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Three loop corrections for the QCD and QED form factors

Results and code interface

4th Workstop/Thinkstart: Towards N^{3}LO for $\gamma^* \to \ell\ell$ | IPPP Durham | August 3, 2022

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Outline

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- Results for QED
- `C/Fortran` code of the Form Factors
- Hadronic contributions at $O(\alpha^3)$?

Renormalization and IR subtraction

- UV renormalization in the on-shell scheme. Mass and wave function renormalization.
  Melnikov, van Ritbergen, Phys.Lett.B 482 (2000) 99;
  Chetyrkin, Steinhauser, Nucl.Phys.B 573 (2000) 617-651
- Coupling constant renormalized in $\overline{\mathrm{MS}}$. We use $\alpha_s^{(n_l)}$ as expansion parameter.
- Structure of IR poles is universal

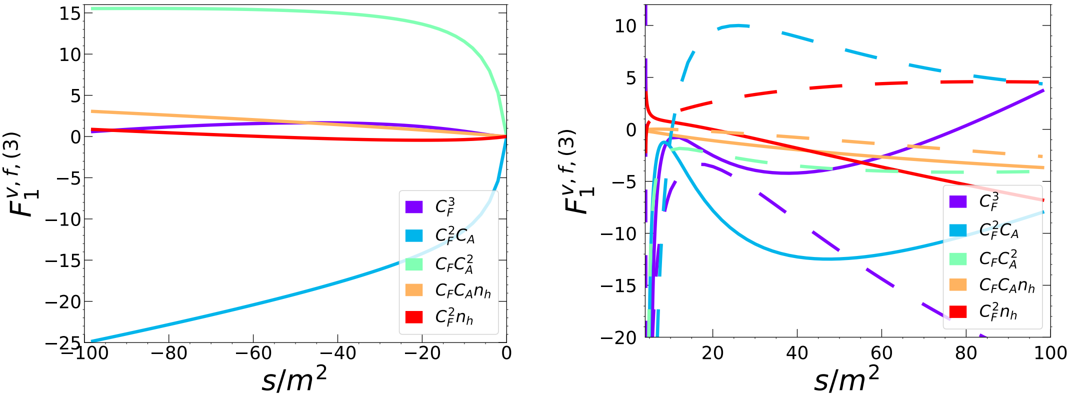
$$F_i^{\mathrm{UV\,ren}}(s) = Z_{\mathrm{IR}} F_i^{\mathrm{f}}(s)$$

with Z_{IR} given in term of the $\Gamma \equiv \Gamma_{\mathrm{cusp}}$ anomalous dimension

$$\begin{aligned}\log Z_{\mathrm{IR}} &= -\frac{\alpha_s(\mu)}{\pi}\frac{\Gamma^{(1)}_{\mathrm{cusp}}}{2\epsilon} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(\frac{\beta_0 \Gamma^{(1)}_{\mathrm{cusp}}}{16\epsilon^2} - \frac{\Gamma^{(2)}_{\mathrm{cusp}}}{4\epsilon}\right) \\ &\quad + \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left(-\frac{\beta_0^2 \Gamma^{(1)}_{\mathrm{cusp}}}{96\epsilon^3} + \frac{\beta_1 \Gamma^{(1)}_{\mathrm{cusp}} + 4\beta_0 \Gamma^{(2)}_{\mathrm{cusp}}}{96\epsilon^2} - \frac{\Gamma^{(3)}_{\mathrm{cusp}}}{6\epsilon}\right)\end{aligned}$$

Grozin, Henn, Korchemsky, Marquard, Phys.Rev.Lett. 114 (2015) 6, 062006; JHEP 01 (2016) 140.

Results in QCD



Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 172003, hep-ph/2207.00027.

