$$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac} + s_{bc}} k_c, \qquad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac} + s_{bc}} k_c,$$

We can now redefine soft and collinear limits to include the re-parametrisation. Explicitly

$$\overline{\mathbf{S}}_i \, R(\{k\}) = -\mathcal{N}_1 \sum_{l, m} \delta_{f_i g} \frac{s_{lm}}{s_{il} \, s_{im}} B_{lm}\left(\{\bar{k}\}^{(ilm)}\right),$$

$$\overline{\mathbf{C}}_{ij} \, R(k) = \frac{\mathcal{N}_1}{s_{ij}} \left[P_{ij} \, B\left(\{\bar{k}\}^{(ijr)}\right) + Q_{ij}^{\mu\nu} \, B_{\mu\nu}\left(\{\bar{k}\}^{(ijr)}\right) \right],$$

$$\overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij} \, R(\{k\}) = 2\mathcal{N}_1 \, C_{f_j} \, \delta_{f_i g} \frac{s_{jr}}{s_{ij} \, s_{ir}} B\left(\{\bar{k}\}^{(ijr)}\right),$$

Note that we have assigned parametrisation triplets differently in different terms. Then

$$\overline{K} = \sum_{i,j\neq i} \overline{K}_{ij}, \qquad \overline{K}_{ij} \equiv \left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i \, \overline{\mathbf{C}}_{ij}\right) R \, \mathcal{W}_{ij},$$

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The pattern of cancellations is more intricate at higher orders

$$
\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \lim_{d \to 4} \left\{ \int d\Phi_n \, VV_n \, \delta_n(X) + \int d\Phi_{n+1} \, RV_{n+1} \, \delta_{n+1}(X) \right.
$$

$$
\left. + \int d\Phi_{n+2} \, RR_{n+2} \, \delta_{n+2}(X) \right\},
$$

More counterterm functions need to be defined

$$
K_{n+2}^{(1)} = \mathbf{L}^{(1)} \, RR_{n+2} \,, \qquad K_{n+2}^{(2)} = \mathbf{L}^{(2)} \, RR_{n+2} \,, \qquad K_{n+2}^{(12)} = \mathbf{L}^{(1)} \mathbf{L}^{(2)} \, RR_{n+2} \,, \qquad K_{n+1}^{(\mathbf{RV})} = \widetilde{\mathbf{L}}^{(1)} \, RV_{n+1} \,.
$$

$$
I_{n+1}^{(1)} = \int d\Phi_{\mathrm{r},1}^{n+2} \, K_{n+2}^{(1)} \,, \quad I_{n+1}^{(12)} = \int d\Phi_{\mathrm{r},1}^{n+2} \, K_{n+2}^{(12)} \,, \quad I_n^{(2)} = \int d\Phi_{\mathrm{r},2}^{n+2} \, K_{n+2}^{(2)} \,, \quad I_n^{(\mathbf{RV})} = \int d\Phi_{\mathrm{r},1}^{n+1} \, K_{n+1}^{(\mathbf{RV})} \,.
$$

A finite expression for the observable in d=4 must combine several ingredients

$$
\frac{d\sigma_{\mathrm{NNLO}}}{dX} = \int d\Phi_n \left[VV_n + I_n^{(2)} + I_n^{(\mathbf{RV})} \right] \delta_n(X)
$$

$$
+ \int d\Phi_{n+1} \left[\left(RV_{n+1} + I_{n+1}^{(1)} \right) \delta_{n+1}(X) - \left(K_{n+1}^{(\mathbf{RV})} + I_{n+1}^{(12)} \right) \delta_n(X) \right]
$$

$$
+ \int d\Phi_{n+2} \left[RR_{n+2} \, \delta_{n+2}(X) - K_{n+2}^{(1)} \, \delta_{n+1}(X) - \left(K_{n+2}^{(2)} - K_{n+2}^{(12)} \right) \delta_n(X) \right]
$$

A systematic generalisation to higher orders is possible. At three loops one finds

$$
\begin{aligned}
\frac{d\sigma_{\mathrm{N3LO}}}{dX} =\ & \int d\Phi_n \left[VVV_n + I_n^{(3)} + I_n^{(\mathbf{RVV})} + I_n^{(\mathbf{RRV},\mathbf{2})} \right] \delta_n(X) \\
& + \int d\Phi_{n+1} \left[\left(RVV_{n+1} + I_{n+1}^{(2)} + I_{n+1}^{(\mathbf{RRV},\mathbf{1})} \right) \delta_{n+1}(X) \right. \\
& \qquad\qquad\qquad \left. - \left(K_{n+1}^{(\mathbf{RVV})} + I_{n+1}^{(\mathbf{23})} + I_{n+1}^{(\mathbf{RRV},\mathbf{12})} \right) \delta_n(X) \right] \\
& + \int d\Phi_{n+2} \left\{ \left(RRV_{n+2} + I_{n+2}^{(1)} \right) \delta_{n+2}(X) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{1})} + I_{n+2}^{(\mathbf{12})} \right) \delta_{n+1}(X) \right. \\
& \qquad\qquad\qquad \left. - \left[\left(K_{n+2}^{(\mathbf{RRV},\mathbf{2})} + I_{n+2}^{(\mathbf{13})} \right) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{12})} + I_{n+2}^{(\mathbf{123})} \right) \right] \delta_n(X) \right\} \\
& + \int d\Phi_{n+3} \left[RRR_{n+3}\, \delta_{n+3}(X) - K_{n+3}^{(1)}\, \delta_{n+2}(X) - \left(K_{n+3}^{(2)} - K_{n+3}^{(\mathbf{12})} \right) \delta_{n+1}(X) \right. \\
& \qquad\qquad\qquad \left. - \left(K_{n+3}^{(3)} - K_{n+3}^{(\mathbf{13})} - K_{n+3}^{(\mathbf{23})} + K_{n+3}^{(\mathbf{123})} \right) \delta_n(X) \right] ,
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A general formula for N^kLO subtraction is available, involving $p = 2^{(k+1)} - 2 - k$ counterterms.

A systematic generalisation to higher orders is possible. At three loops one finds

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A general formula for N^{k}LO subtraction is available, involving $p = 2^{(k+1)} - 2 - k$ counterterms.

LASST status

- So far the formalism is developed for massless partons.

- At NLO we have a full-fledged subtraction formalism, and simple integrals.

- NLO numerical implementation is under way, including optimisation with damping factors.

- At NNLO Local Analytic Subtraction has been achieved for final state radiation.

  - A complete set of NNLO sector functions with the desired sum rules is available.

  - Flexible phase space mappings for single and double unresolved limits exist.

