Scattering Amplitudes and conservative binary dynamics for Spinning Black Holes

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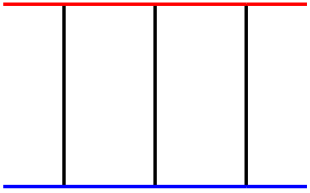
Work in collaboration with: F. Febres Cordero, G. Lin, M. S. Ruf, M. Zeng

HP^2 2022 – Newcastle
21. September 2022



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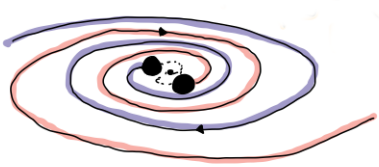
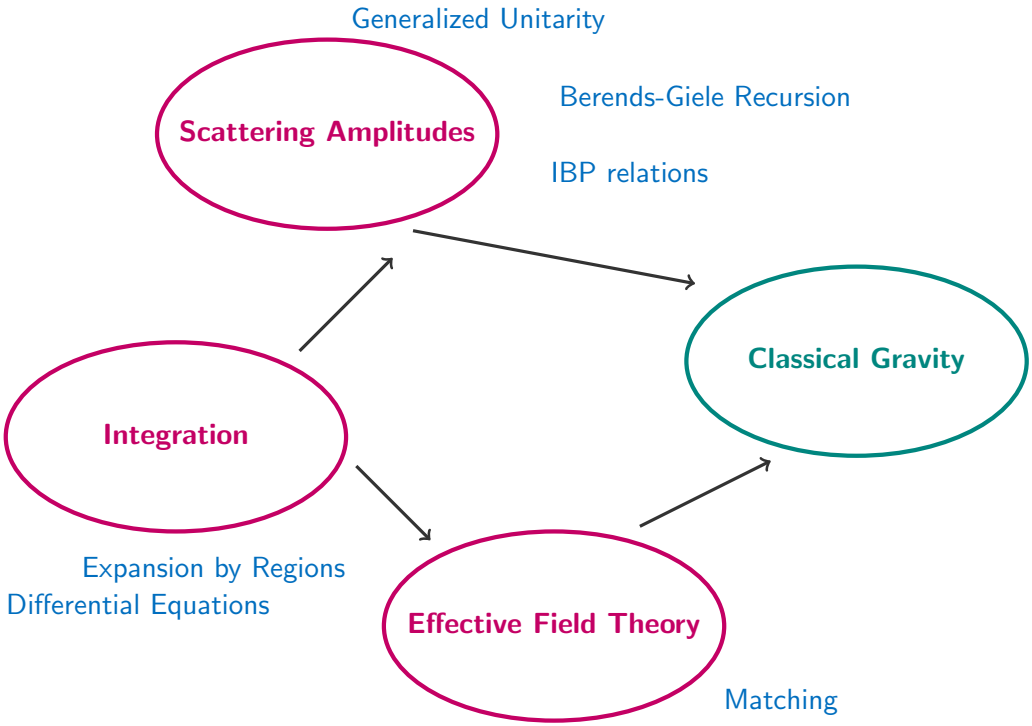
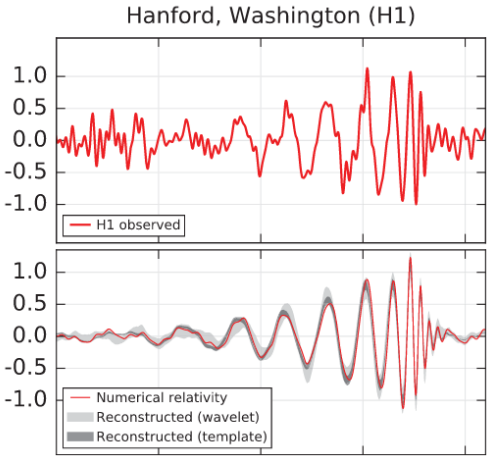


The Landscape of methods



Gravitational Wave Astronomy

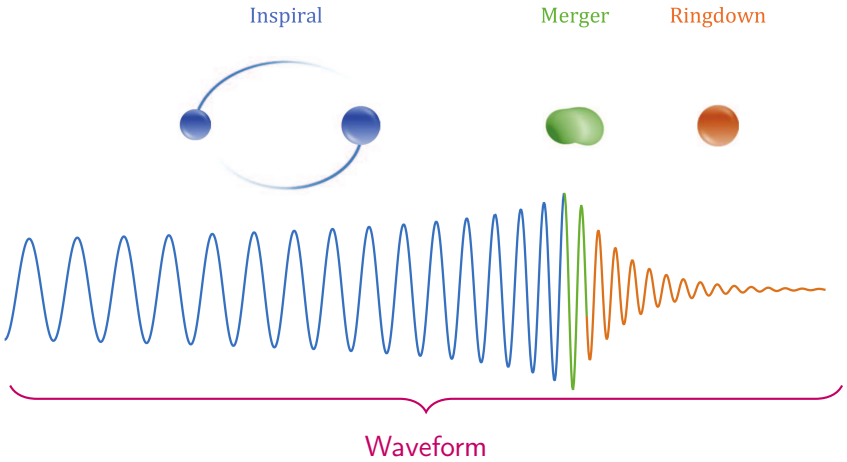


[LIGO & Virgo, arXiv:1602:03837]

- The era of gravitational wave astronomy has just begun ...
- ... and high-precision calculations are required to exploit the full physics potential of current and future GW observatories!