QEDfication

- $C_F \to 1$

$$(T^A_{\mathrm{fun}} T^A_{\mathrm{fun}})_{ij} = C_F \delta_{ij}$$

- $C_A \to 0$

$$f^{ACD} f^{BCD} = \mathrm{Tr}(T^A_{\mathrm{adj}} T^B_{\mathrm{adj}}) = C_A \delta_{AB}$$

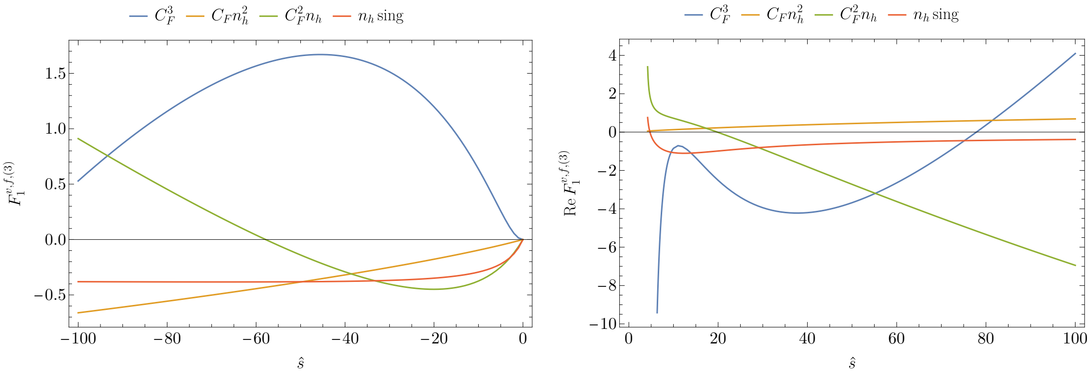
- $T_F \to 1$

$$\mathrm{Tr}(T^A_{\mathrm{fun}} T^B_{\mathrm{fun}}) = T_F \delta_{AB}$$

- Number of massless quarks: $n_l \to 0$ (or massification).
- Massive quark (equal internal and external): $n_h \to 1$.
- $\alpha_s^{(n_l)}(\mu_s) \to \overline{\alpha}(\mu_s)$.

on-shell $\overline{\mathrm{MS}}$ relation for QED: Baikov, Chetyrkin, Kuhn, Sturm, Nucl.Phys.B 867 (2013) 182

Results for QED



Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 172003, hep-ph/2207.00027.

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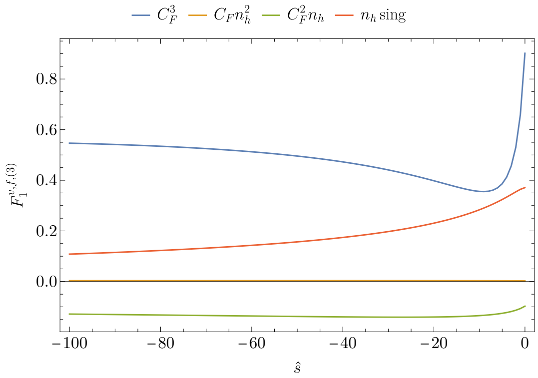

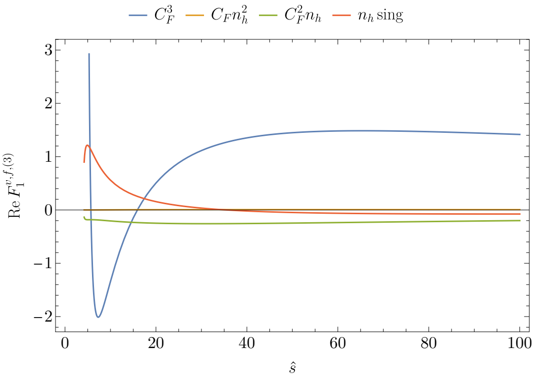

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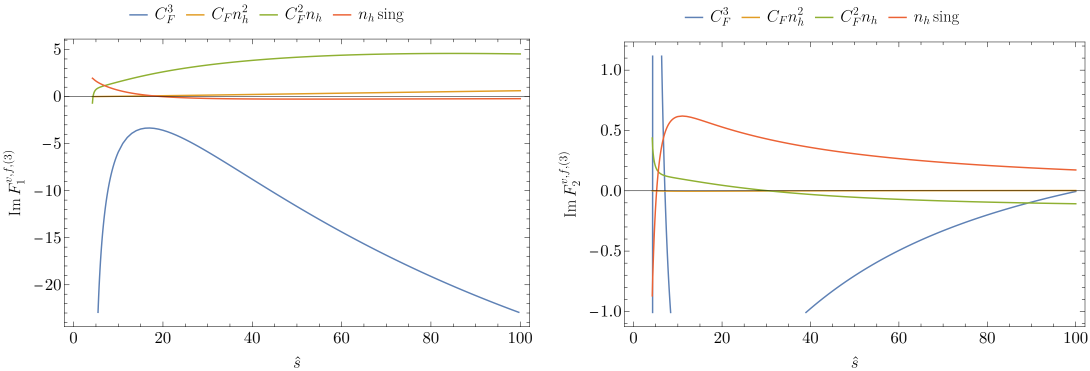

Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 172003, hep-ph/2207.00027.

