Two-loop Amplitudes via the AIDA framework

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In collaboration with:

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4th Workstop/Thinkstart

Towards N^3LO for $\gamma^* \to \bar{l}l$

August 03-05, 2022

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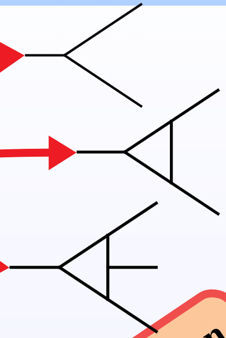
Cross Section and Scattering Amplitudes in pQFT

Target observable: **cross section**

$$\sigma(1 \to 2) = \alpha^2 \left[\sum_{j=0}^{n} \left(\frac{\alpha}{\pi}\right)^j \sigma_{\mathrm{N^jLO}} + O(\alpha^{n+1}) \right]$$

related to the **scattering amplitude**

$$\mathcal{A} = 4\pi\mu^{-2\epsilon} \sum_j \left(\frac{\alpha}{\pi}\right)^j \mathcal{A}_j$$

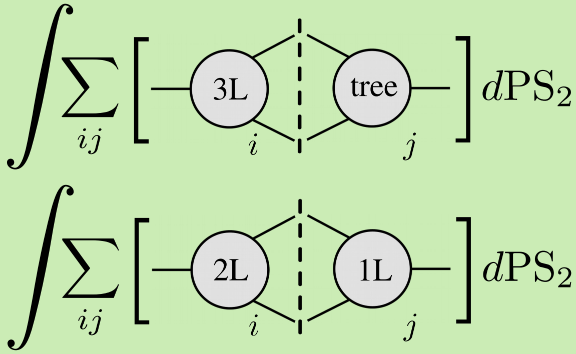


Feynman diagrams

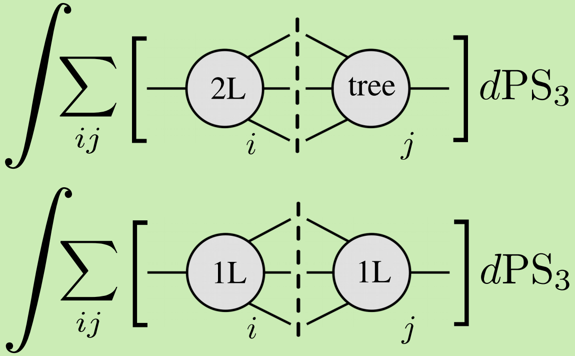
$\sigma_{\mathrm{N^3LO}}$

gets four contributions

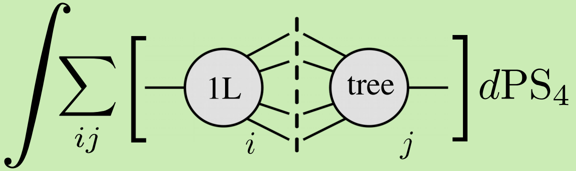
VVV



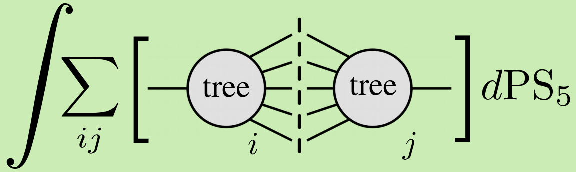
RVV



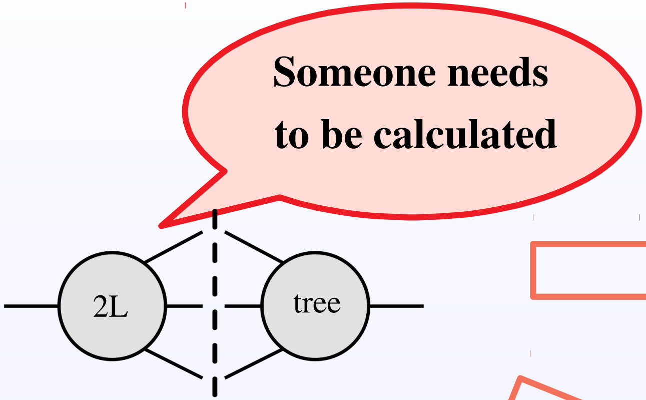
RRV



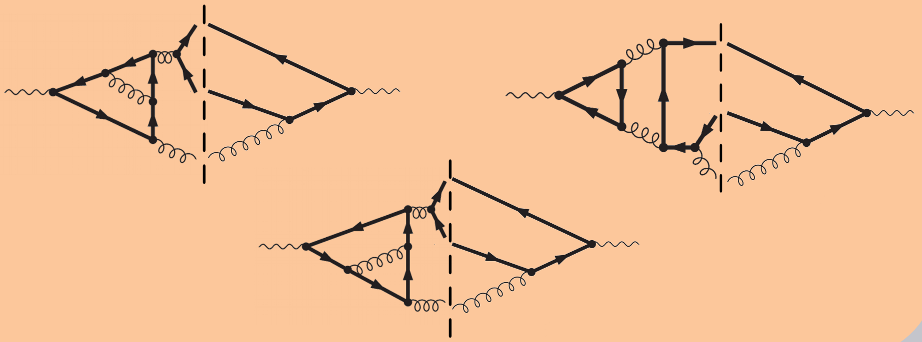
RRR



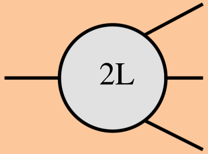
RVV Interferences



Direct calculation of interferences?

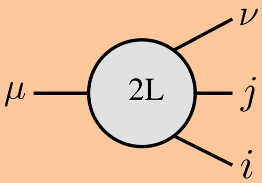


Helicity amplitudes?



$$= \sum_{h_1 h_2 p_1 p_2} \epsilon^{\mu}_{h_1} \epsilon^{\nu}_{h_2} \overline{\psi}_{p_1} \mathcal{M}_{\mu\nu} \psi_{p_2}$$

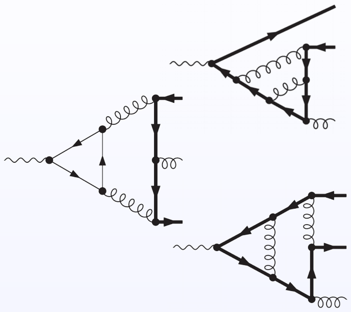
Projection onto form factors?



$$= \sum_{n} \mathrm{FF}_n S^{\mu\nu}_{n,ij}$$

Anatomy of $\gamma^* \to q\bar{q}g$ two-loop NNLO QED contributions

$$\mathcal{M}^{(2)} = \overline{\sum} 2\mathrm{Re}[\mathcal{A}^{(0)*}\mathcal{A}^{(n)}]$$
$$= (S_\epsilon)^2 \int d^d k_1 d^d k_2 \sum_G \frac{\mathcal{N}_G(\mathbf{k},\mathbf{l})}{\prod_{\sigma\in G} D_\sigma(\mathbf{k},\mathbf{l})}$$



$\in G$

Two-loop Feynman integrals

- 4-point kinematics
- Up to 4-point integrals
- Mass-scales
  - 2 Mandelstam s_{12}, s_{23}
  - Masses m_γ, m_l
  - Additional virtuality m_v

Analytically known
integrals for
$m_l = 0$
from the **QCD**
$e^+e^- \to$ 3jets

[Gehrmann, Remiddi (2001)] x2

How many integrals are known for the **massive lepton** case?

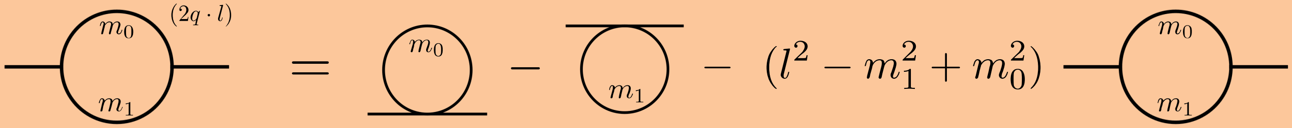
Feynman integrals are (in general) UV and IR divergent

Using **Dimensional Regularization**: space-time is treated as a free parameter d

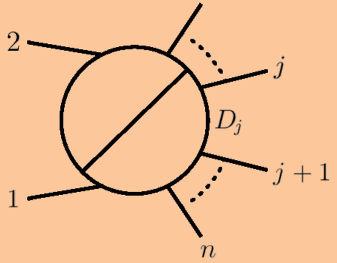
Integrand decomposition of Feynman Integrals

$$\mathcal{I}(l) = \int_k \frac{\mathcal{N}(k,l)}{D_1(k,l)\cdots D_n(k,l)}$$

@1L



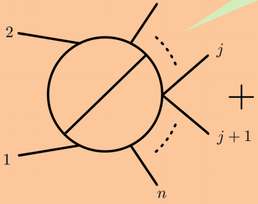
[Ossola, Papadopoulos, PiSau (2006)]
[Ellis, Giele, Kunszt, Melnikov (2007)]
[Mastrolia, Ossola, Papadopoulos, PiSau (2008)]



$$= \frac{\mathcal{N}(k,l)}{D_1 \cdots D_j \cdots D_n}$$

Polynomial division modulo Gröbner basis

$$= \sum_{j=1}^{n}$$



$$+ \frac{\Delta_{\hat{j}}(k,l)}{D_1 \cdots D_j \cdots D_n}$$

Iterating...