The Stages of a Binary Black Hole Merger

[Antelis,Moreno EPJP 132 (2017) 1, 10]



Inspiral — Weak-field approximation

$$r \gg r_s = \frac{2G(m_1+m_2)}{c^2}$$

Merger — Numerical Relativity - ab initio, but very computing intense

Ringdown — Black Hole Perturbation Theory

The Classical Hamiltonian

During the Inspiral the dynamics are described by classical Hamiltonians

$$H = \sum_{i=1}^{2} \sqrt{\boldsymbol{p}^2 + m_i^2} + V^{(1)}(\boldsymbol{r}^2, \boldsymbol{p}^2) + V^{(2)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{\boldsymbol{L}\cdot\boldsymbol{S}}{\boldsymbol{r}^2}$$
$$+ V^{(3)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{(\boldsymbol{p}\cdot\boldsymbol{S})^2}{\boldsymbol{r}^2} + V^{(4)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{\boldsymbol{S}^2}{\boldsymbol{r}^2} + V^{(5)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{(\boldsymbol{r}\cdot\boldsymbol{S})^2}{\boldsymbol{r}^4} + \cdots$$

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0PN (1687)	1PN (1938)	2PN (1973)	3PN (2000)	4PN (2014)	5PN (2020)	6PN (2020)
Newton	Einstein, Infeld Hoffmann	Kimura et al	Damour et al Blanchet, Faye	Damour et al	Mastrolia et al Blümlein et al	Blümlein et al

$$\begin{array}{rcccccccccccc}
G(& \boxed{1} & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + & v^{10} & + \ldots) \\
G^2(& & & 1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + \ldots) \\
G^3(& & & & & 1 & + & v^2 & + & v^4 & + & v^6 & + \ldots) \\
G^4(& & & & & & & 1 & + & v^2 & + & v^4 & + \ldots) \\
G^5(& & & & & & & & & 1 & + & v^2 & + \ldots) \\
G^6(& & & & & & & & & & & 1 & + \ldots)
\end{array}$$

$$\frac{v^2}{c^2} \sim \frac{GM}{r} \ll 1$$

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$G(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + \ldots)$	1PM	
$G^2(1 + v^2 + v^4 + v^6 + v^8 + \ldots)$	2PM (1985)	Westphal
$G^3(1 + v^2 + v^4 + v^6 + \ldots)$	3PM (2019)	Bern, Cheung, Roiban Shen, Solon, Zeng
$G^4(1 + v^2 + v^4 + \ldots)$	4PM (2021)	Bern,Parra-Martinez Roiban,Ruf,Shen Solon, Zeng
$G^5(1 + v^2 + \ldots)$		
$G^6(1 + \ldots)$		

$$\frac{GM}{r} \ll 1$$

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Spin Hamilitonian much less known!

Post Newtonian

- $\boldsymbol{S}$: 3PN $\mathcal{O}(G^4)$ [Levi et al arXiv:2208.14949, Mastrolia et al arXiv:2209.00611]
- $\boldsymbol{S}^2$: 5PN $\mathcal{O}(G^4)$ [Kim,Levi,Yin arXiv:2112.01509]
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Post Minkowskian

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- $\boldsymbol{S}^5$: 2PM [Bern et al arXiv:2203.06202]
- $\boldsymbol{S}^\infty$: 2PM [Aoude,Haddad,Helset arXiv:2203.06197,arXiv:2205.02809]

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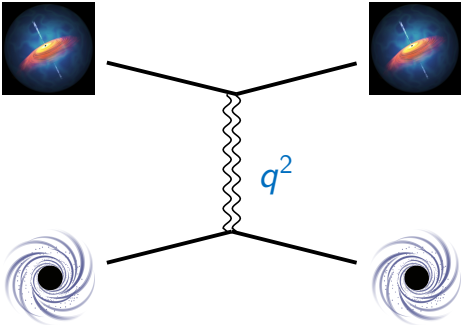
Idea in a Nutshell



$$M = \frac{4\pi\ G}{E_1 E_2}\ \frac{m_1^2 m_2^2 (1-2\sigma^2)}{q^2} + \cdots$$

$$\sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

The classical potential is then given by

$$V^{(0)}(\boldsymbol{r}^2, \boldsymbol{p}^2) = \int \frac{d^3 q}{(2\pi)^3}\ M e^{i\vec{q}\cdot\vec{r}} = -\frac{G m_1 m_2}{r}\left(\frac{m_1 m_2}{E_1 E_2}(2\sigma^2 - 1)\right)$$

$$= -\frac{G m_1 m_2}{r} + \mathcal{O}(\mathrm{v}^2)$$

- Scattering amplitude approach provides all-order v^2 corrections!
- Loop amplitudes provide higher-order $\mathcal{O}(G^n)$ corrections!

