Karlsruhe Institute of Technology

Institute for Theoretical Particle Physics

Three loop corrections for the QCD and QED form factors - master integrals

4th Workstop / Thinkstart: Towards N^{3}LO for $\gamma \to \ell\bar{\ell}$ | August 3 – 5, 2022

Kay Schönwald

in collaboration with Matteo Fael, Fabian Lange, Matthias Steinhauser | August 3, 2022

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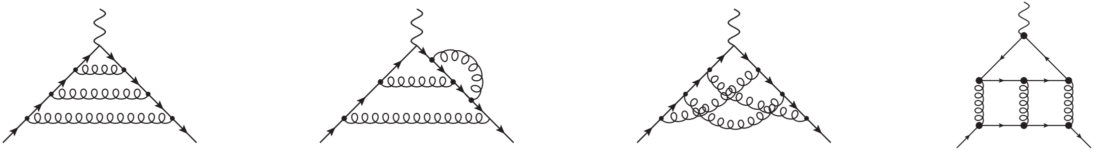
Technical Details



- Reduce the scalar integrals to masters with `Kira`.
  [Klappert, Lange, Maierhöfer, Usovitsch, Uwer '17,'20]
  - We ensure a good basis where denominators factorize in ϵ and $\hat{s}$ with `ImproveMasters.m`. [Smirnov, Smirnov '20]
- Establish differential equations in variable $\hat{s}$ using `LiteRed`. [Lee '12,'14]
- Starting point:

$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}}\vec{M}(s,\epsilon) = \underline{\underline{A}}(s,\epsilon)\cdot\vec{M}(s,\epsilon)$$

	non-singlet	singlet
diagrams	271	66
families	34	17
masters	422	316

Algorithm to Solve Master Integrals

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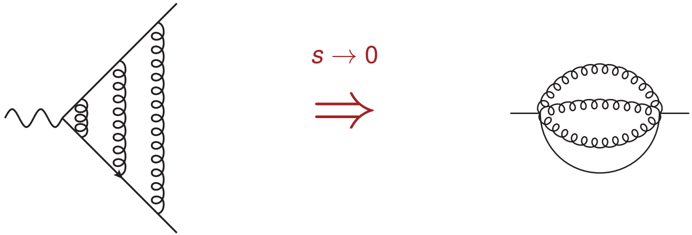
- Repeat the procedure for the next point.

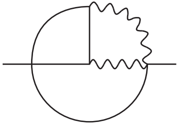
Calculation of Boundary Conditions



- For $s = 0$ the non-singlet master integrals reduce to 3-loop on-shell propagators:
  - These integrals are well studied in the literature. [Laporta, Remiddi '96; Melnikov, Ritbergen '00; Lee, Smirnov '10]
- The reduction introduces high inverse powers in ϵ, which require some integrals up to weight 9.
- We calculate the needed terms with `SummerTime.m` [Lee, Mingulov '15] and `PSLQ` [Ferguson, Bailey '92].
- The singlet master integrals need a proper asymptotic expansion around $s = 0$, which we implemented with the help of `Asy.m` [Jantzen, Smirnov, Smirnov '12].

Calculation of Boundary Conditions

E.g. extension of G_{66} (given up to and including $\mathcal{O}(\epsilon^3)$ in [Lee, Smirnov '10]):



$$= \cdots + \epsilon^4 \Big(-4704 s_6 - 9120 s_{7a} - 9120 s_{7b} - 547 s_{8a} + 9120 s_6 \ln(2) + 28 \ln^4(2) + \frac{112 \ln^5(2)}{3} - \frac{808}{45} \ln^6(2)$$
$$- \frac{347}{9} \ln^8(2) + 672 \mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{5552}{3} \ln^4(2) \mathrm{Li}_4\left(\frac{1}{2}\right) - 22208 \mathrm{Li}_4\left(\frac{1}{2}\right)^2 - 4480 \mathrm{Li}_5\left(\frac{1}{2}\right) - 12928 \mathrm{Li}_6\left(\frac{1}{2}\right) + \ldots \Big)$$
$$+ \epsilon^5 \Big(14400 s_6 - \frac{377568 s_{7a}}{7} - \frac{93984 s_{7b}}{7} - 2735 s_{8a} + 7572912 s_{9a} - 3804464 s_{9b} - \frac{5092568 s_{9c}}{3} - 136256 s_{9d}$$
$$+ 681280 s_{9e} + 272512 s_{9f} + \frac{377568}{7} s_6 \ln(2) - \frac{32465121}{20} s_{8a} \ln(2) - 10185136 s_{8b} \ln(2) + 136256 s_{7b} \ln^2(2) + \ldots \Big)$$
$$+ \mathcal{O}(\epsilon^6)$$