Estimation of the accuracy

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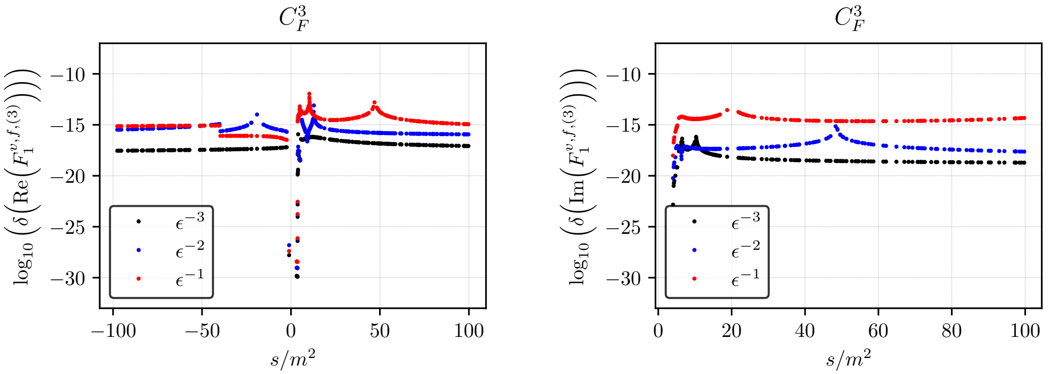
- Bare amplitude up to α_s^2 written in term of HPLs (`ginac`). Exact UV and IR counter terms.
- We estimate the precision from the numerical pole cancellations of the renormalized and infrared-subtracted form factor:

$$\delta\left(F^{f,(3)}\Big|_{\epsilon^i}\right) = \frac{F^{(3)}\Big|_{\epsilon^i} + F^{(\mathrm{CT}+Z)}\Big|_{\epsilon^i}}{F^{(\mathrm{CT}+Z)}\Big|_{\epsilon^i}}$$

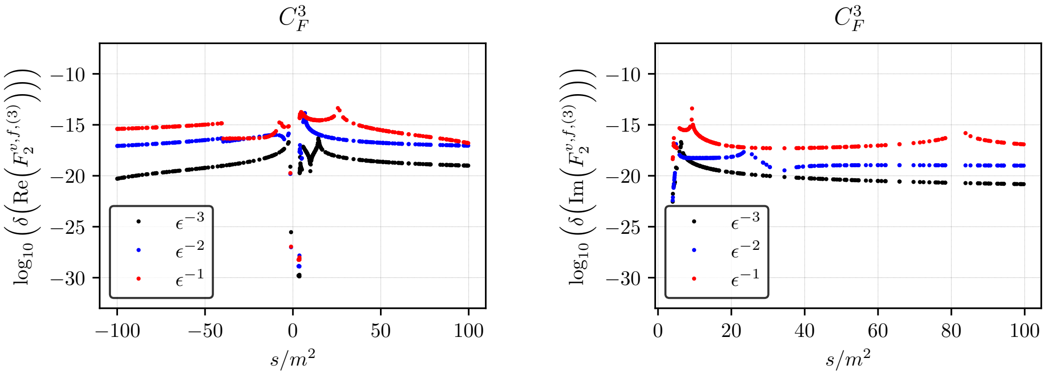
- Close to the zeros of the form factors, the relative precision deteriorates.
- We declare a minimal precision by removing all points whose coefficient size is smaller than 2.5 % of the average in each range
  - $s/m^2 < 0$
  - $0 < s/m^2 < 3.95$
  - $4.05 < s/m^2 < 16$
  - $s/m^2 > 16$
- n_h singlet is finite.