General Relativity

- Consider Einstein-Hilbert gravity coupled to massive scalar and vector particles

$$S[\phi, A^\mu, g_{\mu\nu}] = \int d^4x \, \sqrt{-g} \Big[-\frac{2}{\kappa^2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m_\phi^2 \phi^2 - \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} F_{\mu\nu} + \frac{1}{2} m_A^2 g^{\mu\nu} A_\mu A_\nu \Big]$$

- Expand spacetime metric around flat space

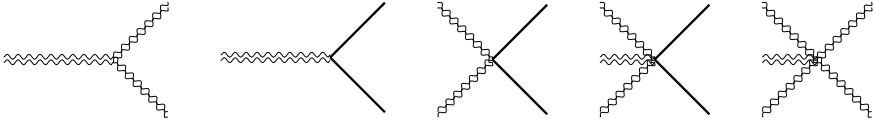
$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \, , \qquad \kappa = \sqrt{32\pi G}$$

- General relativity is non-linear

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda} h^{\nu}{}_{\lambda} + \mathcal{O}(\kappa^3)$$

$$\sqrt{-g} = 1 + \frac{\kappa}{2} h + \frac{\kappa^2}{8} \left(h^2 - 2 h_{\mu\nu} h^{\mu\nu} \right) + \mathcal{O}(\kappa^3)$$

- Leads to quantum EFT of gravity with **infinite** many interaction terms



Scattering Amplitudes - I

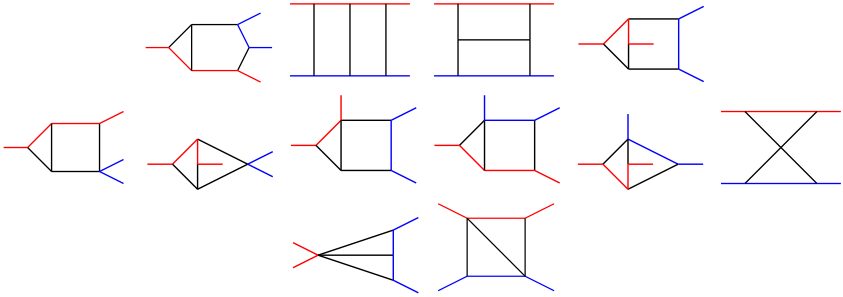
$$A(p_1, \epsilon_1) + \phi(p_2) \to \phi(p_3) + A(p_4, \epsilon_4)$$

- only subset of the full amplitude contributes to classical physics

  $|\ell_1| \sim |\ell_2| \sim |q| \ll m_A, m_\phi, \sqrt{s}$ Spin: $|\boldsymbol{S}| \sim \frac{1}{|q|}$ $q = p_4 - p_1$

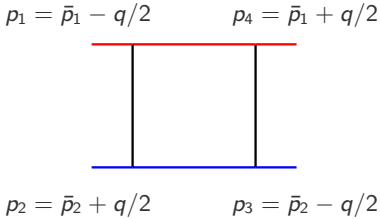
- 1-loop:

- 2-loop:



- Relate: Qunatum Spin $\Leftrightarrow$ Classical Spin

Form factor decomposition:



$$\mathcal{M}_{\lambda_1\lambda_4} = \sum_{n=1}^{5} M_n \, \epsilon_\mu(p_1,\lambda_1)\, T_n^{\mu\nu} \epsilon_\nu^\star(p_4,\lambda_4)$$

with

$$T_1 = (\epsilon_1 \cdot \epsilon_4^\star) \qquad T_2 = (\epsilon_1 \cdot q)(\epsilon_4^\star \cdot q)$$

$$T_3 = q^2(\epsilon_1 \cdot \bar{p}_2)(\epsilon_4^\star \cdot \bar{p}_2) \qquad T_4 = (\epsilon_1 \cdot \bar{p}_2)(\epsilon_4^\star \cdot q) - (\epsilon_1 \cdot q)(\epsilon_4^\star \cdot \bar{p}_2)$$

- Expand relativistic $\epsilon^\mu(p)$ around rest-frame: $\hat{\epsilon}_s^\mu = (0, \hat{\boldsymbol{\epsilon}}_s)$

$$\epsilon_s^\mu(p) = \Lambda^\mu{}_\nu(p)\hat{\epsilon}_s^\nu \,, \qquad \Lambda^\mu{}_\nu(p) = \begin{pmatrix} \sqrt{1+v^2} & \boldsymbol{v}^T \\ \boldsymbol{v} & \delta^{ij} + \frac{1}{v^2}(\sqrt{1+v^2}-1)v^i v^j \end{pmatrix},$$