$$\vdots$$

$$= \sum_{j=1}^{n} \sum_{i_1 \cdots i_j}^{n} \frac{\Delta_{i_1 \cdots i_j}(k,l)}{D_{i_1} \cdots D_{i_j}}$$

Contain Irreducible Scalar Products

- **Generalizable at n-loop**
- **Works with helicity amplitudes**
- **Possible automation**

[Mastrolia, Ossola (2011)]
[Zhang (2012-2016)]
[Badger, Frellesvig, Zhang (2012-2013)]
[Mastrolia, Mirabella, Ossola, Peraro (2012)]

Adaptive Integrand Decomposition (AID)

$$\mathcal{I}(l) = \int_k \frac{\mathcal{N}(k,l)}{D_1(k,l)\cdots D_n(k,l)}$$

[Collins (1984)]
[van Neerven and Vermaseren (1984)]
[Kreimer (1992)]

Idea

$$d = d_{||} + d_{\perp}$$
$$k = k_{||} + k_{\perp}$$

$$k_j^\mu = k_{||j}^\mu + \lambda_j^\mu$$
$$k_i \cdot k_j = k_{||i} \cdot k_{||j} + \lambda_{ij}$$

$$\mathcal{I}(l) = \int d^{d_{||}} k_{||} d^{d_\perp} k_\perp \frac{\mathcal{N}(k,l)}{D_1(k,l)\cdots D_n(k,l)}$$
$$= \int d^{d_{||}} k_{||} \int \prod_{ij} G(\lambda_{ij}) d\lambda_{ij} k_\perp d\Theta_\perp \frac{\mathcal{N}(k_{||},\lambda_{ij},\Theta_\perp)}{D_1(k_{||},\lambda_{ij})\cdots D_n(k_{||},\lambda_{ij})}$$

Transverse direction can be integrated out

$$\int d\Theta_\perp = \int_{-1}^{1} \prod_{i=1}^{4-d_{||}} \prod_{j=1}^{L} d\cos\theta_{i+j-1,j} (\sin\theta_{i+j-1,j})^{d_\perp - i - j - 1}$$

$$\int_{-1}^{1} dx (1-x^2)^{\alpha-\frac{1}{2}} C_n^{(\alpha)}(x) C_m^{(\alpha)}(x) = \delta_{nm} \frac{2^{1-2\alpha}\pi\Gamma(n+2\alpha)}{n!(n+\alpha)\Gamma^2(\alpha)}$$

Gegenbauer polynomials

[Mastrolia, Peraro, Primo (2016)]
[Mastrolia, Peraro, Primo, Torres Bobadilla (2016)]

Divide, Integrate, Divide again

$$\int dk \frac{\mathcal{N}(k,l)}{D_1(k,l)\cdots D_n(k,l)}$$

Divide

$$\sum_{j=1}^{n}\sum_{i_1\cdots i_j}^{n}\int dk_{||}d\lambda d\Theta_{\perp}\frac{\Delta_{i_1\cdots i_j}(k_{||},\lambda,\Theta_{\perp})}{D_{i_1}(k_{||},\lambda)\cdots D_{i_j}(k_{||},\lambda)}$$

Integrate

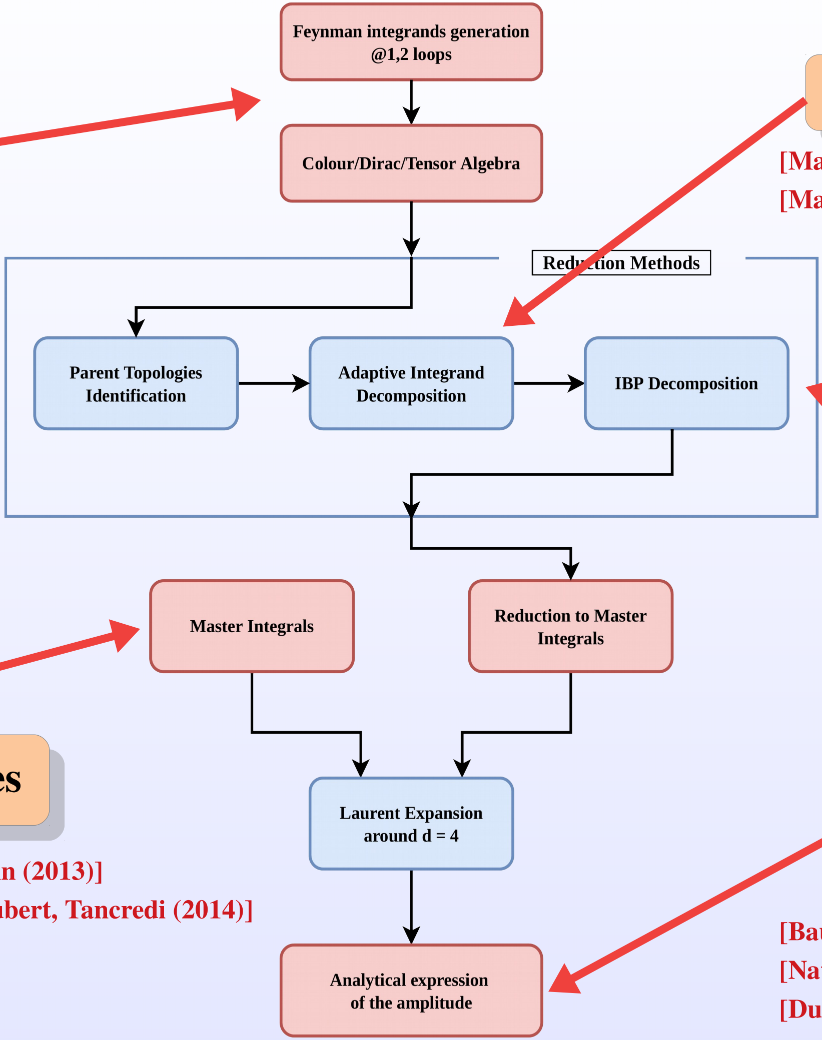
$$\sum_{j=1}^{n}\sum_{i_1\cdots i_j}^{n}\int dk_{||}d\lambda\frac{\Delta^{\text{int}}_{i_1\cdots i_j}(k_{||},\lambda)}{D_{i_1}(k_{||},\lambda)\cdots D_{i_j}(k_{||},\lambda)}$$

Divide

$$\sum_{j=1}^{n}\sum_{i_1\cdots i_j}^{n}\int dk_{||}d\lambda\frac{\Delta'_{i_1\cdots i_j}(k_{||})}{D_{i_1}(k_{||},\lambda)\cdots D_{i_j}(k_{||},\lambda)}$$

- **Numerator reduced in terms of ISPs**
- **No need for integral identities at 1L**
- **At 2L, amplitude ready for IBPs**

The AIDA framework



FeynCalc, FeynArts

[Mertig, Böhm, Denner (1991)]
[Shtabovenko, Mertig, Orellana (2016)]
[Hahn (2001)]

AIDA

[Mastrolia, Peraro, Primo, Torres (2016)]
[Mastrolia, Peraro, Primo, Torres, JR (To be published)]

Reduze, Kira

[Chetyrkin, Tkachev (1981)] [Laporta (2000)]
[von Manteuffel, Studerus (2012)]
[Maierhöfer, Usovitsch, Uwer (2018)]

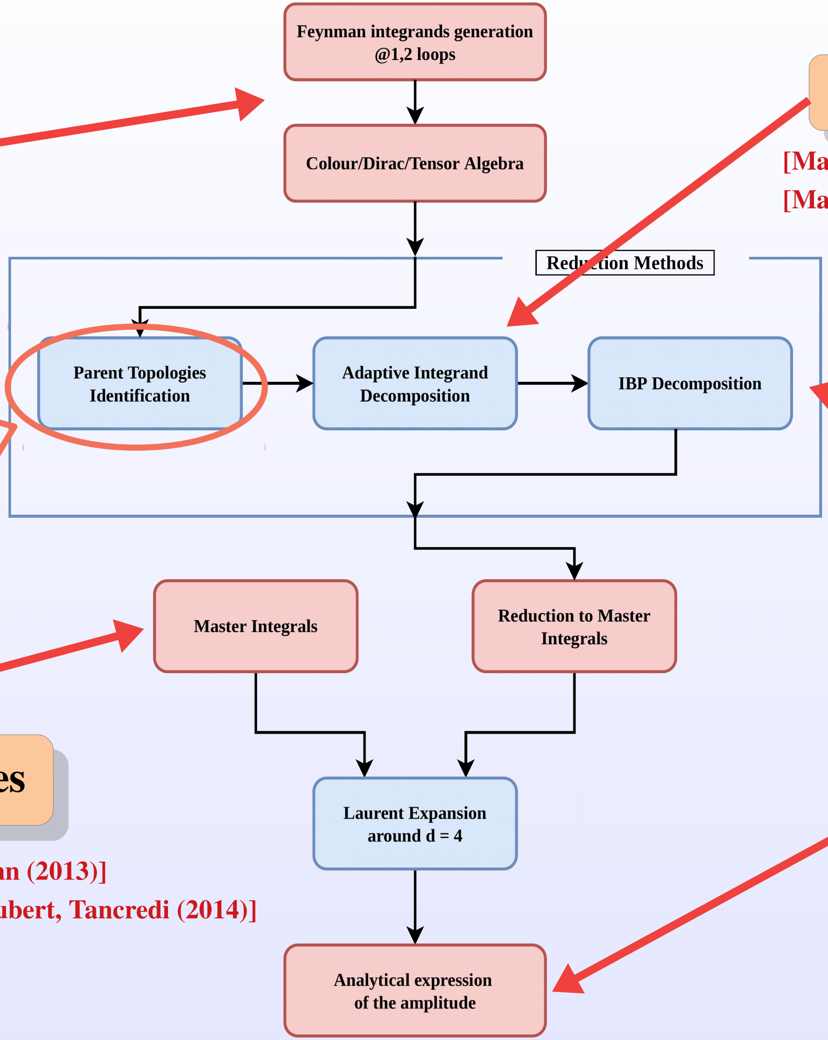
In-house Mathematica routines

[Kotikov (1990)] [Gehrmann, Remiddi (1999)] [Henn (2013)]
[Argeri, Di Vita, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi (2014)]