Pole cancellation in QED



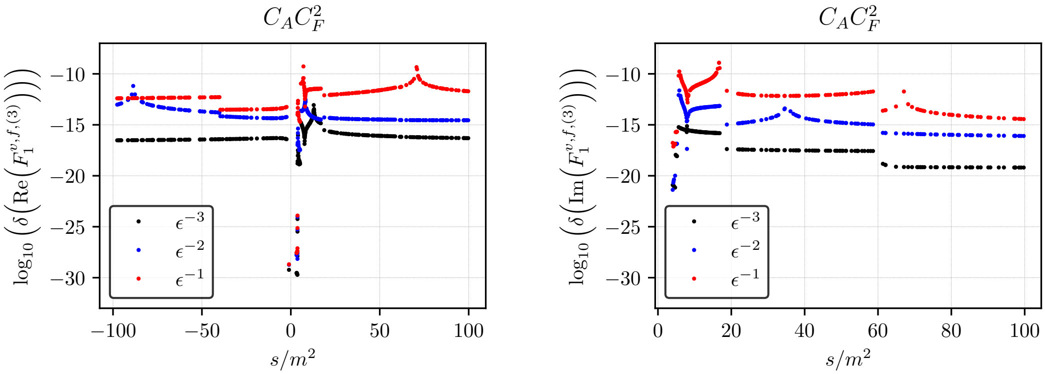
Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 172003, hep-ph/2207.00027.

Pole cancellation in QED



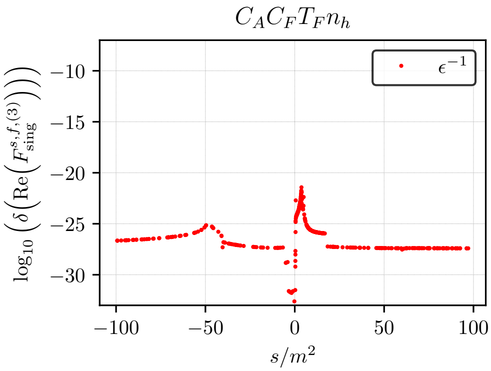
Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 172003, hep-ph/2207.00027.

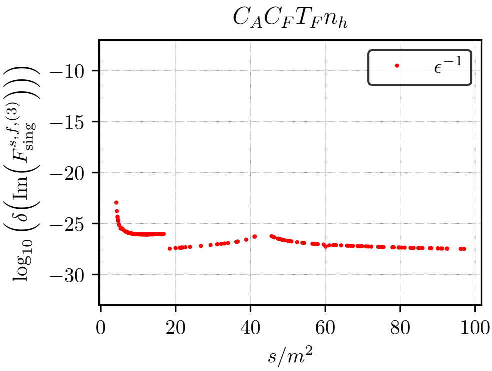
Pole cancellation in QCD



Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 172003, hep-ph/2207.00027.

Pole cancellation for singlet contribution





Lange, MF, Schönwald, Steinhauser, Phys.Rev.Lett. 128 (2022) 172003, hep-ph/2207.00027.

Towards N^3LO Monte Carlo

Experiment	process	F^ℓ	$\sqrt{s}$ [GeV]	Range	
MUonE	$\mu e \to \mu e$	e	0.405	$t \in [0.14, -10^{-3}]\,\mathrm{GeV}^2$	$t/m_e^2 \in [-5 \times 10^5, -4 \times 10^3]$
		μ	0.405	$t \in [0.14, -10^{-3}]\,\mathrm{GeV}^2$	$t/m_\mu^2 \in [-12.5, -0.1]$
BESIII	$ee \to \ell\ell$	e	~ 4 GeV	$s = 16$ GeV^2	$s/m_e^2 \simeq 6 \times 10^7$
		μ	~ 4 GeV	$s = 16$ GeV^2	$s/m_\mu^2 \simeq 1400$
		τ	~ 4 GeV	$s = 16$ GeV^2	$s/m_\tau^2 \simeq 5$
P2	$ep \to ep$	e	1.08 GeV	$t \in [-13, -5] \times 10^{-3}\,\mathrm{GeV}^2$	$t/m_e^2 \in [-5, -2] \times 10^4$