  Expand Lorentzboost for $p = \bar{p}_1 \pm q/2$ for $q \sim 0$
- Relate rest-frame vectors to Spin: $(\boldsymbol{S}_i)_{jk} = -i\epsilon_{ijk}$

$$\langle 1, m_2, |\mathbb{1}|1, m_1\rangle = (\hat{\boldsymbol{\epsilon}}_{m_1} \cdot \hat{\boldsymbol{\epsilon}}_{m_2}^\star) \,,$$

$$i\epsilon_{ijk} \langle 1, m_2|\boldsymbol{S}_k|1, m_1\rangle = \hat{\boldsymbol{\epsilon}}_{m_2}^{\star\, i}\hat{\boldsymbol{\epsilon}}_{m_1}^{j} - \hat{\boldsymbol{\epsilon}}_{m_1}^{i}\hat{\boldsymbol{\epsilon}}_{m_2}^{\star\, j} \,,$$

$$\langle 1, m_2|\boldsymbol{S}_i\boldsymbol{S}_j|1, m_1\rangle = \delta_{ij}(\hat{\boldsymbol{\epsilon}}_{m_1} \cdot \hat{\boldsymbol{\epsilon}}_{m_2}^\star) - \hat{\boldsymbol{\epsilon}}_{m_1}^{i}\hat{\boldsymbol{\epsilon}}_{m_2}^{\star\, j} \,,$$

$$\langle 1, m_2|\boldsymbol{S}^2|1, m_1\rangle = 2(\hat{\boldsymbol{\epsilon}}_{m_1} \cdot \hat{\boldsymbol{\epsilon}}_{m_2}^\star)$$

After expansion of Lorentz Boost:

$$T_1 = -\langle O_1 \rangle + \frac{1}{m_A^2(\gamma_1+1)}\left[\langle O_2 \rangle - \langle O_3 \rangle + \frac{\langle O_4 \rangle}{2m_A^2(\gamma_1+1)}\right] + \mathcal{O}(\boldsymbol{q}^2)$$

$$T_2 = \langle O_3 \rangle + \mathcal{O}(\boldsymbol{q}^2)$$

$$T_3 = -\frac{E^2}{m_A^2}\langle O_4 \rangle + \mathcal{O}(\boldsymbol{q}^2)$$

$$T_4 = -\frac{E}{m_A}\langle O_2 \rangle + \frac{E-m_A}{m_A}\langle O_3 \rangle - \frac{E}{m_A^3(\gamma_1+1)}\langle O_4 \rangle + \mathcal{O}(\boldsymbol{q}^2)$$

with

$$O_1 = \mathbb{1}$$

$$O_2 = -i(\boldsymbol{q}\times\boldsymbol{p})\cdot\boldsymbol{S}$$

$$O_3 = \frac{1}{2}\boldsymbol{q}^2\boldsymbol{S}^2 - (\boldsymbol{q}\cdot\boldsymbol{S})^2$$

$$O_4 = \boldsymbol{q}^2\left(\frac{1}{2}\boldsymbol{p}^2\boldsymbol{S}^2 - (\boldsymbol{p}\cdot\boldsymbol{S})^2\right)$$

Spin multipoles!

Scattering Amplitudes - IV

- Form factors have decomposition: $M_n = \sum_k c_{n,k}\ I_k$
  $\rightarrow$ Use Numerical Unitarity method to compute $c_{n,k}$
- Resolve ϵ dependence of $c_{n,k}$ in tHV scheme
- Reconstruct univariate slices:
  $$c_{n,k} = c_{n,k}(\epsilon, q^2) \text{ for fixed } \{m_A, m_\phi, \sigma\}$$

Caravel

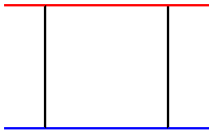
[extended for Masses & Gravity]

- apply constrained IBP reductions to master integrals (classical power counting)
- Expansion by Regions: Potential Region [Parra-Martinez,Ruf,Zeng 2020]
  $\rightarrow$ expand IBPs and master integrals in q^2
  $\rightarrow$ keep only terms $\frac{1}{q^2}$, $\frac{1}{\sqrt{-q^2}}$ and $(q^2)^0$
- After expansion: reconstruct dependence on $\{m_A, m_\phi, \sigma\}$

90 univariate slices to extract analytical expressions for classical terms

Expansion by Regions - I

$p_1 = \bar{m}_A u_1 - q/2 \qquad p_4 = \bar{m}_A u_1 + q/2$



$p_2 = \bar{m}_\phi u_2 + q/2 \qquad p_3 = \bar{m}_\phi u_2 - q/2$

[Beneke,Smirnov '98]

Soft region:

$$|\ell_i| \sim |q| \ll m_A, m_\phi, \sqrt{s}$$

Kinematics:

$$u_i^2 = 1 \ , u_i \cdot q = 0 \ , u_1 \cdot u_2 = y$$

Expanding Matter propagators:

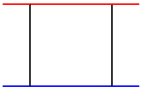
$$\frac{1}{(\ell + p_1) - m_A^2} = \frac{1}{2\bar{m}_A u_1 \cdot \ell} - \frac{\ell^2 - \ell \cdot q}{(2\bar{m}_A u_1 \cdot \ell)^2} + \cdots$$

Graviton propagators unaffected:

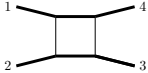
$$\frac{1}{\ell^2} \ , \qquad \frac{1}{(\ell - q)^2}$$