HandyG, GiNaC, PolyLogTools

[Bauer, Frink, Kreckel (2000)][Vollinga,Weinzierl(2005)]
[Naterop, Signer, Ulrich (2019)]
[Duhr, Dulat (2019)]

The AIDA framework



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Something that can be launched quickly

Reduze, Kira

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HandyG, GiNaC, PolyLogTools

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Applications

Double Virtuals for:

$$e^+e^- \to \mu^+\mu^- \text{ @NNLO QED}$$

[Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Schubert, Torres Bobadilla, Tramontano, JR (2021)]

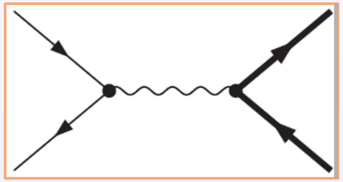
$$q\bar{q} \to t\bar{t} \text{ @NNLO QCD}$$

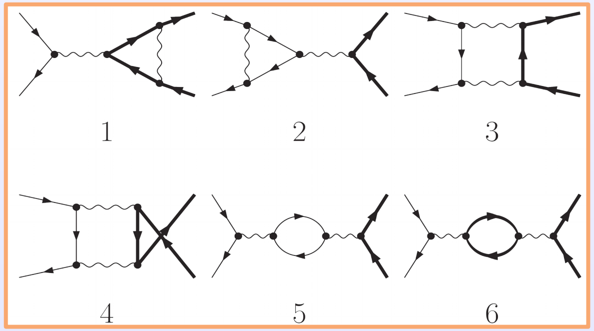
[Mandal, Mastrolia, Torres Bobadilla, JR (2022)]

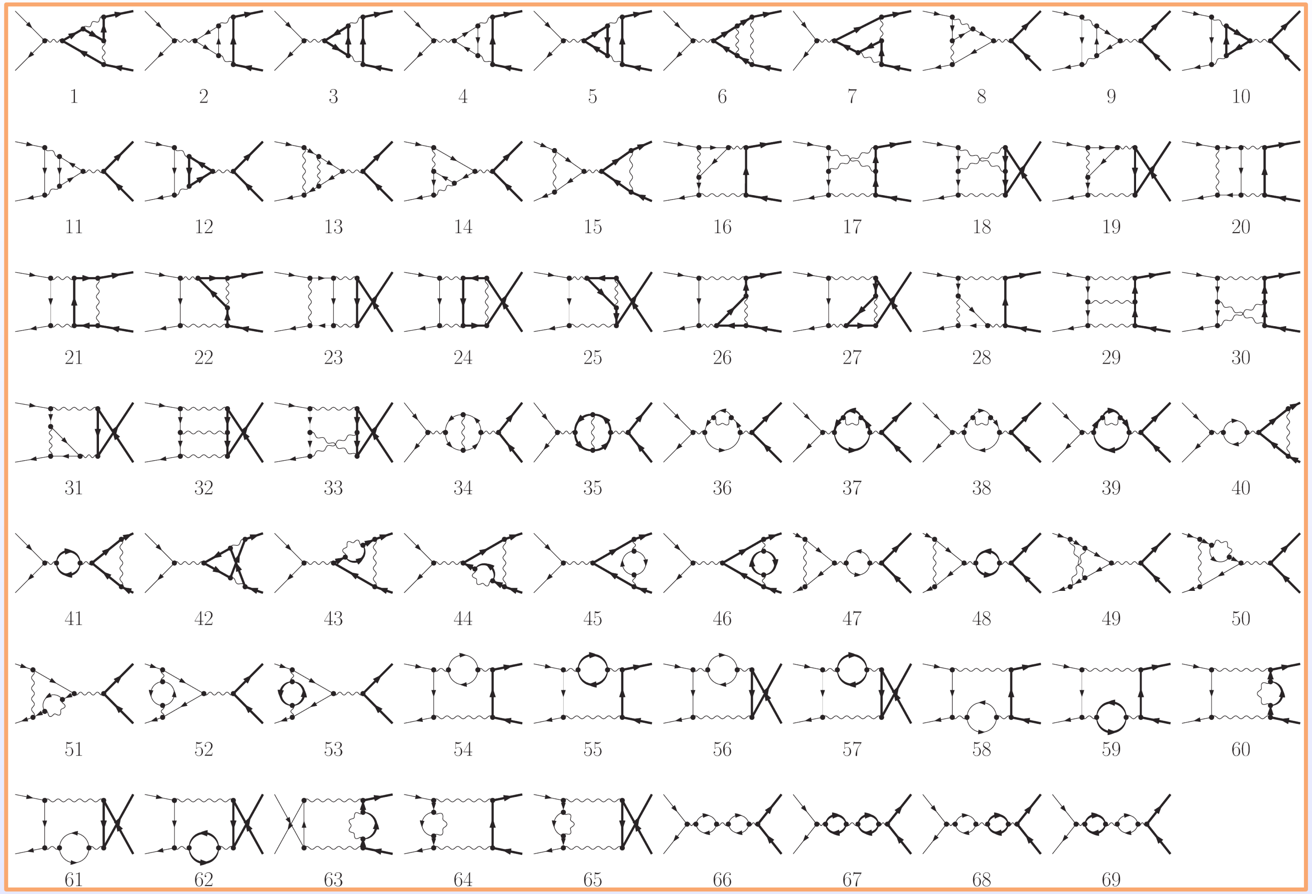
Di-muon production in QED: Feynman Diagrams

$$\mathcal{M}_b^{(1)} = A^{(1)} + n_l B_l^{(1)} + n_h C_h^{(1)}$$

$$\mathcal{M}_b^{(2)} = A^{(2)} + n_l B_l^{(2)} + n_h C_h^{(2)} + n_l^2 D_l^{(2)} + n_l n_h D_{lh}^{(2)} + n_h^2 F_h^{(2)}$$







[Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Schubert, Torres Bobadilla, Tramontano, JR (2021)]

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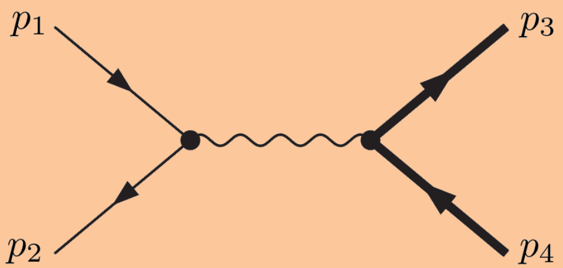


$$m_e = 0$$
$$m_\mu = M$$

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_2 - p_3)^2$$

$$s + t + u = M^2$$

$O(10^4)$ terms
max rank = 4

$$\mathcal{M}^{(2)} = (S_\epsilon)^2 \int d^d k_1 d^d k_2 \sum_G \frac{\mathcal{N}_G(\mathbf{k}, \mathbf{l})}{\prod_{\sigma \in G} D_\sigma(\mathbf{k}, \mathbf{l})}$$

AID

$O(10^4)$ terms
max rank = 2

$$\mathcal{M}^{(2)} = \sum_{j=1}^{n} \sum_{i_1 \cdots i_j}^{n} (S_\epsilon)^2 \int d^d k_1 d^d k_2 \frac{\Delta'_{i_1 \cdots i_j}(\mathbf{k}, \mathbf{l})}{D_{i_1} \cdots D_{i_j}}$$

IBPs

$O(10^2)$ terms
max rank = 2

$$\mathcal{M}^{(2)} = \sum_{j=1}^{N_{\mathrm{MI}}} c_j(s, t, M; d) I_j^{(2)}(s, t, M; d)$$

Master Integrals

Master Integrals for $\mu e \to \mu e$ are known in literature.

$$\mathbf{I}^{(2)}_{\mu e \to \mu e} = \mathbf{I}^{(2)}_{ee \to \mu\mu}\big|_{s \leftrightarrow t}$$

