Quantum Field Theory - Thursday Problems

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1.1 What is the normal ordered product $:\hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{q})\hat{a}(\mathbf{r})\hat{a}^\dagger(\mathbf{s}):$?

1.2 After normal ordering, the conserved three-momentum $P^i = \int d^3x T^{0i}$ takes the form

$$:\hat{P}^i: = \int \frac{d^3p}{(2\pi)^3} p^i \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) \; .$$

Prove the commutator relation

$$\left[:\hat{P}^i:, \hat{a}^\dagger(\mathbf{k})\right] = k^i \hat{a}^\dagger(\mathbf{k}) \; .$$

1.3 Write down the general result for $\left[:\hat{P}^\mu:, \hat{a}^\dagger(\mathbf{k})\right]$ in terms of k^μ and $\hat{a}^\dagger(\mathbf{k})$. Hence show that

$$:\hat{P}^\mu: \; \hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle = (k_1^\mu + k_2^\mu)\, \hat{a}^\dagger(\mathbf{k}_2)\hat{a}^\dagger(\mathbf{k}_1)|0\rangle. \tag{1}$$

Interpret the physics of this result.

1.4 The number operator is

$$\hat{N} = \int \frac{d^3p}{(2\pi)^3}\, \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p})$$

Prove by induction that

$$\int \frac{d^3p}{(2\pi)^3}\, \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) \; \underbrace{|\mathbf{k}, \ldots, \mathbf{k}\rangle}_{n \text{ momenta}} = n \; \underbrace{|\mathbf{k}, \ldots, \mathbf{k}\rangle}_{n \text{ momenta}} \; .$$

[**Hint:** induction proceeds in two steps. *i)* show that the statement is true for some starting value of n; *ii)* show that if the statement holds for some general n, then it also holds for $n+1$.]

1.5 Show that $\hat{N}$ is a constant of motion when

$$\hat{H} = \int \frac{d^3p}{(2\pi)^3} E_p \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) \; .$$

1.6 We normalise our momentum eigenstates such that $\langle \mathbf{p}|\mathbf{k}\rangle = 2E_p(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})$. Show that the combination $E_p\, \delta^3(\mathbf{p}-\mathbf{k})$ is Lorentz invariant.