Expansion by Regions - II

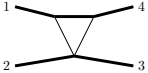
Allows systematic expansion of integrals: [Parra-Martinez,Ruf,Zeng arXiv:2005.04236]



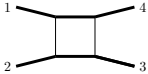
$$= \frac{1}{\epsilon^2 \bar{m}_A \bar{m}_\phi \sqrt{y^2-1}} \frac{1}{(-q^2)}$$



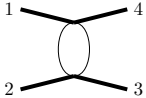
$$+ \frac{\bar{m}_A + \bar{m}_\phi}{\epsilon \bar{m}_A^2 \bar{m}_\phi^2 (y-1)} \frac{1}{\sqrt{-q^2}}$$



$$- \frac{(1+2\epsilon)(2\bar{m}_A \bar{m}_\phi y + \bar{m}_A^2 + \bar{m}_\phi^2)}{8\epsilon^2 \bar{m}_A^3 \bar{m}_\phi^3 (y^2-1)^{3/2}}$$



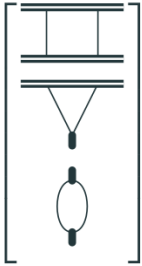
$$+ \frac{(1+2\epsilon)(y(\bar{m}_A^2 + \bar{m}_\phi^2) + 2\bar{m}_A \bar{m}_\phi)}{8\epsilon \bar{m}_A^3 \bar{m}_\phi^3 (y^2-1)}$$



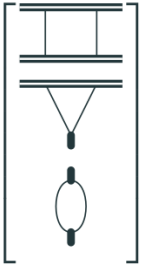
$$+ \mathcal{O}(q^2)$$

Linearized master integrals obtained from differential equations:

$$\frac{\partial}{\partial y}$$



$$= \epsilon \frac{\partial}{\partial y} \underbrace{\log\left(y - \sqrt{y^2+1}\right)}_{=-\operatorname{arcosh} y} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Structure of Scattering Amplitudes

$$\mathcal{M}^{\text{tree}} = G\left(\frac{c_0^{\text{cl}}}{\boldsymbol{q}^2} + \dots\right)$$

$$\mathcal{M}^{\text{1-loop}} = G^2\left(\frac{c_1^{\text{scl}}}{\boldsymbol{q}^2} + \frac{c_1^{\text{cl}}}{|\boldsymbol{q}|} + c_1^{\text{Q}}\log(\boldsymbol{q}^2) + \dots\right)$$

$$\mathcal{M}^{\text{2-loop}} = G^3\left(\frac{c_2^{\text{sscl}}}{\boldsymbol{q}^2} + \frac{c_2^{\text{scl}}}{|\boldsymbol{q}|} + c_2^{\text{cl}}\log(\boldsymbol{q}^2) + \dots\right)$$

- **Classical** terms at *any* loop order
- **Super-classical** contributions have to be subtracted/cancelled
- **Quantum corrections** are suppressed by powers of $\hbar$

How to systematically extract a classical potential from that?

Effective Field Theory - I

[Cheung,Rothstein,Solon arXiv:1808.02489]

Non-relativistic, non-local, classical 3D EFT [Bern,Luna,Roiban,Shen,Zeng arXiv:2005.03071]

$$L_{\mathrm{EFT}} = \int_{\boldsymbol{k}} \hat{\phi}^\dagger(-\boldsymbol{k}) \left(i\partial_t - \sqrt{\boldsymbol{k}^2 + m_\phi^2} \right) \hat{\phi}(\boldsymbol{k}) + \int_{\boldsymbol{k}} \hat{A}^{\dagger,i}(-\boldsymbol{k}) \left(i\partial_t - \sqrt{\boldsymbol{k}^2 + m_A^2} \right) \hat{A}^i(\boldsymbol{k})$$
$$- \int_{\boldsymbol{k},\boldsymbol{k}'} V_{ij}(\boldsymbol{k},\boldsymbol{k}') \hat{A}^{\dagger,i}(\boldsymbol{k}') \hat{A}^j(\boldsymbol{k}) \hat{\phi}^\dagger(-\boldsymbol{k}') \hat{\phi}(-\boldsymbol{k})$$

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[Cheung,Rothstein,Solon arXiv:1808.02489]

Non-relativistic, non-local, classical 3D EFT [Bern,Luna,Roiban,Shen,Zeng arXiv:2005.03071]

$$L_{\mathrm{EFT}} = \int_{\boldsymbol{k}} \hat{\phi}^\dagger(-\boldsymbol{k}) \left(i\partial_t - \sqrt{\boldsymbol{k}^2 + m_\phi^2} \right) \hat{\phi}(\boldsymbol{k}) + \int_{\boldsymbol{k}} \hat{A}^{\dagger,i}(-\boldsymbol{k}) \left(i\partial_t - \sqrt{\boldsymbol{k}^2 + m_A^2} \right) \hat{A}^i(\boldsymbol{k})$$
$$- \int_{\boldsymbol{k},\boldsymbol{k}'} V_{ij}(\boldsymbol{k},\boldsymbol{k}') \hat{A}^{\dagger,i}(\boldsymbol{k}') \hat{A}^j(\boldsymbol{k}) \hat{\phi}^\dagger(-\boldsymbol{k}') \hat{\phi}(-\boldsymbol{k})$$

$$V_{ij}(\boldsymbol{k},\boldsymbol{k}') \sim \sum_{n=1}^{4} \sum_{L}^{\infty} |\boldsymbol{q}|^{L-2} \left(\frac{\mu^2}{\boldsymbol{q}^2} \right)^{L\epsilon} c_L^{(n)}(\boldsymbol{k}^2)\ O_n^{ij}$$