Representation through **Generalized PolyLogarithms**

$$I_j^{(2)}(s,t,M;d) = \sum_i C_i(\{w\}_{ji}, d) G(\{w\}_{ji}, 1)$$

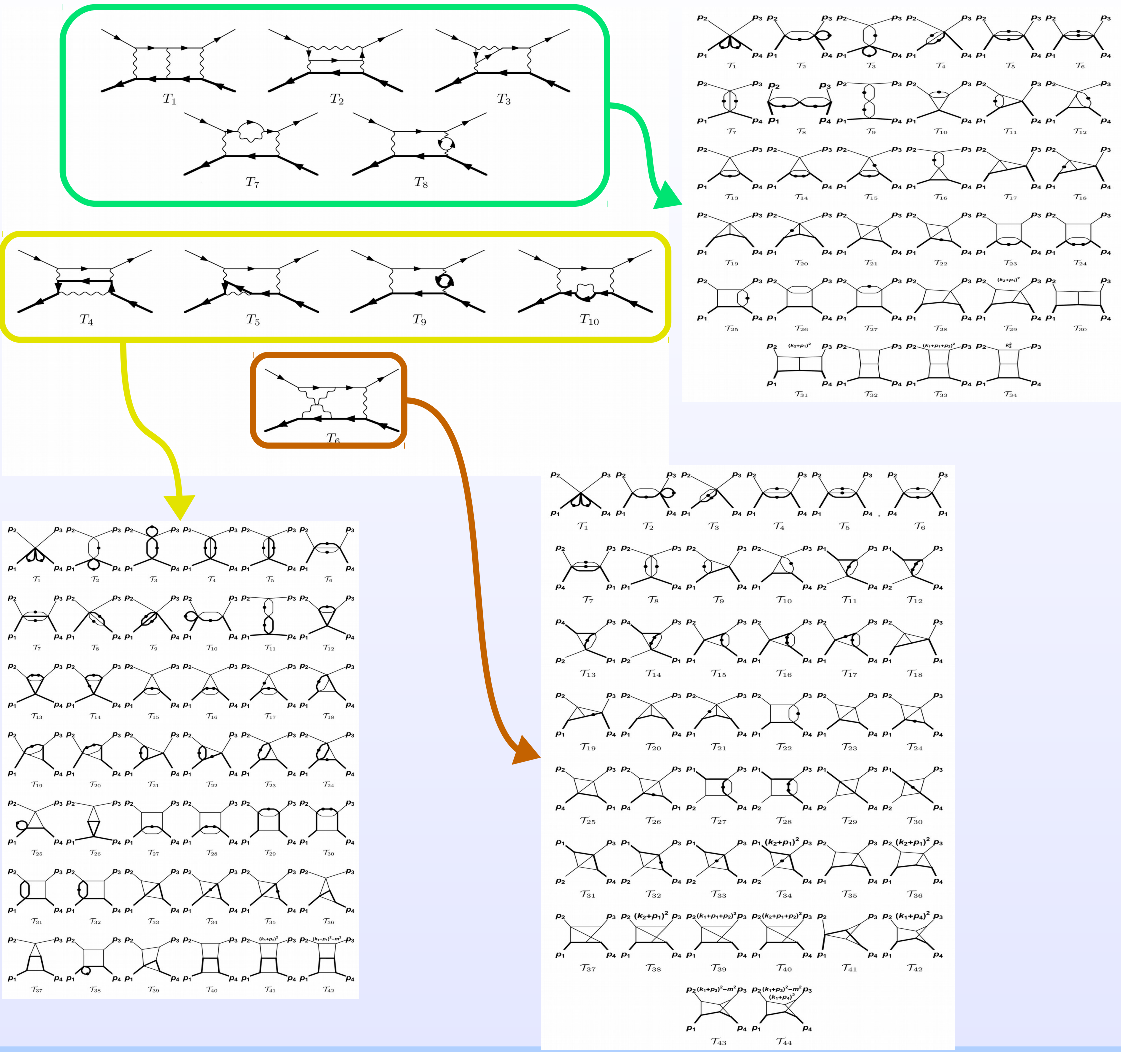
where

$$G(w_n, \ldots, w_1; \tau) = \int_0^\tau \frac{dt}{t - w_n} G(w_{n-1}, \ldots, w_1; t)$$

Letters
$w_j = w_j(s,t,M)$

- O(4000) GPLs
- GPLs up to weight 4
- 18 letters

[Kotikov (1990)] [Gehrmann, Remiddi (1999)] [Henn (2013)]
[Argeri, Di Vita, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi (2014)]
[Mastrolia, Passera, Primo, Schubert (2017)]
[Di Vita, Laporta, Mastrolia, Primo, Schubert (2018)]



Checks

$$\mathcal{M}_b^{(1)} = A^{(1)} + n_l B_l^{(1)} + n_h C_h^{(1)}$$

$$\mathcal{M}_b^{(2)} = A^{(2)} + n_l B_l^{(2)} + n_h C_h^{(2)} + n_l^2 D_l^{(2)} + n_l n_h D_{lh}^{(2)} + n_h^2 F_h^{(2)}$$

Evaluating the interferences with **HandyG** and **GiNaC**

$$\frac{s}{M^2} = 5, \quad \frac{t}{M^2} = -\frac{5}{4}, \quad \mu = M$$

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}	ϵ
$\mathcal{M}^{(0)}$	-	-	-	-	$\frac{181}{100}$	-2
$A^{(1)}$	-	-	$-\frac{181}{100}$	1.99877525	22.0079572	−11.7311017
$B_l^{(1)}$	-	-	-	-	−0.069056030	4.94328573
$C_h^{(1)}$	-	-	-	-	−2.24934027	2.54943566
$A^{(2)}$	$\frac{181}{400}$	−0.499387626	−35.4922919	19.4997261	49.0559119	-
$B_l^{(2)}$	-	$-\frac{181}{400}$	0.785712779	−16.1576674	−3.75247701	-
$C_h^{(2)}$	-	-	1.12467013	−9.50785825	−25.8771503	-
$D_l^{(2)}$	-	-	-	-	−3.96845688	-
$E_{hl}^{(2)}$	-	-	-	-	−4.88512563	-
$F_h^{(2)}$	-	-	-	-	−0.158490810	-

[Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Schubert, Torres Bobadilla, Tramontano, JR (2021)]

How do we check these terms?

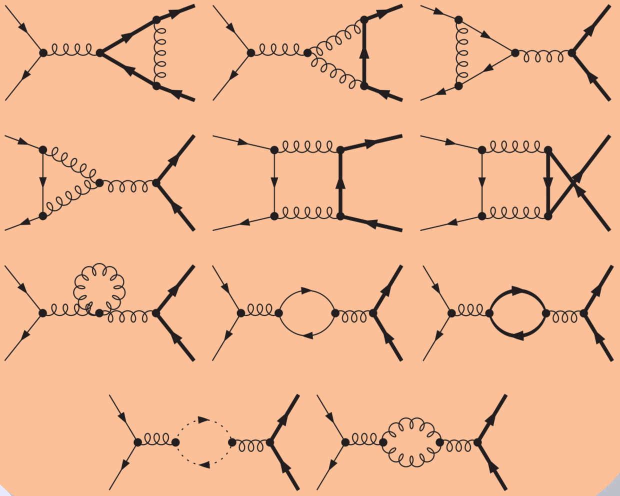
Checks :: Literature

$q\bar{q} \to t\bar{t}$ **in QCD** ⟹ $e^- e^+ \to \mu^- \mu^+$ **in QED**

- Top-pair production admit a color decomposition
- 1-loop and 2-loop corrections already known in literature
- Abelian part get contributions from *QED-like diagrams* only

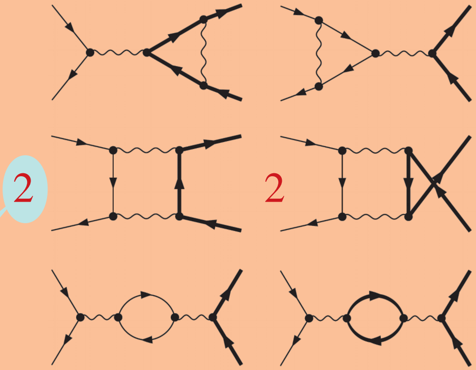
[Czakon(2008)]
[Bonciani, Ferroglia, Gehrmann, Maitre, Studerus (2008)]
[Bärnreuther, Czakon, Fiedler (2014)]

$q\bar{q} \to t\bar{t}$ @one-loop in QCD



Abelian, Color-stripped

$e^- e^+ \to \mu^- \mu^+$@one-loop in QED



Remainder of color factors

Full agreement with the abelian part of top-pair production

Two-loop IR poles from **one-loop and tree** (renormalised) **contributions**

$$\mathcal{M}^{(1)}\Big|_{\text{poles}} = \frac{1}{2} Z_1^{\text{IR}}\, \mathcal{M}^{(0)}\Big|_{\text{poles}}$$

$$\mathcal{M}^{(2)}\Big|_{\text{poles}} = \frac{1}{8}\Big[\Big(Z_2^{\text{IR}} - \big(Z_1^{\text{IR}}\big)^2\Big)\,\mathcal{M}^{(0)} + 2\, Z_1^{\text{IR}}\,\mathcal{M}^{(1)}\Big]\Big|_{\text{poles}}$$

[Becher, Neubert (2009)]
[Hill (2017)]

IR Renormalisation Factor

Beta function

$$\gamma_i = \sum_{j=0}^{n} \left(\frac{\alpha}{\pi}\right)^{j+1} \gamma_i^{(j)} + O\left(\alpha^{n+1}\right)$$

$$\ln Z_{\text{IR}} = \frac{\alpha}{4\pi}\left(\frac{\Gamma_0'}{4\epsilon^2} + \frac{\Gamma_0}{2\epsilon}\right) + \left(\frac{\alpha}{4\pi}\right)^2 \left(-\frac{3\beta_0\Gamma_0'}{16\epsilon^3} + \frac{\Gamma_1' - 4\beta_0\Gamma_0}{16\epsilon^2} + \frac{\Gamma_1}{4\epsilon}\right) + \mathcal{O}\left(\alpha^3\right)$$