Series Expansions

- Special points:

$s = 0$	$s = 4m^2$	$s = \pm\infty$
$x = 1$	$x = -1$	$x = 0$
static limit	2-particle threshold	high energy limit

- Every expansion point needs a different ansatz:

$$M_n(\epsilon, \hat{s} = 0) = \sum_{i=-3}^{\infty} \sum_{j=0}^{j_{\max}} c_{ij}^{(n)} \, \epsilon^i \, \hat{s}^j$$

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$$M_n(\epsilon, \hat{s} = 4) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\text{min}}}^{j_{\text{max}}} \sum_{k=0}^{i+3} c_{ijk}^{(n)} \, \epsilon^i \left[\sqrt{4 - \hat{s}}\right]^j \ln^k \left(\sqrt{4 - \hat{s}}\right)$$

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$s = 0$	$s = 4m^2$	$s = \pm\infty$
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- Every expansion point needs a different ansatz:

$$M_n(\epsilon, \hat{s} \to \pm\infty) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{j_{\max}} \sum_{k=0}^{\boxed{i+6}} c_{ijk}^{(n)} \, \epsilon^i \, \hat{s}^{-j} \, \ln^k(\hat{s})$$

Series Expansions

- Special points:

$s = 0$	$s = 4m^2$	$s = \pm\infty$	$s = 16m^2$
$x = 1$	$x = -1$	$x = 0$	$x = 4\sqrt{3} - 7$
static limit	2-particle threshold	high energy limit	4-particle threshold

- Every expansion point needs a different ansatz:

$$M_n(\epsilon, \hat{s} = 16) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{j_{\max}} \sum_{k=0}^{i+3} c_{ijk}^{(n)} \epsilon^i \left[\sqrt{16 - \hat{s}}\right]^j \ln^k \left(\sqrt{16 - \hat{s}}\right)$$

Series Expansions

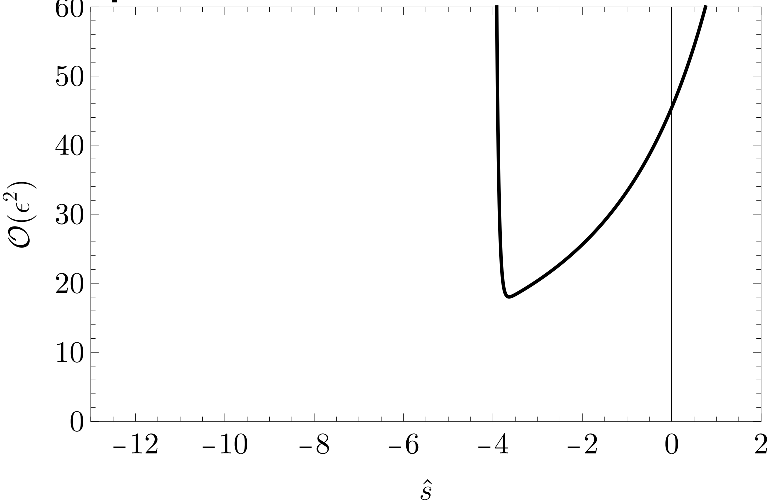
- Special points:

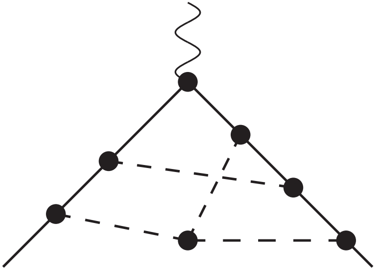
$s = 0$	$s = 4m^2$	$s = \pm\infty$	$s = 16m^2$
$x = 1$	$x = -1$	$x = 0$	$x = 4\sqrt{3} - 7$
static limit	2-particle threshold	high energy limit	4-particle threshold

- Every expansion point needs a different ansatz.
- We construct expansions with $j_{\text{max}} = 50$ around:

$$\hat{s} = \{-\infty, -32, -28, -24, -16, -12, -8, -4, 0, 1, 2, 5/2, 3, 7/2, 4, \\ 9/2, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 19, 22, 28, 40\}$$

Example





- Expansion around $\hat{s} = 0$.