Effective Field Theory - I

[Cheung,Rothstein,Solon arXiv:1808.02489]

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Matching coefficient

$$V_{ij}(\boldsymbol{k},\boldsymbol{k}') \sim \sum_{n=1}^{4} \sum_{L}^{\infty} |\boldsymbol{q}|^{L-2} \left(\frac{\mu^2}{\boldsymbol{q}^2} \right)^{L\epsilon} c_L^{(n)}(\boldsymbol{k}^2)\ O_n^{ij}$$

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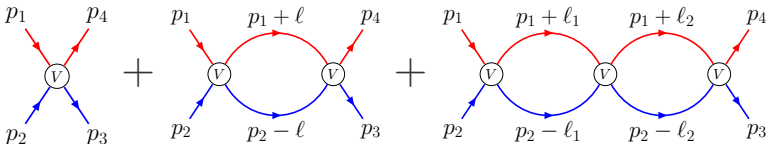
Spin Operators

$$O_1 = \mathbb{1}\,, \qquad O_2 = -i(\boldsymbol{q} \times \boldsymbol{p}) \cdot \boldsymbol{S}\,,$$
$$O_3 = \frac{1}{2}\boldsymbol{q}^2 \boldsymbol{S}^2 - (\boldsymbol{q} \cdot \boldsymbol{S})^2\,, \qquad O_4 = \boldsymbol{q}^2 \left(\frac{1}{2}\boldsymbol{p}^2 \boldsymbol{S}^2 - (\boldsymbol{p} \cdot \boldsymbol{S})^2 \right)$$

Effective Field Theory - II

- EFT given by iterated Bubble diagrams [Cheung,Rothstein,Solon arXiv:1808.02489]



- Form factor decomposition: $\mathcal{M}_{\mathrm{EFT}} = \sum_{n=1}^{4} M_{\mathrm{EFT}}^{(n)}\, O_n \to$ Projectors
- Follow stragegy:
  - integrate ℓ_n^0 by contours

$$\Delta(\ell) = i \int \frac{d\ell^0}{2\pi} \frac{1}{(E_1+\ell^0) - \sqrt{E_1^2+Y_1}} \frac{1}{(E_2-\ell^0) - \sqrt{E_2^2+Y_1}}$$
$$= \frac{1}{E_1+E_2-\sqrt{E_1^2+Y_1}-\sqrt{E_2^2+Y_1}}$$

$$Y_1 = (\boldsymbol{\ell}+\boldsymbol{p})^2 - \boldsymbol{p}^2$$

  - Expand around $Y_n \ll 1$

$$\Delta(\ell) = -\frac{2E_1E_2}{E_1+E_2}\frac{1}{Y_1} + \mathcal{O}(Y_1^0)$$

- Perform Potential Region expansion: $|\boldsymbol{\ell}_i| \sim |\boldsymbol{q}| \ll |\boldsymbol{p}|, m_A, m_\phi$
- IBP reduction to linearized master integrals [Parra-Martinez,Ruf,Zeng 2020]

Conservative Potential: Scalar Term

- Expand full amplitude tensors: $T_n = \sum_k A_{nk}\ O_k$ and then match

$$\mathcal{M}^{(L)}_{\mathrm{EFT}} \stackrel{!}{=} \frac{\mathcal{M}^{(L)}}{4E_1E_2}$$

- $O_1 = \mathbb{1}$ yields spinless Hamiltonian

$$c_1^{(1)}(\boldsymbol{k}^2) = \frac{m_A^2 m_\phi^2}{E_1E_2}\left(1-2\sigma^2\right)\,, \qquad c_2^{(1)}(\boldsymbol{k}^2) = \frac{3(m_\phi+m_A)m_\phi^2 m_A^2}{4E_1E_2}(1-5\sigma^2)\,,$$

$$\begin{aligned} c_3^{(1)}(\boldsymbol{k}^2) = \frac{m_A^2 m_\phi^2}{E_1E_2}\Bigg[&-\frac{2}{3}m_A m_\phi\left(\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\left(-12\sigma^4+36\sigma^2+9\right)+22\sigma^3-19\sigma\right) \\ &-2(m_\phi^2+m_A^2)\left(6\sigma^2+1\right)\Bigg] + \frac{3Em_A^2m_\phi^2}{4E_1E_2}(m_A+m_\phi)\frac{(1-2\sigma^2)(1-5\sigma^2)}{(\sigma^2-1)} \\ &-\frac{3m_A^4m_\phi^4}{E_1E_2\boldsymbol{k}^2}\,, \end{aligned}$$