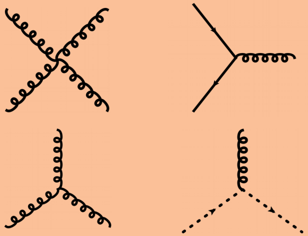
Anomalous dimension $\Gamma = \gamma_{\text{cusp}}(\alpha)\ln\left(-\frac{s}{\mu^2}\right) + 2\gamma_{\text{cusp}}(\alpha)\ln\left(\frac{t-M^2}{u-M^2}\right) + \gamma_{\text{cusp,M}}(\alpha, s) + 2\gamma_h(\alpha) + 2\gamma_\psi(\alpha)$

☑ **Full agreement** with the direct calculation of the two-loop contribution

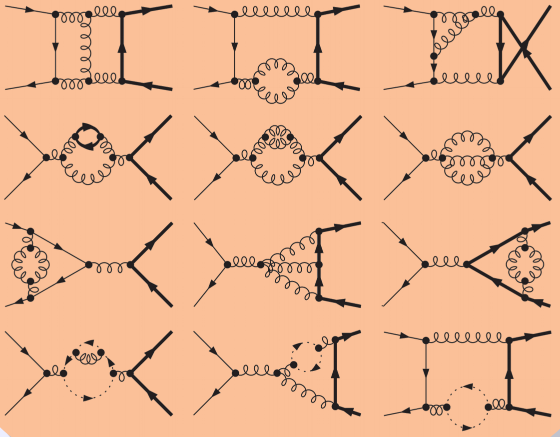
Extension to Two-loop top-pair production @ NNLO QCD

Full two-loop $q\bar{q} \to t\bar{t}$ @NNLO QCD calculation available only **numerical**

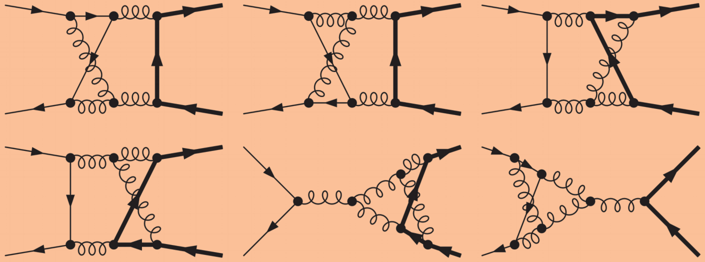
More particles and interactions



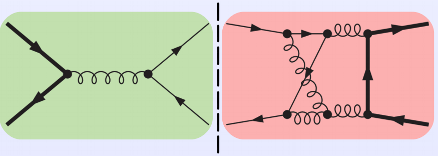
More diagrams (~220)



New topologies occur



BUT



$$\propto T^a_{ij} T^a_{lk} \, f^{abc} (T^d T^c)_{kl} (T^b T^d T^a)_{ji} = 0 \; !$$

No additional Master Integrals are required

Extension to two-loop top-pair production @ NNLO QCD

$$\mathcal{M}^{(1)} = 2(N_c^2 - 1)\Big(N_c A^{(1)} + B^{(1)} + n_l C_l^{(1)} + n_h C_h^{(1)}\Big)$$

$$\mathcal{M}^{(2)} = 2(N_c^2 - 1)\Big(N_c^2 A^{(2)} + B^{(2)} + \frac{C^{(2)}}{N_c^2} + n_l N_c D_l^{(2)} + n_h N_c D_h^{(2)} + n_l \frac{E_l^{(2)}}{N_c} + n_h \frac{E_h^{(2)}}{N_c} + n_l^2 F_l^{(2)} + n_l n_h F_{lh}^{(2)} + n_h^2 F_h^{(2)}\Big)$$

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$A^{(1)}$	-	-	$-\frac{181}{400}$	0.1026418456757775	1.356145770566065	2.230403451742140
$B^{(1)}$	-	-	$\frac{181}{400}$	-0.3180868339485723	-5.763132746701004	2.913169881363488
$C_l^{(1)}$	-	-	0	0	-0.01726400752682416	1.235821434465827
$C_h^{(1)}$	-	-	0	0	-0.5623350683773134	0.6373589172648111
$A^{(2)}$	$\frac{181}{800}$	1.391733154324222	-2.298174307221209	-4.145752448999165	17.37136598564062	-
$B^{(2)}$	$-\frac{181}{400}$	-1.323646320375650	8.507455541210568	6.035611156200398	-35.12861106350758	-
$C^{(2)}$	$\frac{181}{800}$	-0.06808683394857230	-18.00716652035224	6.302454931016090	3.524044912826756	-
$D_l^{(2)}$	0	- $\frac{181}{800}$	0.2605057338631945	-0.7250180282219092	-1.935417246635768	-
$D_h^{(2)}$	0	0	0.5623350683773134	0.1045606449242690	-1.704747997587188	-
$E_l^{(2)}$	0	$\frac{181}{800}$	-0.3323207299541260	7.904121951420471	2.848697836597635	-
$E_h^{(2)}$	0	0	-0.5623350683773134	4.528240788258799	12.73232424278180	-
$F_l^{(2)}$	0	0	0	0	-1.984228442234312	-
$F_{lh}^{(2)}$	0	0	0	0	-2.442562819239786	-
$F_h^{(2)}$	0	0	0	0	-0.07924540546146283	-

[Mandal, Mastrolia, Torres Bobadilla, JR (2022)]

First fully-analytical calculation

Full agreement with the literature

[Czakon(2008)]
[Bonciani, Ferroglia, Gehrmann, Maitre, Studerus (2008)]
[Bärnreuther, Czakon, Fiedler (2014)]
[Badger, Hartanto, Zoia (2021)]

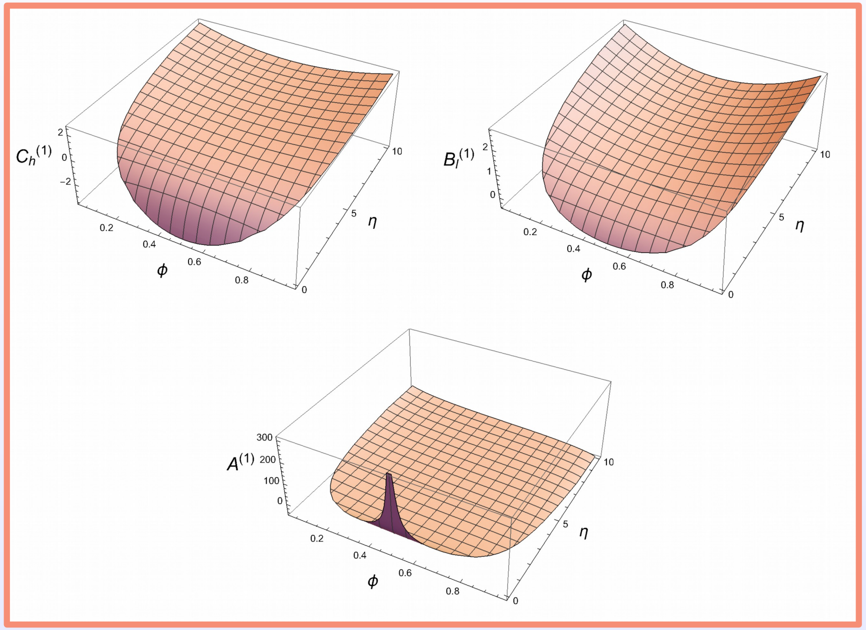
Results: One-loop di-muon production @ NLO QED

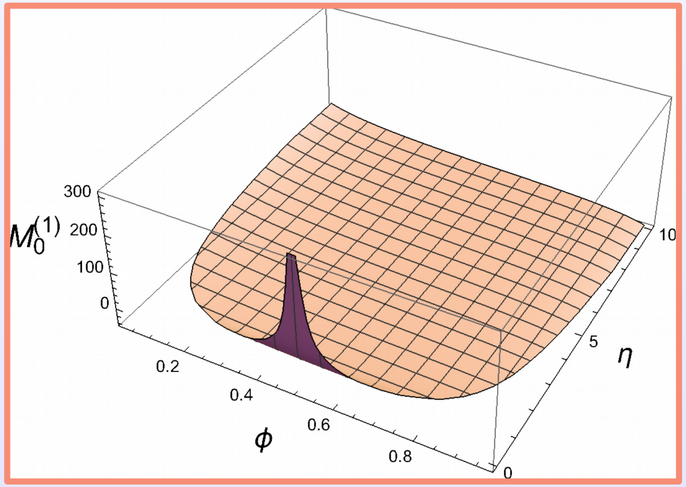
$$\mathcal{M}_0^{(1)} = A^{(1)} + n_l B_l^{(1)} + n_h C_h^{(1)} \Big|_{\text{finite}}$$

$$\eta = \frac{s}{4M^2} - 1, \quad \phi = -\frac{(t - M^2)}{s}$$

Production region

$$\eta > 0, \quad \frac{1}{2}\left(1 - \frac{\eta}{1+\eta}\right) < \phi < \frac{1}{2}\left(1 + \frac{\eta}{1+\eta}\right)$$

