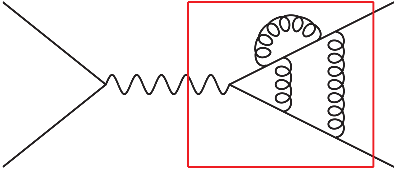
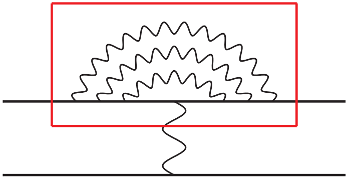
Three loop corrections for the QCD and QED form factors - setup and reduction

4th Workstop / Thinkstart: Towards N^{3}LO for $\gamma \to \ell\bar{\ell}$ | August 3 – 5, 2022

Fabian Lange
in collaboration with Matteo Fael, Kay Schönwald, Matthias Steinhauser | August 3, 2022

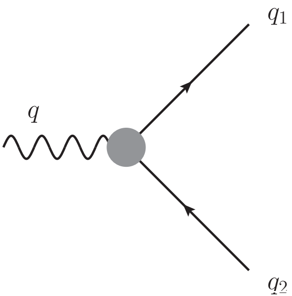
Motivation





- Form factors are basic building blocks for many physical observables:
  - $f\bar{f}$ production at hadron and e^+e^- colliders
  - μe scattering
  - Higgs production and decay
  - ...
- Form factors exhibit an universal infrared behavior which is interesting to study
- Aim of this workstop: $\gamma^* \to \ell\bar{\ell}$ at N^{3}LO
- This session: virtual three-loop corrections

The process



$$X(q) \to Q(q_1) + \bar{Q}(q_2)$$

$$q_1^2 = q_2^2 = m^2, \quad q^2 = s = \hat{s} \cdot m^2$$

vector : $j_\mu^{\mathrm{v}} = \overline{\psi}\gamma_\mu\psi, \quad \Gamma_\mu^{\mathrm{v}} = F_1^{\mathrm{v}}(s)\gamma_\mu - \frac{i}{2m}F_2^{\mathrm{v}}(s)\sigma_{\mu\nu}q^\nu$

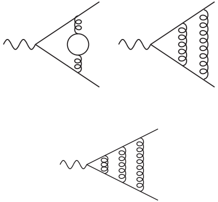
axial-vector : $j_\mu^{\mathrm{a}} = \overline{\psi}\gamma_\mu\gamma_5\psi, \quad \Gamma_\mu^{\mathrm{a}} = F_1^{\mathrm{a}}(s)\gamma_\mu\gamma_5 - \frac{1}{2m}F_2^{\mathrm{a}}(s)q_\mu\gamma_5$

scalar : $j^{\mathrm{s}} = m\overline{\psi}\psi, \quad \Gamma^{\mathrm{s}} = mF^{s}(s)$

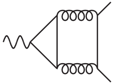
pseudo-scalar : $j^{\mathrm{p}} = \mathrm{i}m\overline{\psi}\gamma_5\psi, \quad \Gamma^{\mathrm{p}} = \mathrm{i}mF^{\mathrm{p}}(s)\gamma_5$

History of massive form factors

non-singlet:



singlet:



$F_i^{(2)}$ (**NNLO**):

- Non-singlet fermionic contributions [Hoang, Teubner 1997]
- QED for vector current [Mastrolia, Remiddi 2003; Bonciani, Mastrolia, Remiddi 2003]
- Complete QCD [Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi 2004 - 2005]

$F_i^{(3)}$ (**NNNLO**) non-singlet:

- Large N_c [Henn, Smirnov, Smirnov, Steinhauser 2016; Lee, Smirnov, Smirnov, Steinhauser 2 $\times$ 2018; Ablinger, Blümlein, Marquard, Rana, Schneider 2 $\times$ 2018]
- n_l [Lee, Smirnov, Smirnov, Steinhauser 2018; Ablinger, Blümlein, Marquard, Rana, Schneider 2 $\times$ 2018]
- n_h with $N_c = 3$ [Blümlein, Marquard, Rana, Schneider 2019]

This session: full (numerical) results at NNNLO

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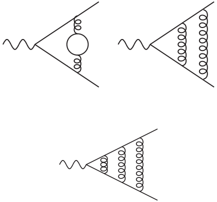

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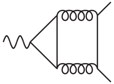