Agrees with [Bern, Cheung, Roiban, Shen, Solon, Zeng arXiv:1901.04424]

Conservative Potential: Spin-Orbit Term

- $O_2 = -i(\boldsymbol{q} \times \boldsymbol{p}) \cdot \boldsymbol{S}$ yields the spin-orbit coupling.

$$c_i^{(2)}(\boldsymbol{k}^2) = c_{i,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) + c_{i,\mathrm{iter}}^{(2)}(\boldsymbol{k}^2) + \frac{c_i^{(1)}(\boldsymbol{k}^2)}{m_A^2(\gamma_1+1)}$$

$$c_{1,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) = -\frac{2\sigma m_\phi}{E\xi}\,, \qquad c_{2,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) = \frac{m_\phi(4m_A+3m_\phi)\sigma(5\sigma^2-3)}{4E\xi(\sigma^2-1)}\,,$$

$$c_{3,\mathrm{red}}^{(2)}(\boldsymbol{k}^2) = \frac{m_\phi}{E\xi(\sigma^2-1)^2}\Big[-2m_A^2\sigma(3-12\sigma^2+10\sigma^4) - \left(\tfrac{83}{6}+27\sigma^2-52\sigma^4+\tfrac{44}{3}\sigma^6\right)m_A m_\phi - m_\phi^2\sigma\left(\tfrac{7}{2}-14\sigma^2+12\sigma^4\right)$$
$$+\frac{(4m_A+3m_\phi)E}{4}\sigma(2\sigma^2-1)(5\sigma^2-3)+4m_A m_\phi\sigma(\sigma^2-6)(2\sigma^2+1)\sqrt{\sigma^2-1}\,\mathrm{arccosh}(\sigma)\Big]\,,$$

$$c_{1,\mathrm{iter}}^{(2)}(\boldsymbol{k}^2) = 0\,, \qquad c_{2,\mathrm{iter}}^{(2)}(\boldsymbol{k}^2) = E\xi c_1^{(2)}\frac{\partial c_1^{(1)}}{\partial \boldsymbol{k}^2} + c_1^{(1)}\left(E\xi\frac{\partial c_1^{(2)}}{\partial \boldsymbol{k}^2} + \frac{c_1^{(2)}\left(\frac{2E^2\xi}{\boldsymbol{k}^2}+\frac{1}{\xi}-3\right)}{2E}\right)\,,$$

$$c_{3,\mathrm{iter}}^{(2)}(\boldsymbol{k}^2) = \left(c_1^{(1)}\right)^2\left(-\frac{2}{3}E^2\xi^2\frac{\partial^2 c_1^{(2)}}{\partial(\boldsymbol{k}^2)^2} + \left(\xi\left(3-\frac{E^2\xi}{\boldsymbol{k}^2}\right)-1\right)\frac{\partial c_1^{(2)}}{\partial \boldsymbol{k}^2} + c_1^{(2)}\left(\frac{\frac{1}{2\xi}-2}{E^2}+\frac{3\xi-1}{\boldsymbol{k}^2}\right)\right)$$
$$+c_1^{(1)}\left(c_1^{(2)}\left(\left(-\frac{3E^2\xi^2}{\boldsymbol{k}^2}+6\xi-2\right)\frac{\partial c_1^{(1)}}{\partial \boldsymbol{k}^2} - \frac{4}{3}E^2\xi^2\frac{\partial^2 c_1^{(1)}}{\partial(\boldsymbol{k}^2)^2}\right)\right.$$
$$\left.+\frac{4}{3}E\xi\left(\frac{\partial c_2^{(2)}}{\partial \boldsymbol{k}^2} - 2E\xi\frac{\partial c_1^{(1)}}{\partial \boldsymbol{k}^2}\frac{\partial c_1^{(2)}}{\partial \boldsymbol{k}^2}\right) + \frac{E^2\xi^2\left(c_1^{(2)}\right)^2}{2\boldsymbol{k}^2} + c_2^{(2)}\left(\frac{\frac{2}{3\xi}-2}{E}+\frac{E\xi}{\boldsymbol{k}^2}\right)\right) - \frac{1}{6}E^2\xi^2\left(c_1^{(2)}\right)^3$$
$$+c_1^{(2)}\left(\frac{2}{3}E\xi\left(\frac{\partial c_2^{(1)}}{\partial \boldsymbol{k}^2} - 2E\xi\left(\frac{\partial c_1^{(1)}}{\partial \boldsymbol{k}^2}\right)^2\right) + \frac{c_2^{(1)}\left(\frac{3E^2\xi}{\boldsymbol{k}^2}+\frac{1}{\xi}-3\right)}{3E}\right) + \frac{2}{3}E\xi c_2^{(1)}\frac{\partial c_1^{(2)}}{\partial \boldsymbol{k}^2} + \frac{4}{3}E\xi c_2^{(2)}\frac{\partial c_1^{(1)}}{\partial \boldsymbol{k}^2}\,.$$

- O_3 and O_4 more involved

With our Hamilton we reproduced observables at 3PM computed in world-line QFT formalism by [Jakobsen,Mogull arXiv:2201.07778]

Summary & Outlook

First ...