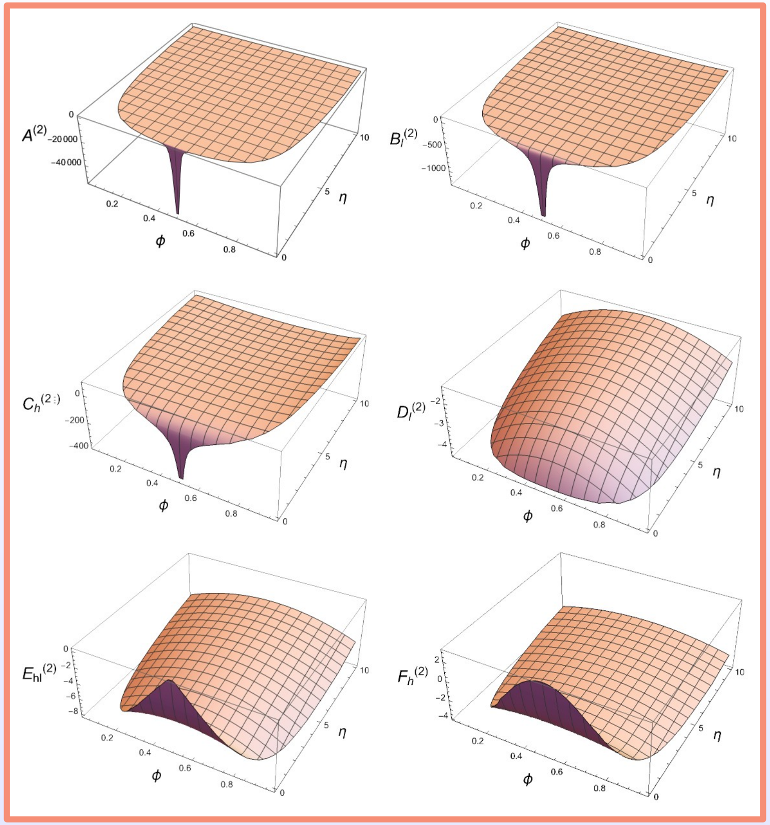




[Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Schubert, Torres Bobadilla, Tramontano, JR (2021)]

Results: Two-loop di-muon production @ NNLO QED

$$\mathcal{M}_0^{(2)} = A^{(2)} + n_l B_l^{(2)} + n_h C_h^{(2)} + n_l^2 D_l^{(2)} + n_l n_h E_{lh}^{(2)} + n_h^2 F_h^{(2)} \Big|_{\text{finite}}$$



$$\eta = \frac{s}{4M^2} - 1, \qquad \phi = -\frac{(t - M^2)}{s}$$

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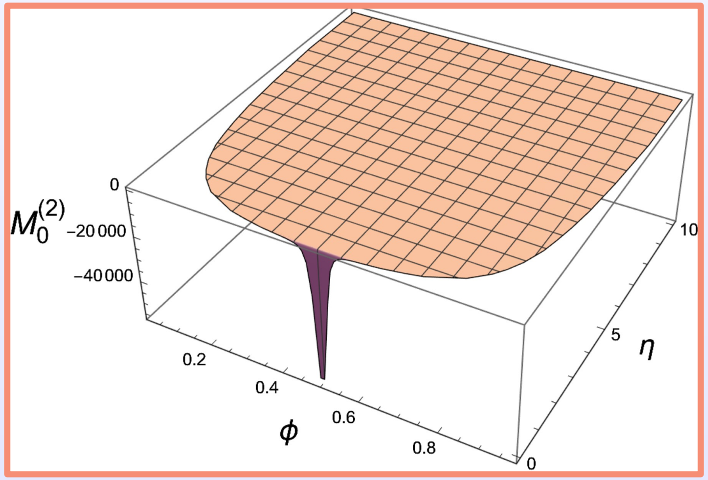
$$\eta > 0, \qquad \frac{1}{2}\left(1 - \frac{\eta}{1+\eta}\right) < \phi < \frac{1}{2}\left(1 + \frac{\eta}{1+\eta}\right)$$



[Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Schubert, Torres Bobadilla, Tramontano, JR (2021)]

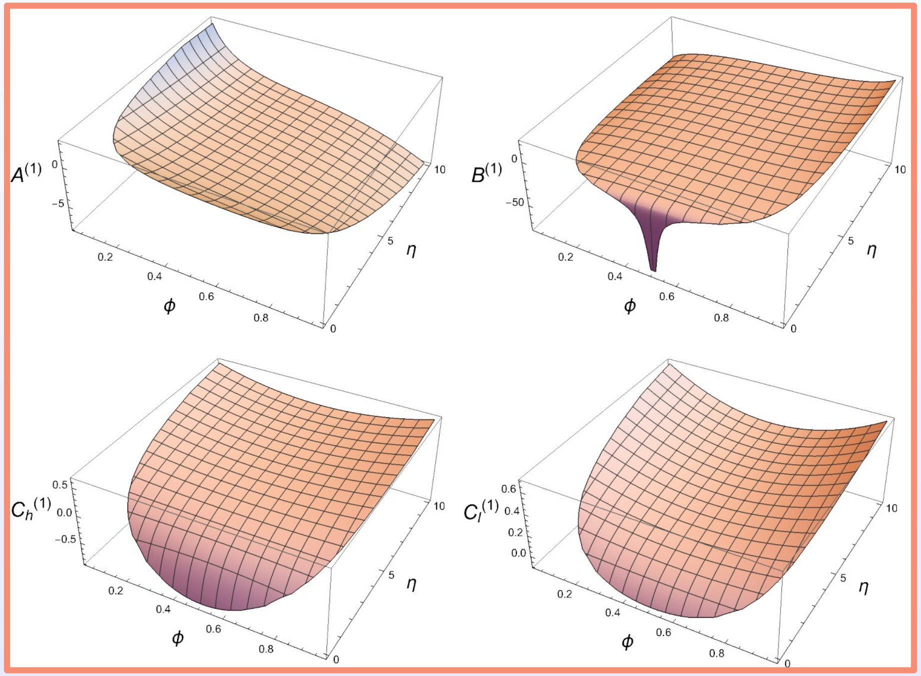
Results: One-loop top-pair production @ NLO QCD

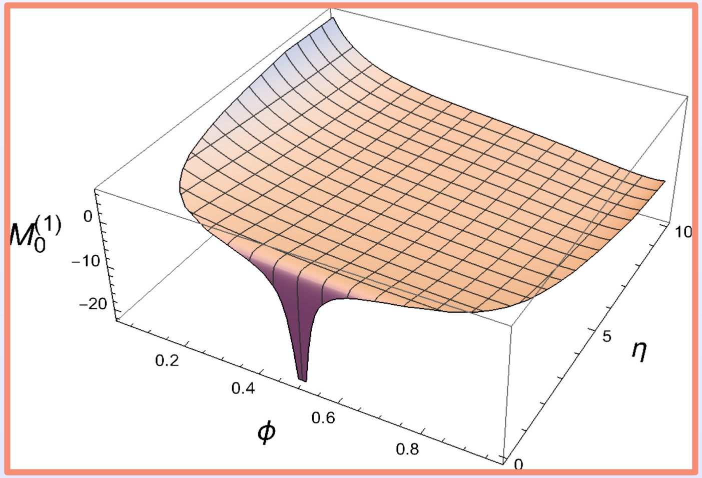
$$\mathcal{M}_0^{(1)} = 2(N_c^2 - 1)\left(N_c A^{(1)} + \frac{B^{(1)}}{N_c} + n_l C_l^{(1)} + n_h C_h^{(1)}\right)\Big|_{\text{finite}}$$

$$\eta = \frac{s}{4M^2} - 1, \quad \phi = -\frac{(t - M^2)}{s}$$

Production region

$$\eta > 0, \quad \frac{1}{2}\left(1 - \frac{\eta}{1+\eta}\right) < \phi < \frac{1}{2}\left(1 + \frac{\eta}{1+\eta}\right)$$

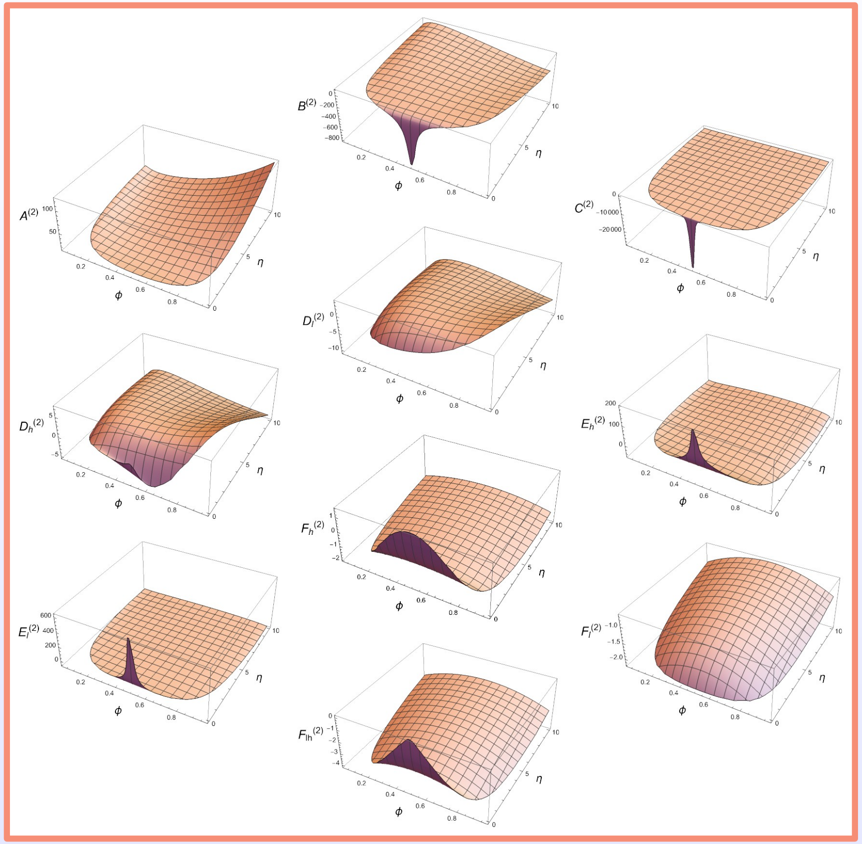




[Mandal, Mastrolia, Torres Bobadilla, JR (2022)]