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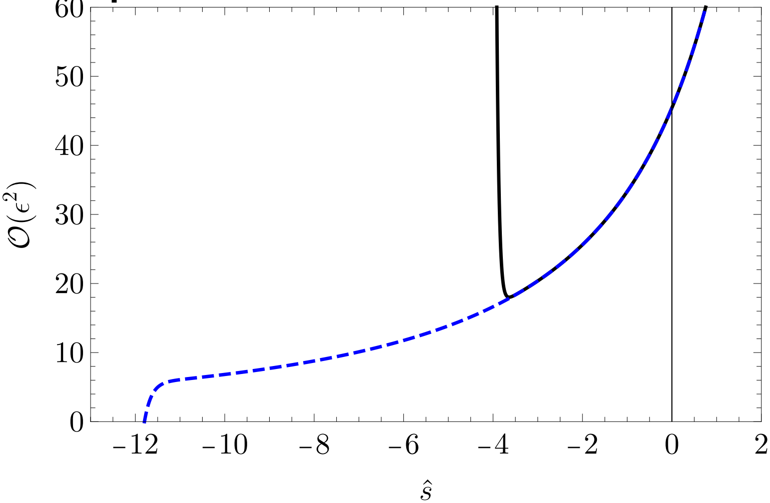

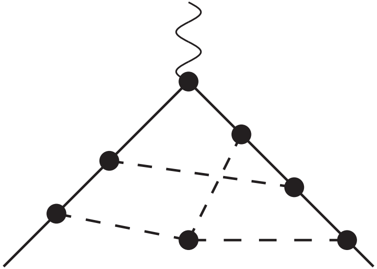

- Expansion around $\hat{s} = 0$.
- Expansion around $\hat{s} = -4$, matched at $\hat{s} = -2$.

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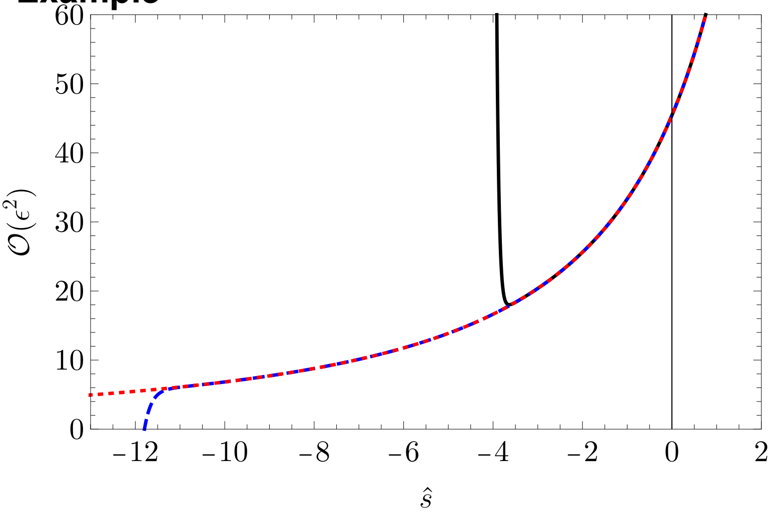

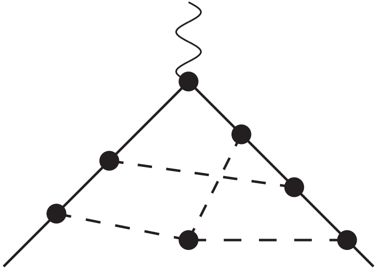

- Expansion around $\hat{s} = 0$.
- Expansion around $\hat{s} = -4$, matched at $\hat{s} = -2$.
- Expansion around $\hat{s} = -8$, matched at $\hat{s} = -6$.