  - Phase space mappings have been checked not to misalign nested limits.

  - All integrations are done analytically, including finite parts, without IBP techniques.

- The numerical implementation at NNLO is the natural next step. Off the shelf?

- Generalisation to initial state radiation requires work (under way) but no new concepts.

- More `interesting' integrals may arise with massive partons.

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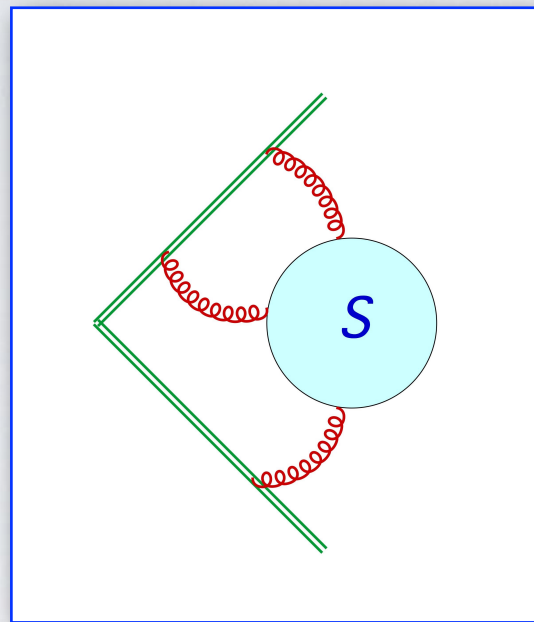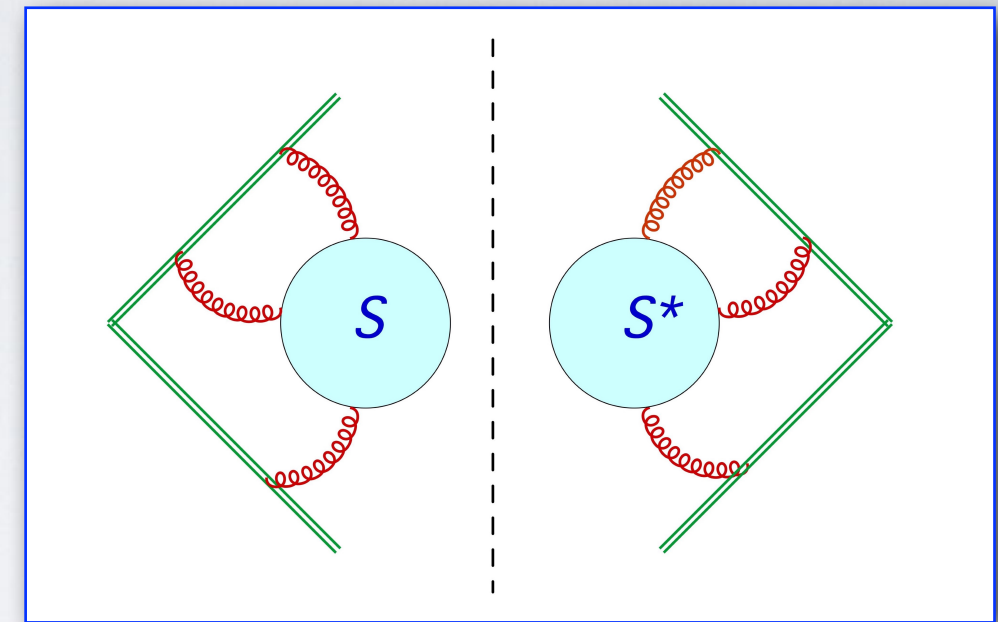
At amplitude level poles factorise and exponentiate.
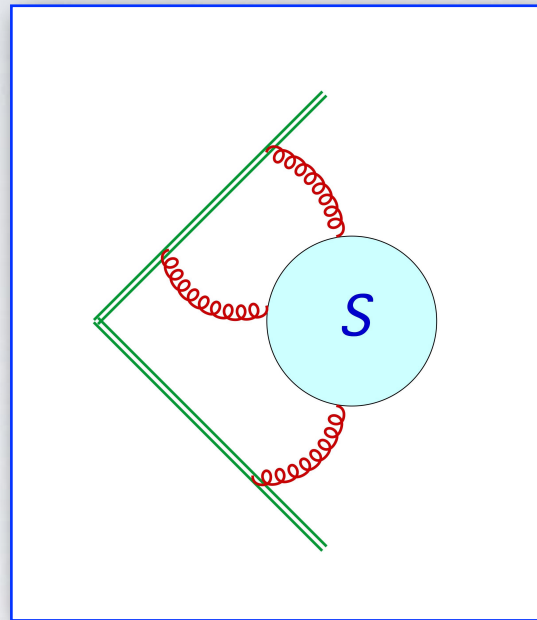
Soft cross sections: pictorial

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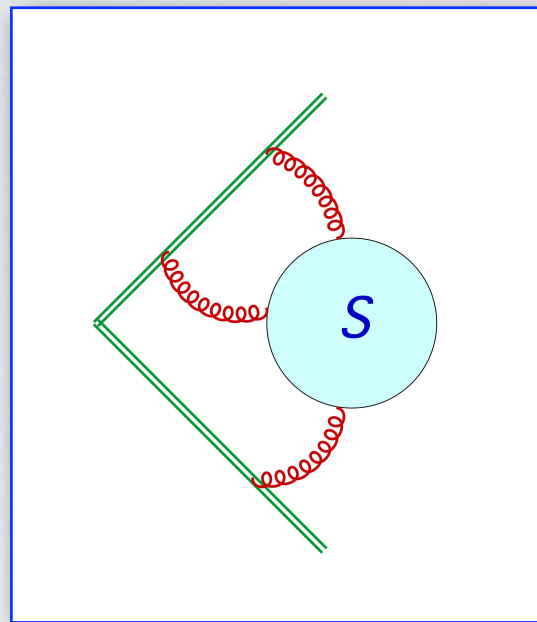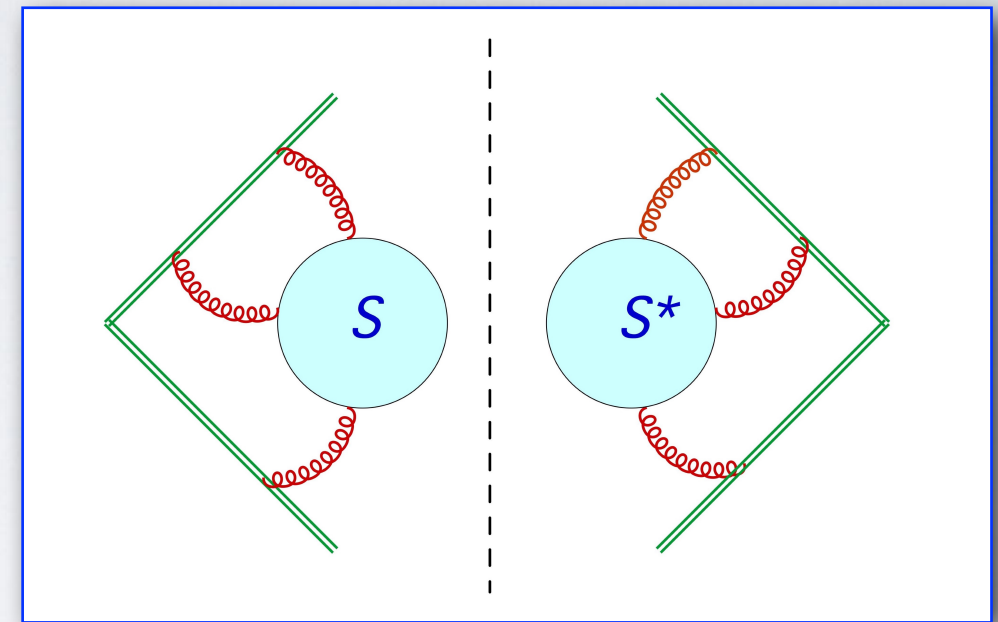At **amplitude level** poles **factorise** and **exponentiate**.