Results: Two-loop top-pair production @ NNLO QCD

$$\mathcal{M}_0^{(2)} = 2(N_c^2-1)\left(N_c^2 A^{(2)} + B^{(2)} + \frac{C^{(2)}}{N_c^2} + n_l N_c D_l^{(2)} + n_h N_c D_h^{(2)} + n_l \frac{E_l^{(2)}}{N_c} + n_h \frac{E_h^{(2)}}{N_c} + n_l^2 F_l^{(2)} + n_l n_h F_{lh}^{(2)} + n_h^2 F_h^{(2)}\right)\Big|_{\text{finite}}$$

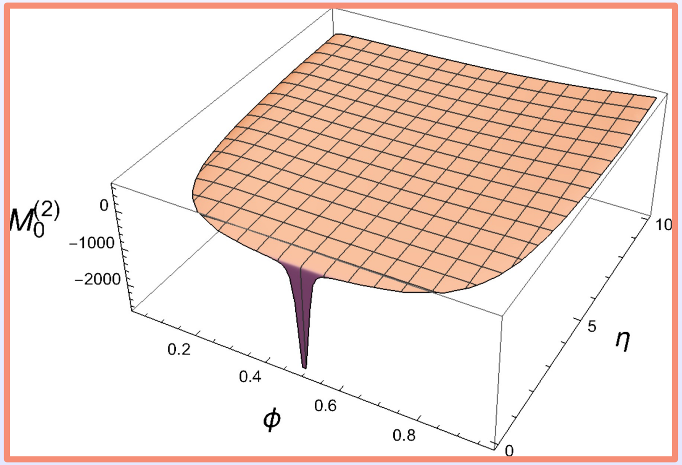


[Mandal, Mastrolia, Torres Bobadilla, JR (2022)]

$$\eta = \frac{s}{4M^2} - 1, \quad \phi = -\frac{(t-M^2)}{s}$$

Production region

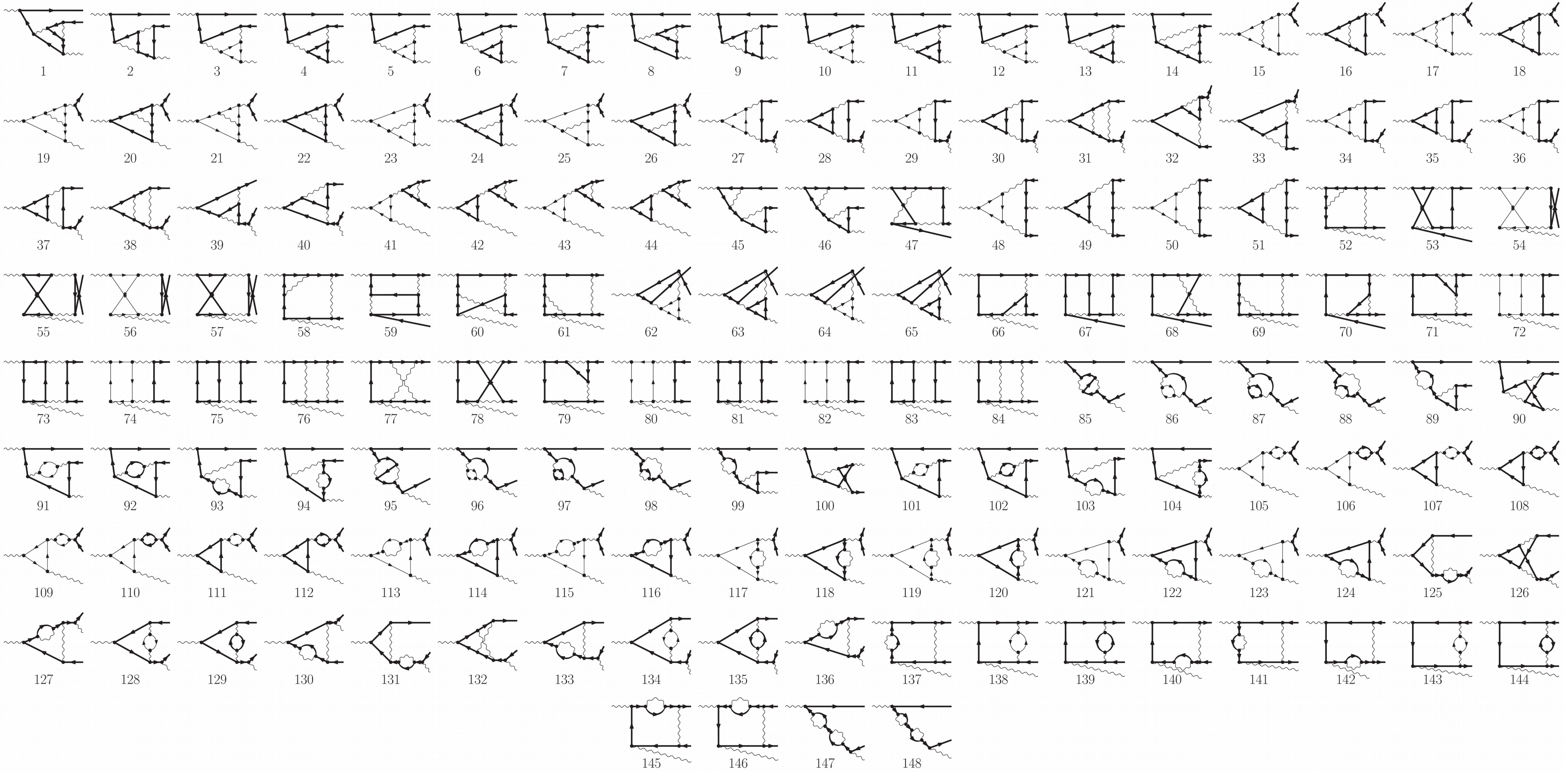
$$\eta > 0, \quad \frac{1}{2}\left(1 - \frac{\eta}{1+\eta}\right) < \phi < \frac{1}{2}\left(1 + \frac{\eta}{1+\eta}\right)$$



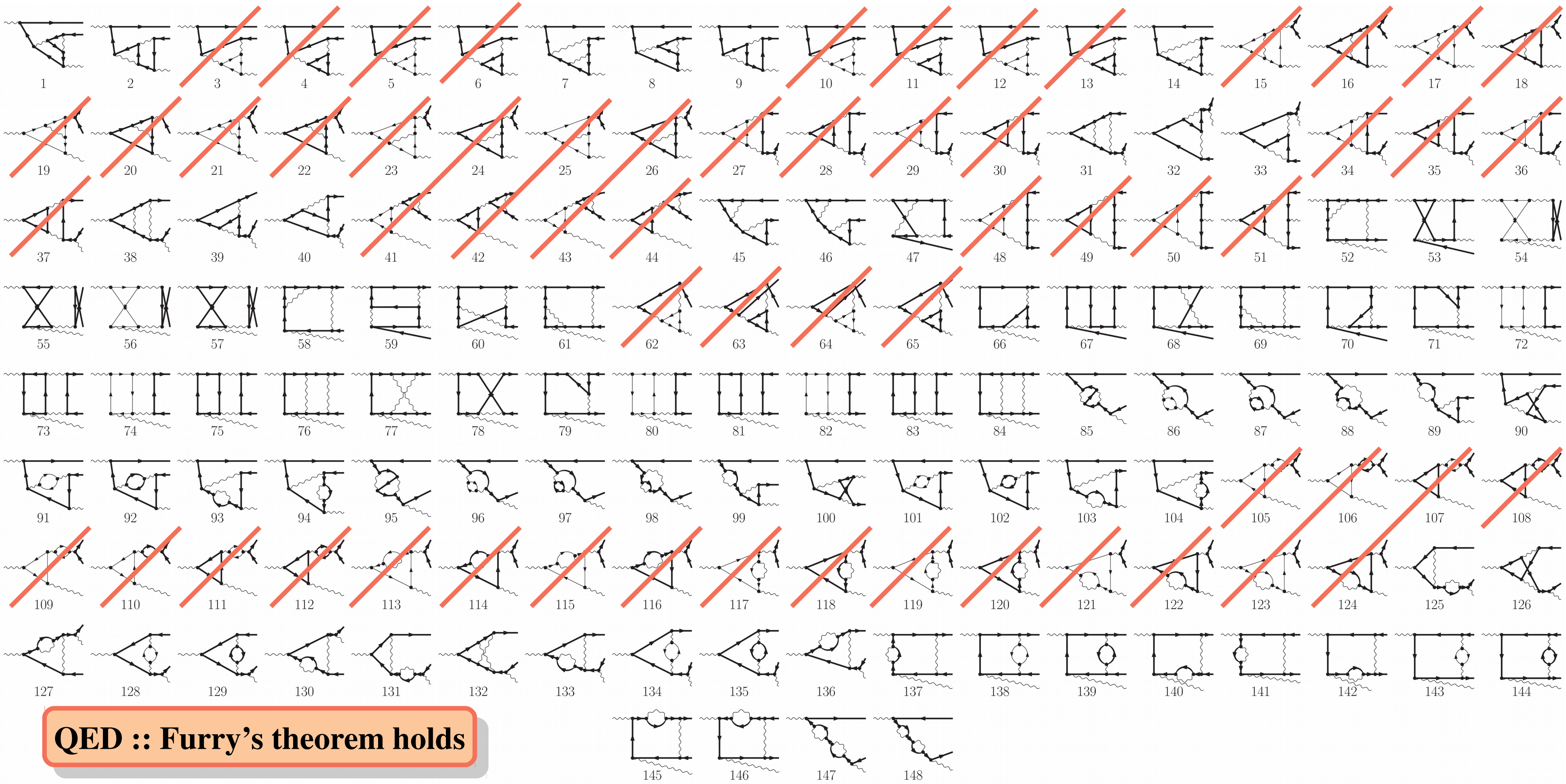
Before concluding...