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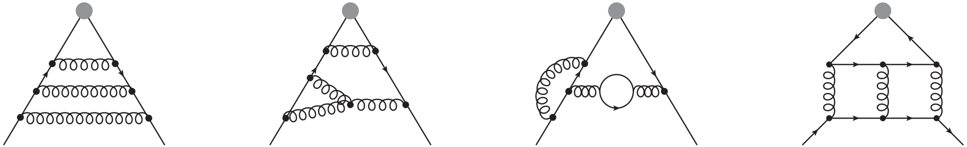
Side notes:

- Massless $F_i^{(4)}$ computed recently [Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser 2022]
- Singlet contributions to $F_a^{(3)}$ with massive quark loop computed in [Chen, Czakon, Niggetiedt 2021]

Setup



- Generate diagrams with `qgraf` [Nogueira 1991]
- Map to predefined integral families with `q2e/exp` [Harlander, Seidensticker, Steinhauser 1998; Seidensticker 1999]
- `FORM` [Vermaseren 2000; Kuipers, Ueda, Vermaseren, Vollinga 2013; Ruijl, Ueda, Vermaseren 2017] for Lorentz, Dirac, and color algebra [van Ritbergen, Schellekens, Vermaseren 1998]
- Reduction to master integrals ⇒ this talk
- Solve master integrals semi-numerically ⇒ Kay's talk

	non-singlet	n_h-singlet	n_l-singlet
diagrams	271	66	66
families	34	17	13
integrals	302671	106883	127980
masters	422	316	158

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Integral reduction

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- Search for good basis of master integrals for every integral family [Smirnov, Smirnov 2020; Usovitsch 2020]
  - Takes about three hours for most expensive families
- Reduce every family separately to good basis with `Kira` [Maierhöfer, Usovitsch, Uwer 2017; Klappert, **FL**, Maierhöfer, Usovitsch 2020] and `Fermat` [Lewis]
  - Most expensive (non-singlet) families: one week on eight cores and 200 GiB of memory
- Reduce good basis of all families to minimal basis (still good) by employing symmetry relations between families with `Kira`:
  - 3131 $\to$ 422 master integrals for non-singlet families
  - Takes about one day for non-singlet families
  - Can be done in parallel to the per-family reductions
- Establish differential equations:
  - Take derivative of master integrals with respect to $\hat{s}$ with `LiteRed` [Lee 2012 + 2013]
  - Reduce resulting integrals again on per-family basis and use known symmetry relations
  - Takes a few hours for most expensive families

Good basis

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- It was shown that it is always possible to find a *good* basis in which d and the kinematic variables in denominators factorizes [Smirnov, Smirnov 2020; Usovitsch 2020] :

$$I = \frac{\dots}{\text{complicated polynomial}} M_{\text{bad}} + \dots \to \frac{\dots}{(d-3)^2(d-4)(s-4m^2)\cdots} M_{\text{good}} + \dots$$

- Strategy to find good basis:
  - Reduce sample integrals (simpler than needed for actual reduction)
  - Search for relation

$$I = \frac{\text{good polynomial}}{\text{bad polynomial}} M_{\text{bad}} + \dots$$

  - Invert and express

$$M_{\text{bad}} = \frac{\text{bad polynomial}}{\text{good polynomial}} I + \dots$$

    everywhere
  - Bad polynomial cancels and I is good master integral
  - If no such relation available, choose least bad polynomial and repeat until basis is good
  - Increase sample size if good basis not found
- We use an improved version of the automatized code from [Smirnov, Smirnov 2020]

Good basis with less poles in ϵ

- Similarly, it was shown that one can always find an ϵ-finite basis without poles in front of master integrals [Chetyrkin, Faisst, Sturm, Tentyukov 2006] :

$$I = \left(\frac{\dots}{\epsilon} + \frac{\dots}{\epsilon^0} + \dots \right) M_{\text{poles}} + \dots \to \left(\frac{\dots}{\epsilon^0} + \dots \right) M_{\epsilon-\text{finite}} + \dots$$

- Search procedure similar:

$$I = \frac{\dots}{\epsilon} M_{\text{poles}} + \dots \quad \to \quad M_{\text{poles}} = \frac{\epsilon}{\dots} I + \dots$$

- It does not seem possible to find basis which is good and ϵ-finite at the same time
- However, one can get rid of some poles in addition to having a good basis
- Easily incorporated into code of [Smirnov, Smirnov 2020] by treating ϵ as least bad denominator