Mathematica Package

- `https://gitlab.com/formfactors3l/formfactors3l`
- What is implemented:
  - Bare and renormalized form factors.
  - Non-singlet and n_h singlet
  - Interpolation grid in the rages $-40 < s/m^2 < 3.75$ and $4.25 < s/m^2 < 60$.
  - In the remaining regions, it uses special series expansion around $s = \pm\infty$ and $s = 4$.
- Complete QCD color structures. For QED
  - $C_F \to 1$, denoted by `cR`
  - $T_F \to 1$, denoted by `I2R`
  - $C_A \to 0$, denoted by `cA`
  - $n_\ell \to 0$, no massless lepton.
  - $n_h = 1$, internal and external leptons have the same mass.
- 10^4 random points, single core, in about 120s (full QCD) for each form factor.

- Compute bare form factor F_1 and F_2 at order ϵ^0 at $s/m^2 = -1$:

```
In[] := Get["PATH/FormFactors3l.m"]

In[] := FormFactorBareNonSing[veF1, 0, -1]
Out[]:= 77.0506 cA^2 cR+95.0634 cA cR^2+0.467466 cR^3-21.9243 cA cR I2R nh
       -11.5582 cR^2 I2R nh+0.751403 cR I2R^2 nh^2-62.6063 cA cR I2R nl
       -45.5408 cR^2 I2R nl+9.35837 cR I2R^2 nh nl+11.8102 cR I2R^2 nl^2

In[] := FormFactorBareNonSing[veF2, 0, -1]
Out[]:= 36.8004 cA^2 cR+21.3251 cA cR^2+31.335 cR^3-10.8947 cA cR I2R nh
       -4.17381 cR^2 I2R nh+0.434916 cR I2R^2 nh^2-26.8267 cA cR I2R nl
       -12.9507 cR^2 I2R nl+3.90787 cR I2R^2 nh nl+4.58439 cR I2R^2 nl^2
```

- n_h-singlet contribution

```
In[] := FormFactorBareNhSing[veF1, 0, -1]
Out[]:= -0.075237 d33 nh/nc
```

- Renormalized form factor

```
In[] := FormFactorRenNonSing[veF1, -1]
Out[]:= 3.10714 cA^2 cR-3.23413 cA cR^2+0.0144347 cR^3+0.0435081 cA cR I2R nh
       -0.0640418 cR^2 I2R nh-0.0107609 cR I2R^2 nh^2-2.59041 cA cR I2R nl
       +1.02032 cR^2 I2R nl+0.000282528 cR I2R^2 nh nl+0.494057 cR I2R^2 nl^2
```

Towards a compiled C/Fortran version

- Use `C`: interpolation routines from `GNU Scientific Libraries`.
- Use `Fortran`: native support for complex numbers.

<u>Features</u>

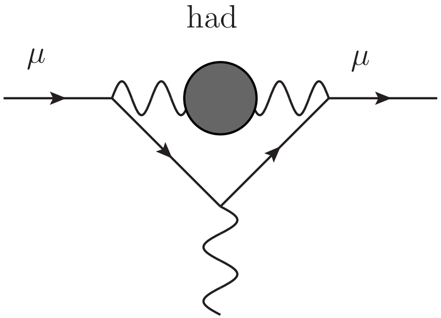
- Should we have a `FormFactors3l.F` or `FormFactors3l_qed.F`?
- For QED we can aim at 10^{-10} relative precision at $O(\epsilon^0)$.
- Implementation of bare and renormalized form factors?
- Standalone implementation of the $s = \pm\infty$ expansion?
- Do you need an exact (but slow) evaluation of the CTs?
- Parallellizable?