We need to build **cross-section level** quantities.

Soft cross sections: pictorial

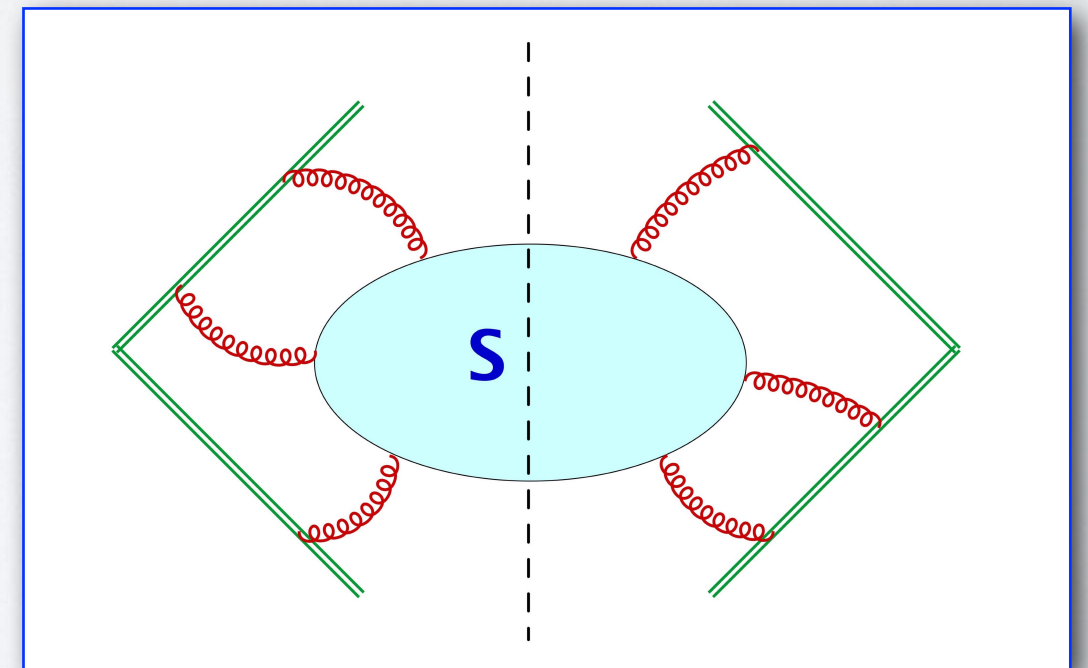Consider first the (academic) case of purely soft final state divergences.

At amplitude level poles factorise and exponentiate.

We need to build cross-section level quantities.

- Inclusive eikonal cross sections are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local soft counterterms.

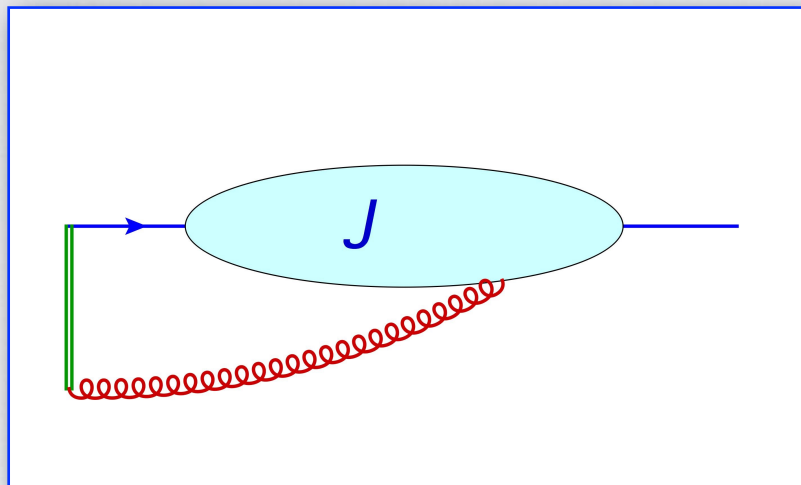Consider next collinear final state divergences. They are associated with individual partons.