The high-energy limit

$$
\begin{aligned}
F_1^{v,f,(3)}\Big|_{s\to-\infty} &= 8.3501C_A^2C_F - 20.762C_AC_F^2 + 10.425C_AC_FT_Fn_h + 4.7318C_F^3 \\
&- 3.2872C_F^2T_Fn_h + \Big[-6.3561C_A^2C_F - 4.0082C_AC_F^2 + 7.6917C_AC_FT_Fn_h + 3.4586C_F^3 \\
&- 2.8785C_F^2T_Fn_h\Big]l_s + \Big[-2.2488C_A^2C_F + 0.51078C_AC_F^2 + 2.2962C_AC_FT_Fn_h \\
&+ 1.4025C_F^3 - 1.8900C_F^2T_Fn_h\Big]l_s^2 + \Big[-0.42778C_A^2C_F + 0.90267C_AC_F^2 \\
&+ 0.33008C_AC_FT_Fn_h + 0.062184C_F^3 - 0.55727C_F^2T_Fn_h\Big]l_s^3 + \Big[-0.035012C_A^2C_F \\
&+ 0.20814C_AC_F^2 + 0.025463C_AC_FT_Fn_h - 0.075860C_F^3 - 0.086806C_F^2T_Fn_h\Big]l_s^4 \\
&+ \Big[0.019097C_AC_F^2 - 0.023438C_F^3 - 0.0069444C_F^2T_Fn_h\Big]l_s^5 + \Big[-0.0026042C_F^3\Big]l_s^6 \\
&+ \frac{m^2}{-s}\Big\{-47.821C_A^2C_F + 123.65C_AC_F^2 - 52.115C_AC_FT_Fn_h - 92.918C_F^3 \\
&- 5.2612C_F^2T_Fn_h + \Big[17.305C_A^2C_F + 2.3223C_AC_F^2 - 25.912C_AC_FT_Fn_h - 10.381C_F^3 \\
&+ 3.3633C_F^2T_Fn_h\Big]l_s + \Big[8.0183C_A^2C_F - 19.097C_AC_F^2 - 7.8739C_AC_FT_Fn_h + 4.9856C_F^3 \\
&+ 8.4570C_F^2T_Fn_h\Big]l_s^2 + \Big[1.9149C_A^2C_F - 6.8519C_AC_F^2 - 1.4464C_AC_FT_Fn_h + 3.0499C_F^3 \\
&+ 2.3758C_F^2T_Fn_h\Big]l_s^3 + \Big[0.24069C_A^2C_F - 0.91213C_AC_F^2 - 0.067130C_AC_FT_Fn_h \\
&+ 0.67172C_F^3 + 0.48843C_F^2T_Fn_h\Big]l_s^4 + \Big[0.0043403C_A^2C_F - 0.051389C_AC_F^2 \\
&- 0.0034722C_AC_FT_Fn_h + 0.13229C_F^3 + 0.0069444C_F^2T_Fn_h\Big]l_s^5 + \Big[-0.00052083C_A^2C_F \\
&- 0.0010417C_AC_F^2 + 0.0041667C_F^3\Big]l_s^6\Big\} + \mathcal{O}\left(\frac{m^4}{(-s)^2}\right) + n_l,\ n_l^2 \text{ and } n_h^2 \text{ terms}, \qquad (35)
\end{aligned}
$$

- We can reconstruct the Sudakov logarithm.
- We can reconstruct the leading logarithm for the mass supressed term, which has been calculated in the literature [Penin, Liu, Zerf '17-'22]
- We correct the expansion for $F_2^{v,f,(3)}$.
- Our method gives many more supressed terms, although with less precision, when going to higher powers.

Algorithm to Solve Master Integrals

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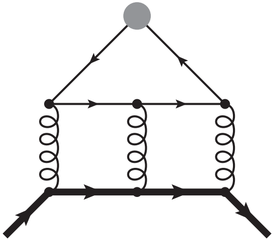
There are other approaches based on expansions:

- `SolveCoupledSystems.m` [Blümlein, Schneider '17]
- `DESS.m` [Lee, Smirnov, Smirnov '18]
- `DiffExp.m` [Hidding '20]
- `AMFlow.m` [Liu, Ma '22]
- `SeaSyde.m` [Armadillo et al '22]
- ...

Our approach ...

