

Quantum Field Theory - Wednesday Problems

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1 Free quantum fields: working with commutators

- 1.1 Consider the Heisenberg equation of motion for the momentum operator \hat{p} of the harmonic oscillator with Hamiltonian

$$\hat{H} = \frac{1}{2} \left(\frac{\hat{p}^2}{m} + m\omega^2 \hat{x}^2 \right),$$

and show that it is equivalent to Newton's law for the position operator \hat{x} .

- 1.2 Starting from the expression

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2E(\mathbf{k})}} \left(e^{ik \cdot x} \hat{a}^\dagger(\mathbf{k}) + e^{-ik \cdot x} \hat{a}(\mathbf{k}) \right),$$

find the corresponding expression for the canonical momentum operator $\hat{\pi}(x) = \partial_0 \hat{\phi}(x)$.

- 1.3 Show that the equal time commutation relations

$$\left[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t) \right] = i\delta^3(\mathbf{x} - \mathbf{x}') \quad , \quad \left[\hat{\phi}(\mathbf{x}, t), \hat{\phi}(\mathbf{x}', t) \right] = \left[\hat{\pi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t) \right] = 0,$$

imply that

$$\left[\hat{a}(\mathbf{k}), \hat{a}(\mathbf{p}) \right] = 0.$$

(A similar method would also show that $\left[\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{p}) \right] = 0$.)

- 1.4 In this problem, we want to show that the scalar field operator $\hat{\phi}(\mathbf{x}, t)$ satisfies the Klein-Gordon equation:

$$\partial_\mu \partial^\mu \hat{\phi}(\mathbf{x}, t) + m^2 \hat{\phi}(\mathbf{x}, t) = 0.$$

We know already that

$$\hat{\pi}(\mathbf{x}, t) = \partial_t \hat{\phi}(\mathbf{x}, t).$$

We now need to find an equation for $\partial_t \hat{\pi}(\mathbf{x}, t)$. This can be done with the Heisenberg equation of motion, which for a general field operator \hat{O} is

$$\frac{\partial}{\partial t} \hat{O} = i[\hat{H}, \hat{O}],$$

where

$$\hat{H} = \frac{1}{2} \int d^3x \left\{ \hat{\pi}^2 + (\nabla \hat{\phi})^2 + m^2 \hat{\phi}^2 \right\}.$$

Assuming the equal time commutation relations

$$\left[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t) \right] = i\delta^3(\mathbf{x} - \mathbf{x}') \quad , \quad \left[\hat{\phi}(\mathbf{x}, t), \hat{\phi}(\mathbf{x}', t) \right] = \left[\hat{\pi}(\mathbf{x}, t), \hat{\pi}(\mathbf{x}', t) \right] = 0,$$

evaluate $[\hat{H}, \hat{\pi}(\mathbf{x}, t)]$ to show that

$$\partial_t \hat{\pi}(\mathbf{x}, t) = \nabla^2 \hat{\phi}(\mathbf{x}, t) - m^2 \hat{\phi}(\mathbf{x}, t).$$

[Hint: You will need to integrate terms involving $(\nabla \hat{\phi})^2$ by parts and assume that the fields vanish at the boundary of space (spatial infinity).]