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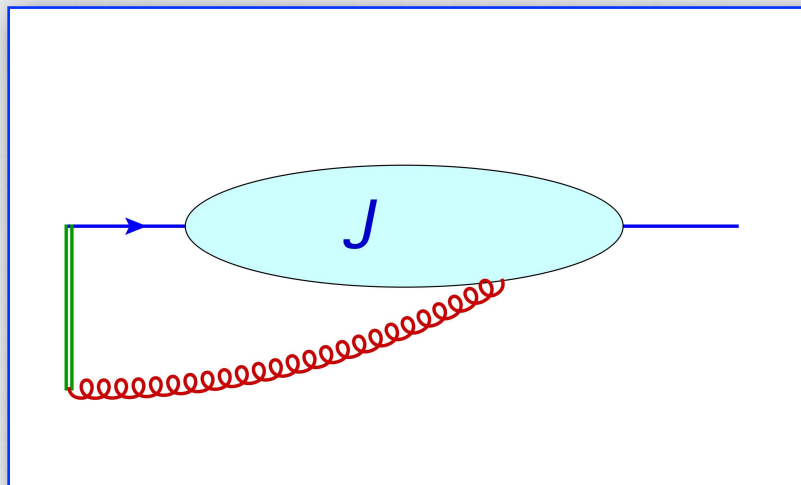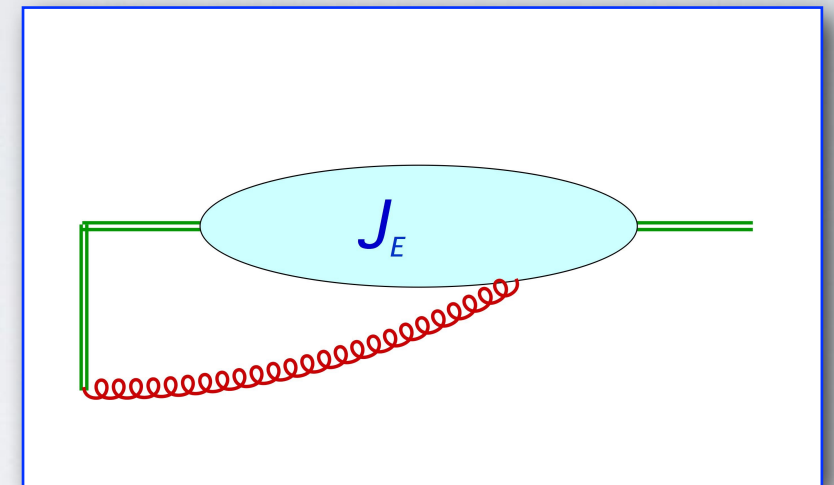

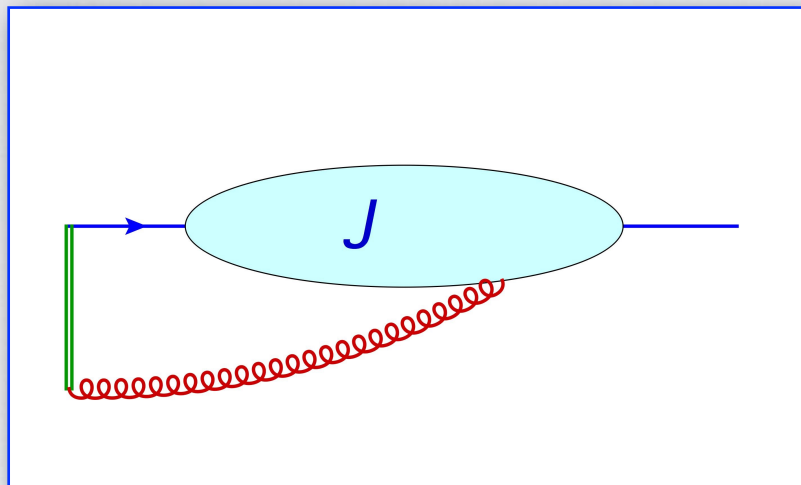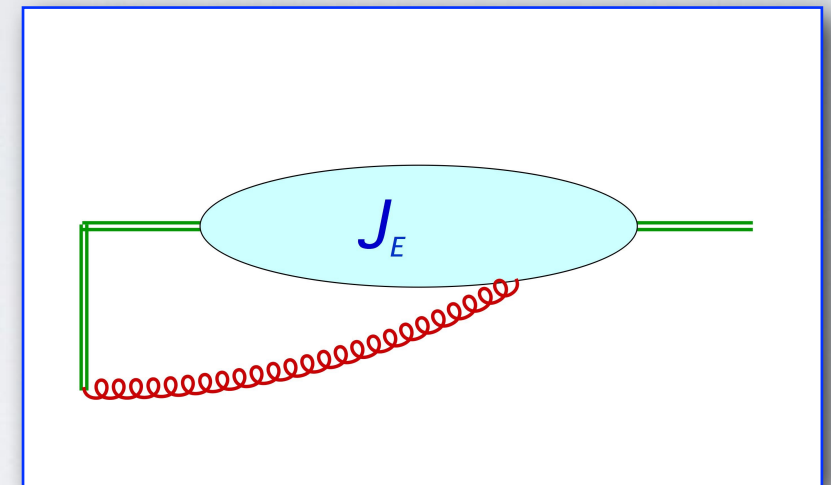Soft-collinear poles can be subtracted

Collinear cross sections: pictorial

Consider next collinear final state divergences. They are associated with individual partons.

At amplitude level poles factorise and exponentiate.

Soft-collinear poles can be subtracted

- Inclusive `jet cross sections' are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local collinear counterterms.
- Eikonal jet cross sections subtract the soft-collinear double counting.
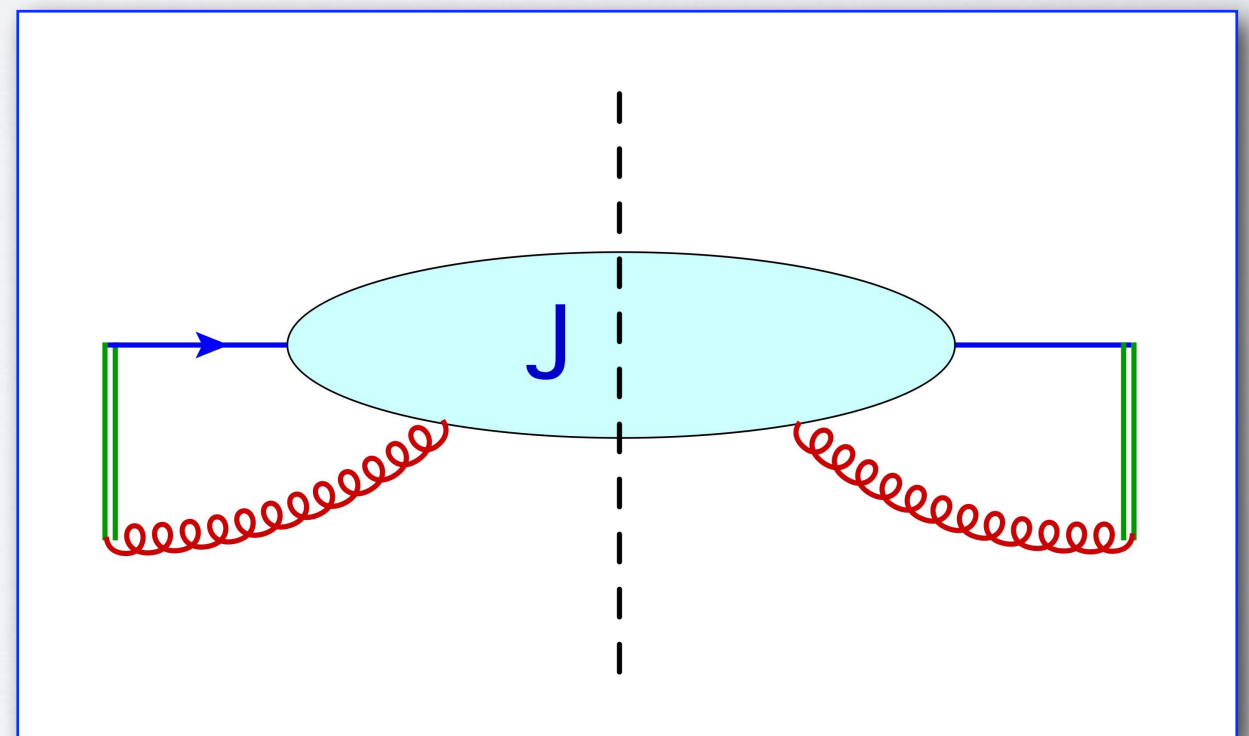
Soft counterterms: all orders