- ... does not require a special form of differential equation.
- ... provides approximation in whole kinematic range.
- ... is applied to physical quantity. [Fael, Lange, KS, Steinhauser '21]

Next Steps



- Calculate the singlet diagrams with coupling to massless quarks/leptons.
  - Calculation of boundary conditions harder.
  - Master integrals can be computed with the same method.
  - Results probably less relevant to MuOnE physics.
  - Oposite mass assignement already calculated [Chen, Czakon, Niggetiedt, '21] .

Next Steps

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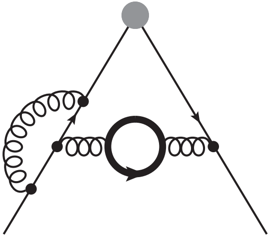

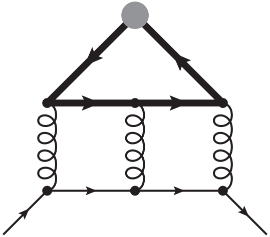

- How can we incorporate a second, heavier massive quark/lepton flavor?
  - Contributions probably very small:

$$m_e/m_\mu \sim 0.005\,, \qquad m_e/m_\tau \sim 0.0003$$

  - Relatively 'easy' to compute since we can do a large mass expansion.
  - Algorithm already automatized in the `q2e/exp` framework [Harlander, Seidensticker, Steinhauser '97, '99] .

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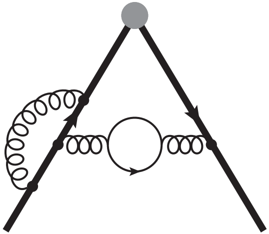
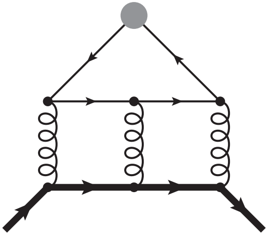
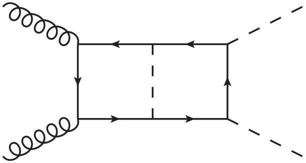




- How can we incorporate a second, lighter massive quark/lepton flavor?
  - Contributions probably bigger.
  - 'Harder' to compute since we have to do a non-trivial asymptotic expansion.
  - With another scale we should think of more numerical approaches.
  - The results from the heavier mass could be used as boundaries for this problem.

Next Steps



- $p_1^2 = p_2^2 = 0$, $p_3^2 = p_4^2 = m_H^2$
- $(p_1 + p_2)^2 = s$, $(p_2 + p_3)^2 = t$, $s + t + u = 2m_H^2$
- High-energy limit: $p_3^2, p_4^2 \sim 0$, $m_t^2 \ll s, t$

- Can we go beyond form factors, i.e. one variable?
  - The method has to be revised, since a matching between two rational functions does not work.
  - Investigate which regions are important and apply appropriate expansions:
    Applied in [Davies, Mishima, Steinhauser, Wellmann, '18; Davies, Mishima, KS, Steinhauser, Zhang '22] for double Higgs production in the high-energy limit, only one deep expansion is needed.

Backup

Renormalization and Infrared Structure

UV renormalization

- On-shell renormalization of mass Z_m^{OS}, wave function Z_2^{OS}, and (if needed) the currents.
  [Chetyrkin, Steinhauser '99; Melnikov, Ritbergen '00]

IR subtraction

- Structure of the infrared poles is given by the cusp anomalous dimension Γ_{cusp}.
  [Grozin, Henn, Korchemski, Marquard '14]
- Define finite form factors $F = Z_{\mathrm{IR}} F^{\mathrm{finite}}$ with the UV renormalized form factor F and

$$Z_{\mathrm{IR}} = 1 - \frac{\alpha_s}{\pi}\frac{1}{2\epsilon}\Gamma^{(1)}_{\mathrm{cusp}} - \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\dots}{\epsilon^2} + \frac{1}{4\epsilon}\Gamma^{(2)}_{\mathrm{cusp}}\right) - \left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{\dots}{\epsilon^3} + \frac{\dots}{\epsilon^2} + \frac{1}{6\epsilon}\Gamma^{(3)}_{\mathrm{cusp}}\right)$$

- $\Gamma_{\mathrm{cusp}} = \Gamma_{\mathrm{cusp}}(x)$ depends on kinematics.
- Γ_{cusp} is universal for all currents.