NLO Hadronic contribution to the form factors

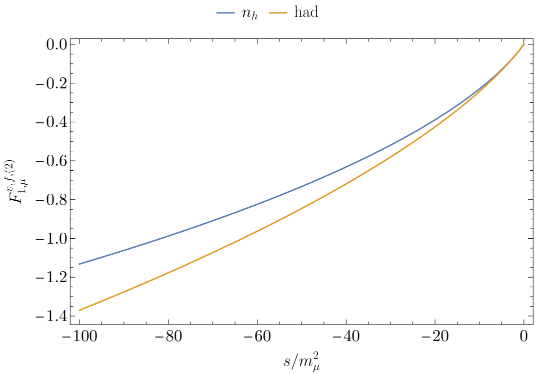
$$F_i^{\ell,\mathrm{vp}}(s) = -\frac{\alpha}{\pi}\int_0^1 \mathrm{d}x\,\Pi\left(\frac{m_\ell^2 x^2}{x-1}\right) f_i(x, s/m_\ell^2)$$

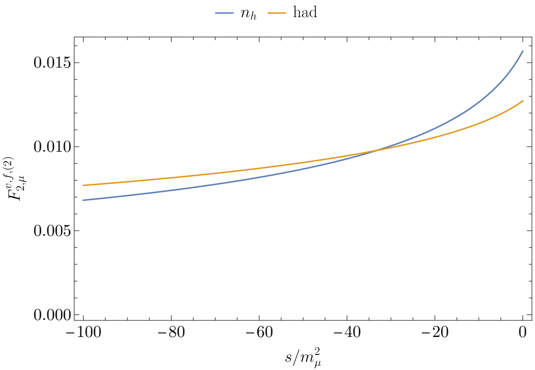
MF, JHEP 02 (2019) 027

- Are hadronic corrections really needed?
- For electron form factors negligible at low energies.
- For muon, n_h and hadronic corrections are comparable.
- NLO corrections are doable.
- Light-by-light appears at NLO also.



Muon form factors



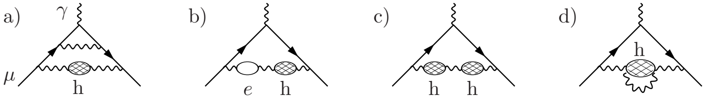


for hadronic vp: Jegerlehner, `alphaQEDc17`

NLO Hadronic corrections

- Dispersion relation

$$F_i^{\ell,vp}(s) = \left(\frac{\alpha}{\pi}\right)^3 \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}z}{z} R(z)\, K(z,s)$$



Jegerlehner, Nyffeler, Phys.Rept. 477 (2009) 1

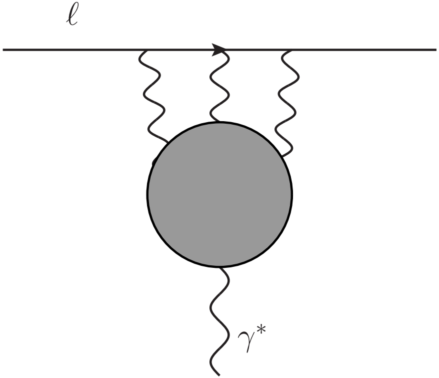
- Diagrams with heavy photon with mass $\sqrt{z}$.
- Expansion in $m_\ell^2 \ll z$.

Example for a_μ^{HNLO}

$$K^{[(a)]}(s/m^2) = \frac{m^2}{s}\left[\frac{223}{54} - 2\zeta(2) - \frac{23}{36}\ln\frac{s}{m^2}\right] + \frac{m^4}{s^2}\left[\frac{8785}{1152} - \frac{37}{8}\zeta(2) - \frac{367}{216}\ln\frac{s}{m^2} + \frac{19}{144}\ln^2\frac{s}{m^2}\right] + \frac{m^6}{s^3}\left[\frac{13072841}{432000} - \frac{883}{40}\zeta(2) - \frac{10079}{3600}\ln\frac{s}{m^2} + \frac{141}{80}\ln^2\frac{s}{m^2}\right] + \ldots$$

Krause, Phys. Lett. B 390 (1997) 392

Light-by-light



Conclusions

- Non-singlet and n_h-singlet contributions to the QED/QCD massive form factors at N^3LO.
- Currents: vector, axial-vector, scalar, pseudoscalar.
- Bare form factors are determined as expansions around certain regular and singular kinematical points.
- We provide a Mathematica package with interpolation grids and expansions around $s = \pm\infty$ and $s = 4$.
- Precision (QCD): at least 7 digits.
- Precision (QED), vector form factors: at least 10 digits.
- We provide on of the building blocks for $\gamma^* \to \ell^+\ell^-$ differential Monte Carlo with dominant N^3LO corrections.