Introduce **eikonal form factors** for the emission of **m soft** partons from **n hard** ones.

$$\mathcal{S}_{n,m}(k_1, \ldots, k_m; \beta_i) \equiv \langle k_1, \lambda_1; \ldots; k_m, \lambda_m | \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0) |0\rangle$$

$$\equiv \epsilon_{\mu_1}^{*(\lambda_1)}(k_1) \ldots \epsilon_{\mu_m}^{*(\lambda_m)}(k_m) \, J_{\mathcal{S}}^{\mu_1 \ldots \mu_m}(k_1, \ldots, k_m; \beta_i)$$

$$\equiv \sum_{p=0}^{\infty} \mathcal{S}_{n,m}^{(p)}(k_1, \ldots, k_m; \beta_i)$$

These matrix elements **define** soft gluon **multiple emission currents**. They are **gauge invariant** and they contain **loop corrections** to all orders.

Existing finite order **calculations** and all-order **arguments** are **consistent** with the **factorisation**

$$\mathcal{A}_{n,m}(k_1, \ldots, k_m; p_i) = \mathcal{S}_{n,m}(k_1, \ldots, k_m; \beta_i) \, \mathcal{H}_n(p_i) + \mathcal{R}_{n,m}(k_1, \ldots, k_m; p_i)$$

with **corrections** that are **finite** in dimensional regularisation, and **integrable** in the soft gluon phase space. It is a **working assumption**: a formal all-order proof is still **lacking**.

Soft counterterms: all orders

The factorisation is reflected at cross-section level, for fixed final state quantum numbers.

$$\sum_{\lambda_i} |\mathcal{A}_{n,m}(k_1,\ldots,k_m;p_i)|^2 \simeq \mathcal{H}_n^\dagger(p_i)\, S_{n,m}(k_1,\ldots,k_m;\beta_i)\, \mathcal{H}_n(p_i)$$

The *cross-section level* "radiative soft functions" are Wilson-line squared matrix elements

$$S_{n,m}\big(\{k_m\},\{\beta_i\}\big) \equiv \sum_{p=0}^{\infty} S_{n,m}^{(p)}\big(\{k_m\},\{\beta_i\}\big)$$

$$\equiv \sum_{\{\lambda_i\}} \langle 0|\, \overline{T}\left[\prod_{i=1}^{n}\Phi_{\beta_i}(0,\infty)\right] |k_1,\lambda_1;\ldots;k_m,\lambda_m\rangle \langle k_1,\lambda_1;\ldots;k_m,\lambda_m|\, T\left[\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\right] |0\rangle\,,$$

These functions provide a complete list of local soft subtraction counterterms, to all orders. Indeed, summing over particle numbers and integrating over the soft phase space one finds

$$\sum_{m=0}^{\infty} \int d\Phi_m\, S_{n,m}\big(\{k_m\};\{\beta_i\}\big) = \langle 0|\, \overline{T}\left[\prod_{i=1}^{n}\Phi_{\beta_i}(0,\infty)\right] T\left[\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\right] |0\rangle\,.$$

"Completeness relation"

This is a finite fully inclusive soft cross section, order by order in perturbation theory.

Collinear counterterms: all orders

For collinear poles, introduce jet matrix elements for the emission of m partons. For quarks

$$
\overline{u}_s(p)\,\mathcal{J}_{q,m}\left(k_1,\ldots,k_m;p,n\right) \equiv \langle p,s;k_1,\lambda_1;\ldots;k_m,\lambda_m|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle
$$

At *cross-section level*, "radiative jet functions" can be defined as Fourier transforms of squared matrix elements, to account for the non-trivial momentum flow. We propose

$$
J_{q,m}\big(\{k_m\};l,p,n\big) \equiv \sum_{p=0}^{\infty} J_{q,m}^{(p)}\big(\{k_m\};l,p,n\big)
$$

$$
\equiv \int d^d x\,\mathrm{e}^{il\cdot x} \sum_{\{\lambda_m\}} \langle 0|\,\overline{T}\Big[\Phi_n(\infty,x)\,\psi(x)\Big]\,|p,s;\{k_m,\lambda_m\}\rangle\,\langle p,s;\{k_m,\lambda_m\}|\,T\Big[\overline{\psi}(0)\,\Phi_n(0,\infty)\Big]\,|0\rangle\,,
$$

These functions provide a complete list of local collinear counterterms, to all orders. Summing over particle numbers and integrating over the collinear phase space one finds

$$
\sum_{m=0}^{\infty} \int d\Phi_{m+1}\,J_{q,m}\big(\{k_m\};l,p,n\big) = \mathrm{Disc}\left[\int d^d x\,\mathrm{e}^{il\cdot x}\,\langle 0|\,T\Big[\Phi_n(\infty,x)\psi(x)\overline{\psi}(0)\Phi_n(0,\infty)\Big]\,|0\rangle\right].
$$

A "two-point function", finite order by order in perturbation theory. Note however

- The collinear limit must still be taken (as $l^2 \to 0$), unlike the case of radiative soft functions.