- application of Numerical Unitarity for massive particles
  - Many applications in QCD & Gravity
- EFT matching for classical binary dynamics fully in dimensional regularization
  - Opens path to even higher-order corrections

- computation of classical Hamiltonian up to $\boldsymbol{S}^2$ terms at $\mathcal{O}(G^3)$

$$H = \sum_{i=1}^{2} \sqrt{\boldsymbol{p}^2 + m_i^2} + V^{(1)}(\boldsymbol{r}^2, \boldsymbol{p}^2) + V^{(2)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{\boldsymbol{L}\cdot\boldsymbol{S}}{\boldsymbol{r}^2}$$
$$+ V^{(3)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{(\boldsymbol{p}\cdot\boldsymbol{S})^2}{\boldsymbol{r}^2} + V^{(4)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{\boldsymbol{S}^2}{\boldsymbol{r}^2} + V^{(5)}(\boldsymbol{r}^2, \boldsymbol{p}^2)\frac{(\boldsymbol{r}\cdot\boldsymbol{S})^2}{\boldsymbol{r}^4} + \cdots$$

Outlook:

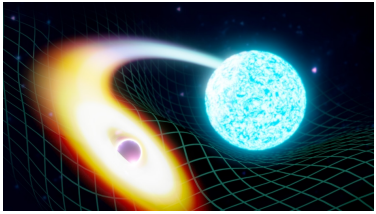
- Extend current framework for
  - higher spin contributions (massive higher-spin representations)
  - finite size effects (higher dim. operators)
  - Radiation effects (external gravitons)
  - higher loops

Backup

Constraining Neutron Star EOS from Neutron Star Black Hole mergers

- Tidal deformation leads to quadrupole moments in GW

$$Q_{ij} \sim \left(\frac{M}{R}\right)^{-5}$$



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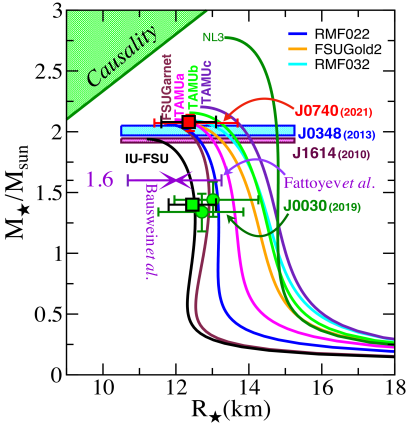

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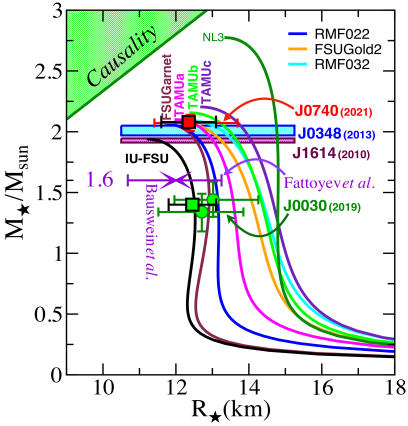
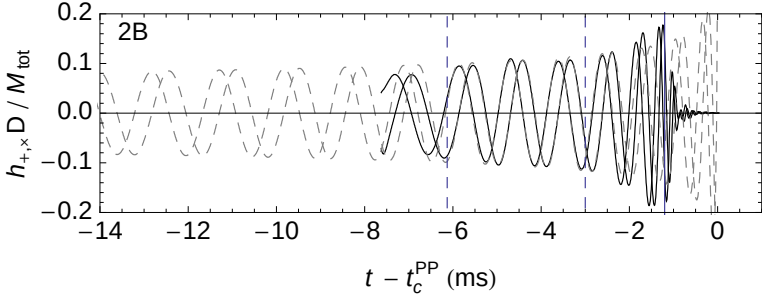


Impact on GW waveform:



dashed: point particle

solid: tidal deformation

Power counting

- Restore $\hbar$ in couplings $G \to G/\hbar$
- Small momentum transfer $|\boldsymbol{q}| \to \hbar|\boldsymbol{q}|$
- Fourier transformation scaling

$$V = \int \frac{d^3q}{(2\pi)^3} \mathcal{M} e^{\frac{i}{\hbar}\vec{q}\cdot\vec{r}} = \hbar^3 \int \frac{d^3p}{(2\pi)^3} \mathcal{M} e^{i\vec{p}\cdot\vec{r}}$$

- Then classical terms scale as $\hbar^{-3}$

$$\frac{G}{\boldsymbol{q}^2} \to \frac{1}{\hbar^3}\frac{G}{\boldsymbol{q}^2}$$
$$\frac{G^2}{|\boldsymbol{q}|} \to \frac{1}{\hbar^3}\frac{G^2}{|\boldsymbol{q}|}$$
$$G^3 \log(|\boldsymbol{q}|) \to \frac{1}{\hbar^3} G^3 \log(|\boldsymbol{q}|)$$