- $n^2 \neq 0$ avoids spurious collinear poles, but is cumbersome ➜ use SCET-like anti-collinear n^μ.

$$A_n(p_i) = \prod_{i=1}^{n} \left[\frac{\mathcal{J}_i(p_i, n_i)}{\mathcal{J}_{E,i}(\beta_i, n_i)} \right] \mathcal{S}_n(\beta_j) \mathcal{H}_n(p_i)$$

NLO subtraction

The outlines of a subtraction procedure emerge. Begin by expanding the virtual matrix element

$$\mathcal{A}_n(p_i) = \left[\mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) \right.$$
$$\left. + \sum_{i=1}^{n} \left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{E,i}^{(1)}(\beta_i) \right) \mathcal{S}_n^{(0)}(\beta_i)\, \mathcal{H}_n^{(0)}(p_i) \right] \left(1 + \mathcal{O}\left(\alpha_s^2\right) \right)$$

From the master formula, get the virtual poles of the cross section in terms of virtual kernels

$$V_n \equiv 2\,\mathbf{Re}\left[\mathcal{A}_n^{(0)*} \mathcal{A}_n^{(1)} \right] \simeq \mathcal{H}_n^{(0)\,\dagger}(p_i) S_{n,0}^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \sum_i \left(J_{i,0}^{(1)}(p_i) - J_{E,i,0}^{(1)}(\beta_i) \right) \left| \mathcal{A}_n^{(0)}(p_i) \right|^2$$

Go through the list of proposed soft and collinear counterterms to collect the relevant ones

$$S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1 \, S_{n,1}^{(0)}(k, \beta_i) = \text{finite}$$

$$J_{i,0}^{(1)}(l, p, n) + \int d\Phi_1 \, J_{i,1}^{(0)}(k; l, p, n) = \text{finite}$$

Construct the appropriate local functions.

↑ Mapping required! ↑

$$K_{n+1}^{\mathrm{NLO,\,s}} = \mathcal{H}_n^{(0)\,\dagger}(p_i)\, S_{n,1}^{(0)}(k, \beta_i)\, \mathcal{H}_n^{(0)}(p_i)$$

$$K_{n+1}^{\mathrm{NLO,\,c}} = \sum_{i=1}^{n} J_{i,1}^{(0)}(k_i; l, p_i, n_i) \left| \mathcal{A}_n^{(0)}(p_1, \dots, p_{i-1}, l, p_{i+1}, \dots, p_n) \right|^2$$

with a similar expression for the anti-subtraction of the soft-collinear region in terms of $\mathcal{J}_{\mathcal{E}}$.

STRONG ORDERING

The tree-level **double soft-gluon current** simplifies considerably in the **strong-ordering** limit

$$\left[J_{\text{CG}}^{(0),\,\text{s.o.}}\right]_{\mu_1\mu_2}^{a_1 a_2}(k_1, k_2; \beta_i) = \left(J_{\mu_2}^{(0)\,a_2}(k_2)\,\delta^{a_1 a} + \text{i}g_s\,f^{a_1 a_2 a}\,\frac{k_{1,\mu_2}}{k_1 \cdot k_2}\right)J_{\mu_1,\,a}^{(0)}(k_1),$$

$$J_{\mu}^{(0)\,a}(k) = g_s \sum_{i=1}^{n}\frac{\beta_{i,\mu}}{\beta_i \cdot k}\,T_i^a$$

One may define a **strongly-ordered soft form factor** by contracting with physical **polarisations**

$$\left[\mathcal{S}_{n;1,1}^{(0)}\right]^{a_1 a_2}(k_1, k_2; \beta_i) = \epsilon^{*\,\mu_1}(k_1)\,\epsilon^{*\,\mu_2}(k_2)\left[J_{\text{CG}}^{(0),\,\text{s.o.}}\right]_{\mu_1\mu_2}^{a_1 a_2}(k_1, k_2; \beta_i).$$

The form factor is given by an interesting **"re-factorisation"** of the **double-radiative soft function**

$$\begin{aligned}
\left[\mathcal{S}_{n;1,1}^{(0)}\right]_{\{d_i e_i\}}^{a_1 a_2}(k_1, k_2; \beta_i) &\equiv \langle k_2, a_2|\,\Phi_{\beta_{k_1}}^{a_1 b}(0, \infty)\prod_{i=1}^{n}\Phi_{\beta_i,\,d_i}^{c_i}(0, \infty)\,|0\rangle\\
&\times \langle k_1, b|\prod_{i=1}^{n}\Phi_{\beta_i,\,c_i e_i}(0, \infty)\,|0\rangle\Big|_{\text{tree}}\\
&= \left[\mathcal{S}_{n+1,1}^{(0)}\right]_{\{d_i c_i\}}^{a_2,\,a_1 b}(k_2; \beta_{k_1}, \beta_i)\left[\mathcal{S}_{n,1}^{(0)}\right]_{b,\,\{c_i e_i\}}(k_1; \beta_i),
\end{aligned}$$

Notice the **non-trivial colour structure**: the product is **ordered**.

The **original** system of **n** Wilson lines **radiates** the **harder** gluon, which then **"Wilsonises"**. The **augmented** system of **(n+1)** Wilson lines **radiates** the **softer** gluon
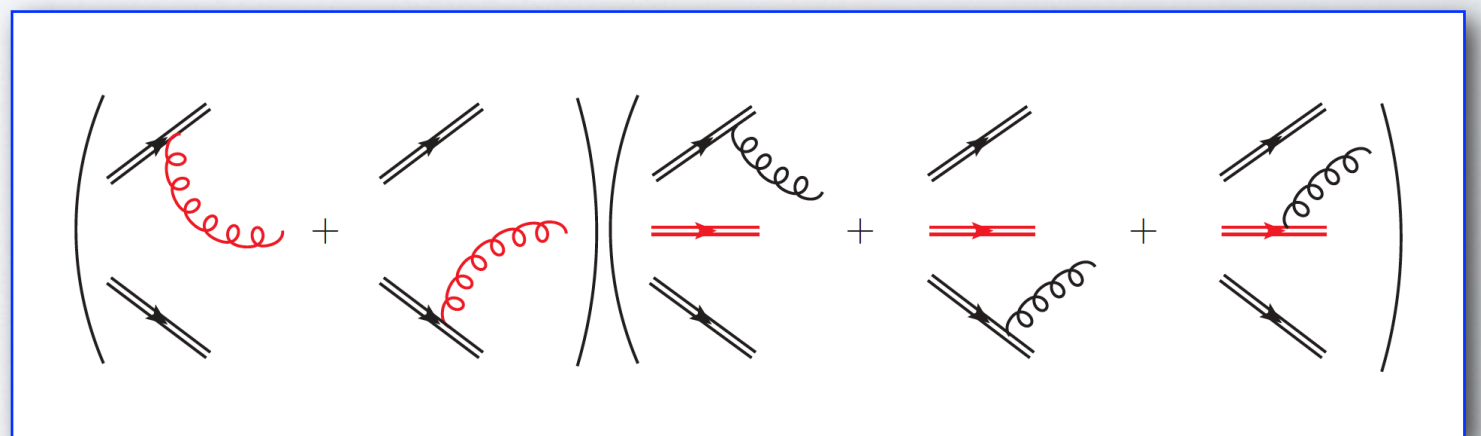
Soft refactorisation: tree level

This framework **generalises** to **arbitrary patterns of strong ordering** for **multiple soft radiation** at **tree level**. For example for strongly-ordered **triple radiation** one can define

$$
\begin{aligned}
\left[\mathcal{S}_{n;1,1,1}^{(0)}\right]_{\{f_i e_i\}}^{a_1 a_2 a_3} (k_1, k_2, k_3; \beta_i) \;\equiv\; & \left[\mathcal{S}_{n+2,1}^{(0)}\right]_{\{f_i d_i\}, a_1 b_1, a_2 b_2}^{a_3} \left[\mathcal{S}_{n+1,1}^{(0)}\right]_{\{d_i c_i\}, b_1 g_1}^{b_2} \left[\mathcal{S}_{n,1}^{(0)}\right]_{\{c_i e_i\}}^{g_1} \\
=\;& \langle k_3, a_3 | \, \Phi_{\beta_{k_1}}^{a_1 b_1}(0, \infty) \, \Phi_{\beta_{k_2}}^{a_2 b_2}(0, \infty) \prod_{i=1}^{n} \Phi_{\beta_i}^{f_i d_i}(0, \infty) |0\rangle \\
& \times \langle k_2, b_2 | \, \Phi_{\beta_{k_1}}^{b_1 g_1}(0, \infty) \prod_{i=1}^{n} \Phi_{\beta_i}^{d_i c_i}(0, \infty) |0\rangle \\
& \times \langle k_1, g_1 | \prod_{i=1}^{n} \Phi_{\beta_i}^{c_i e_i}(0, \infty) |0\rangle \Big|_{\text{tree}},
\end{aligned}
$$

Computing the **form factors**, one **reproduces** the strongly-ordered **limit** of (Catani et al. 2019).

$$
\begin{aligned}
\left[\mathcal{S}_{n;1,1,1}^{(0)}\right]^{a_1 a_2 a_3} \;=\;& \epsilon_{\mu_3}^*(k_3) \, \epsilon_{\mu_2}^*(k_2) \, \epsilon_{\mu_1}^*(k_1) \\
& \times \left[J_{a_3}^{\mu_3}(k_3) \, \delta^{a_1 b_1} \, \delta^{a_2 b_2} + \mathrm{i} g_s \, f^{a_1 a_3 b_1} \, \delta^{a_2 b_2} \frac{k_1^{\mu_3}}{k_1 \cdot k_3} + \mathrm{i} g_s \, f^{a_2 a_3 b_2} \, \delta^{a_1 b_1} \frac{k_2^{\mu_3}}{k_2 \cdot k_3} \right] \\
& \times \left[J_{b_2}^{\mu_2}(k_2) \, \delta^{b_1 c_1} + \mathrm{i} g_s \, f^{b_1 b_2 c_1} \frac{k_1^{\mu_2}}{k_1 \cdot k_2} \right] J_{c_1}^{\mu_1}(k_1),
\end{aligned}
$$

- **Generalising** to strongly-ordered soft radiation of **m gluons** is **natural** (and **tested** for **m=3**).
- **Similar definitions** hold for soft form factors for **multiple ordered subsets of several gluons**.
- **Preliminary evidence** suggests that similar **soft re-factorisations** may hold to **higher orders**.

Strongly-ordered soft counterterms

The **top-down** approach **suggests** an expression for the **soft real-virtual counterterm**

$$K_{n+1}^{(\mathbf{RV}),\mathrm{s}} = \mathcal{H}_n^{(0)\,\dagger} S_{n,1}^{(1)} \mathcal{H}_n^{(0)} + \text{finite}$$

Collinear poles?

The **refactorisation** of strongly-ordered soft radiation **suggests** an expression for the **soft K$^{(12)}$**

$$
\begin{aligned}
K_{n+2}^{(\mathbf{12}),\mathrm{s}} &= \mathcal{H}_n^{(0)\,\dagger} S_{n,1,1}^{(0)} \mathcal{H}_n^{(0)} \\
&= \mathcal{H}_n^{(0)\,\dagger} \left[\mathcal{S}_{n,1}^{b,(0)}(\beta_i; k_1) \right]^{\dagger} \left[\mathcal{S}_{n+1,1}^{a_2, a_1 b\,(0)}(\beta_i, \beta_{k_1}; k_2) \right]^{\dagger} \mathcal{S}_{n+1,1}^{a_2, a_1 c,(0)}(\beta_i, \beta_{k_1}; k_2)\, \mathcal{S}_{n,1}^{c,(0)}(\beta_i; k_1) \mathcal{H}_n^{(0)} \\
&\equiv \mathcal{H}_n^{(0)\,\dagger} \left[\mathcal{S}_{n,1}^{b}(\beta_i; k_1) \right]^{\dagger} S_{n+1,1}^{bc,(0)}(\beta_i, \beta_{k_1}; k_2)\, \mathcal{S}_{n,1}^{c}(\beta_i; k_1) \mathcal{H}_n^{(0)}
\end{aligned}
$$

One can now use the **finiteness of inclusive soft cross sections** to **cancel soft poles** arising from the **phase-space integration** of K$^{(12)}$, using

$$S_{n+1,0}^{bc,(1)}(\beta_i, \beta_{k_1}) + \int d\Phi_1(k_2)\, S_{n+1,1}^{bc,(0)}(\beta_i, \beta_{k_1}; k_2) = \text{finite}$$

"Completeness relation"

This gives a **new expression** for the **real-virtual soft** counterterm

$$K_{n+1}^{(\mathbf{RV}),\mathrm{s}} = \mathcal{H}_n^{(0)\,\dagger} \left[\mathcal{S}_{n,1}^{b,(0)}(\beta_i; k_1) \right]^{\dagger} S_{n+1,0}^{bc,(1)}(\beta_i, \beta_{k_1})\, \mathcal{S}_{n,1}^{c,(0)}(\beta_i; k_1) \mathcal{H}_n^{(0)} + \text{finite}$$

A "bottom-up" approach

The **two** definitions have **identical soft poles**, which was checked with a **non-trivial** calculation.

A top-down approach

This result is better understood by taking more seriously the idea of refactorisation

- The radiative soft function is not a pure counterterm: it has IR poles and finite contributions.
- It can be considered as an amplitude in the presence of sources: virtual IR poles will factorise.

Applying the standard soft-jet-hard factorisation for scattering amplitudes we write

$$\mathcal{A}_n(p_i) = \prod_{i=1}^{n} \left[\frac{\mathcal{J}_i(p_i, n_i)}{\mathcal{J}_{E,i}(\beta_i, n_i)} \right] \mathcal{S}_n(\beta_j) \, \mathcal{H}_n(p_i) \quad \to \quad \mathcal{S}_{n,1}(k; \beta_i) = \frac{\mathcal{J}_g(k, n)}{\mathcal{J}_{E,g}(\beta_k, n)} \, \mathcal{S}_{n+1,0}(\beta_k, \beta_i) \, \mathcal{S}_{n,1}^{\mathrm{fin}}(k; \beta_i)$$

Expanding to one-loop order, the terms containing IR poles are

$$\mathcal{S}_{n,1}^{(1)}(k; \beta_i) = \mathcal{S}_{n+1,0}^{(1)}(\beta_k, \beta_i) \, \mathcal{S}_{n,1}^{(0)}(k; \beta_i) + \left(\mathcal{J}_g^{(1)}(k, n) - \mathcal{J}_{E,g}^{(1)}(\beta_k, n) \right) \mathcal{S}_{n,1}^{(0)}(k; \beta_i)$$

We recognise (upon squaring) the soft contribution to K(RV), plus hard collinear corrections.

This can be explicitly checked against the general expression for the soft limit of RV

$$S_{n,1}^{(1)}(k; \beta_i) = \mathbf{S}_k RV - \frac{\alpha_{\mathrm{s}}^2 \mu^{2\epsilon}}{S_\epsilon} \sum_{i>j}^{n} \frac{\beta_i \cdot \beta_j}{\beta_i \cdot k \, \beta_j \cdot k} \, \mathbf{T}_i \cdot \mathbf{T}_j \left[\sum_{m=1}^{n} \frac{\gamma_m^{(1)}}{\epsilon} + \frac{b_0}{2\epsilon} \right]$$

To match the two calculations, one must subtract the hard-collinear poles of the virtual part.

Collinear refactorisation

The top-down approach suggests an expression for the collinear real-virtual counterterm

$$K_{n+1}^{(\mathbf{RV}),\,c,\,i} = \mathcal{H}_n^{(0)\,\dagger}\,J_{i,1}^{(1)}\,\mathcal{H}_n^{(0)} + \text{finite}$$

Soft poles?

In the bottom-up approach one starts with strongly-ordered collinear kernels, for example

$$\lim_{\theta_{12}\ll\theta_{13}\ll 1} RR_{n+2} = \frac{\mathcal{N}^2}{s_{12}\,s_{[12]3}}\,P_{gq}^{\alpha\beta}\left(z_{[12]},q_\perp\right)d_{\alpha\mu}\left(k_{[12]},n\right)P_{q\bar{q}}^{\mu\nu}\left(\frac{z_1}{z_{[12]}},k_\perp\right)d_{\nu\beta}\left(k_{[12]},n\right),$$

q → q q q̄

This can be directly translated in the language of jet functions. At cross-section level

$$J_{q,1,1}^{(0)}\left(k_1,k_2;k_3,n\right)\Big|_{gg,\,\text{ab.}} = J_{q,1}^{(0)}\left(k_1;k_{[23]},n\right)\,J_{q,1}^{(0)}\left(k_2;k_3,n\right),$$

q → q g g , abelian

One can now use the finiteness of inclusive collinear cross sections to cancel collinear poles arising from the phase-space integration of K$^{(12)}$, using

$$K_{n+2}^{(\mathbf{12}),\,c,\,q} = \mathcal{H}_n^{(0)\,\dagger}\,J_{q,1,1}^{(0)}\,\mathcal{H}_n^{(0)};\qquad J_{q,0}^{(1)}(k_{[23]},n) + \int d\Phi_1(k_3)\,J_{q,1}^{(0)}(k_2;k_3,n) = \text{finite}$$

"Completeness relation"

This gives a new expression for the real-virtual collinear counterterm

$$K_{n+1}^{(\mathbf{RV}),\,c,\,q} = \mathcal{H}_n^{(0)\,\dagger}\,J_{q,1}^{(0)}(k_1;k_{[23]},n)\,J_{q,0}^{(1)}(k_{[23]},n)\,\mathcal{H}_n^{(0)} + \text{finite}$$

Bottom-up approach

The two definitions have identical collinear poles, which again calls for an explanation.

Once again, the result is better understood by means of a refactorisation of the radiative jet

- The radiative jet function has both UV and IR poles, as well as phase-space singularities.
- As before, it is an amplitude in the presence of sources: virtual IR poles will factorise.

Applying the standard soft-jet-hard factorisation for amplitudes we write

$$
\mathcal{J}_{f,1}(k;p,n) = \left[\frac{\mathcal{J}(k,n_k)}{\mathcal{J}_E(\beta_k,n_k)} \frac{\mathcal{J}(p,n_p)}{\mathcal{J}_E(\beta_p,n_p)} \right] \mathcal{S}_3(\beta_k,\beta_p,\beta_n) \, \mathcal{J}_{f,1}^{\mathrm{fin}}(k,p,n)
$$

Expanding to one-loop order, the terms containing IR poles are

$$
\mathcal{J}_{f,1}^{(1)}(k;p,n) = \left[\mathcal{J}^{(1)}(k,n_k) - \mathcal{J}_E^{(1)}(\beta_k,n_k) + \mathcal{J}^{(1)}(p,n_p) - \mathcal{J}_E^{(1)}(\beta_p,n_p) + \mathcal{S}_3^{(1)}(\beta_k,\beta_p,\beta_n) \right] \mathcal{J}_{f,1}^{\mathrm{fin},0}(k,p,n)
$$

One reconstructs (upon squaring) the collinear contribution to K(RV), plus soft corrections.

- For gg radiation, hard collinear terms are identical and phase space provides a factor 1/2.
- The three-point soft function does not affect collinear factorisation: it simplifies to a singlet quantity when the collinear limit is taken.

Detailed checks against collinear limits of RR$_{n+2}$ require implementing phase-space mappings.

OUTLOOK

Outlook

- Infrared subtraction beyond NLO requires understanding all strongly-ordered IR limits.

- Factorisation provides definitions for local soft and collinear counterterms to all orders.

- Soft and collinear kernels are expressed by matrix elements of fields and Wilson lines.

- In strongly ordered limits the kernels re-factorise into lower-order matrix elements.

- Known strongly ordered IR limits at NNLO and N3LO are reproduced by re-factorisation.

- "Completeness relations" link strongly-ordered kernels and real-virtual counterterms.

- Upon implementing phase-space mappings, the cancellation of RV poles can be checked.

- The refactorisation approach to strong-ordering generalises smoothly to higher orders.

- The architecture of infrared subtraction is becoming clear to all orders.

THANKS TO THE ORGANISERS!

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