a glimpse into the amplitude

$\gamma^* \to \bar{l}l$ :: Two-Loop Diagrams

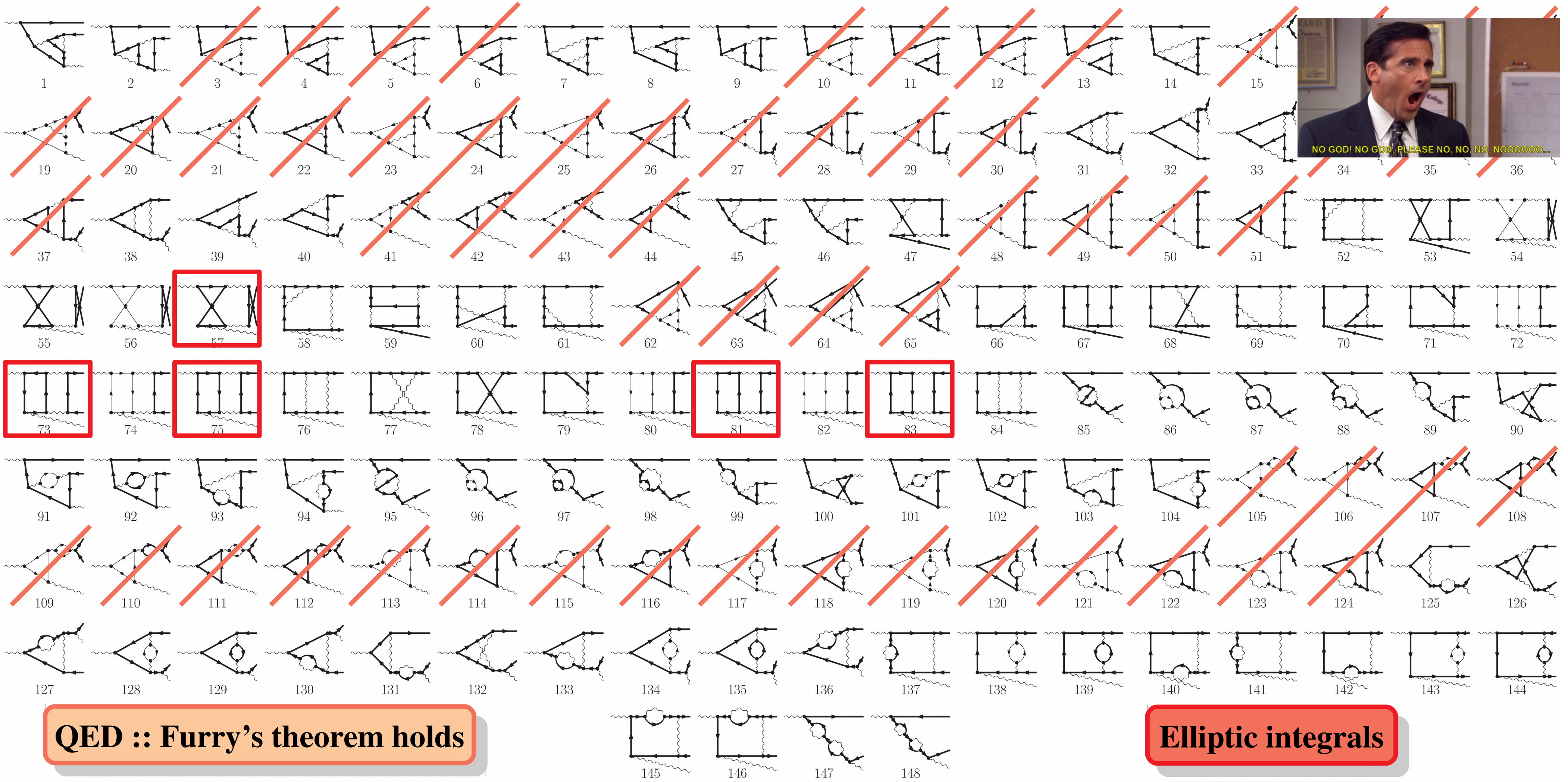


$\gamma^* \to \bar{l}l$ :: Two-Loop Diagrams :: good news



QED :: Furry's theorem holds

$\gamma^* \to \bar{l}l$ :: Two-Loop Diagrams :: bad news



QED :: Furry's theorem holds

Elliptic integrals

Conclusions & Outlook

Target: complete **NNNLO QED** calculation of the **lepton-pair production**.

Crucial ingredient: **Two-loop amplitudes,** for both VVV and for RVV corrections.

- RVV involve (up-to-)**4-point** (massive) **Feynman integrals**.
- Massless lepton approximation calculation can be accessed **immediatly**.
- Massive lepton introduces **elliptic integrals**.

Automated AIDA framework has proven itself of be efficient and appliable to QED and QCD contexts, providing interference terms for:

- VV contribution to **NNLO QED di-muon production via electron-positron annihilation**
- VV contribution to **NNLO QCD top-pair production via quark-antiquark annihilation**
- VV contribution to **NNLO QED Electron-muon elastic scattering**
  - Crucial input for the MuonE experiment
  - Massified by the "Mules"

Questions to be addressed:

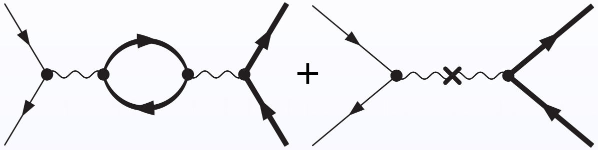
- Does massification over $\gamma^* \to \bar{l}l$ offer a good approximation for the massive case? What about the n_f contributions?
- Missing integrals: Analytical? Numerical? Both?
- If numerical: Grids :: direct implementation of (Feynman) parametrised integrals? DiffExp? AMFlow? SecDec? Others?
- If Analytical: how? [Baglio, Campanario, Glaus, Müllheitner, Spira, JR (2020)]

Lastly: **1Lx1L** can be calculated through public codes; 1L master integrals should not represent a problem.

Thank you

UV Renormalization

$\mathcal{M}_b^{(2)}$ is **UV divergent** $\xrightarrow{\text{Renormalisation}}$ $\mathcal{M}^{(2)}$



= UV finite

$$\mathcal{M} = Z_{2,e} Z_{2,\mu} \mathcal{M}_b(\alpha_b = \alpha_b(\alpha), M_b = M_b(M))$$

where

$$M_b(M) = Z_M M$$

$$\alpha_s S_\epsilon = \alpha(\mu^2)\mu^{2\epsilon} Z_\alpha$$

Renormalisation constants:

$$Z_j = 1 + \left(\frac{\alpha}{\pi}\right) \delta Z_j^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 \delta Z_j^{(2)} + O(\alpha^3)$$

Renormalisation schemes

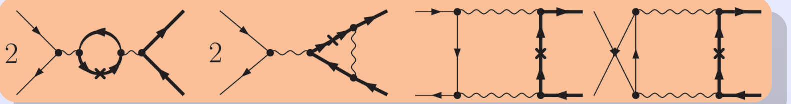
- **On-shell** renormalisation $Z_{2,e}, Z_{2,\mu}, Z_M$
- $\overline{\text{MS}}$ renormalisation Z_α

Renormalised interferences:

$$\mathcal{M}^{(0)} = \mathcal{M}_b^{(0)}$$

$$\mathcal{M}^{(1)} = \mathcal{M}_b^{(1)} + (\delta Z_{2,\mu}^{(1)} + Z_\alpha^{(1)})\mathcal{M}_b^{(0)}$$

$$\mathcal{M}^{(2)} = \mathcal{M}_b^{(2)} + (\delta Z_{2,\mu}^{(1)} + Z_\alpha^{(1)})\mathcal{M}_b^{(1)} + (\delta Z_{2,\mu}^{(2)} + \delta Z_{2,e}^{(2)} + Z_\alpha^{(2)} + \delta Z_{2,\mu}^{(1)} Z_\alpha^{(1)})\mathcal{M}_b^{(0)} + \delta Z_M^{(1)} \mathcal{M}_{\text{massCT}}^{(1)}$$



Reduction of Feynman Integrals

$$\mathcal{M}_b^{(2)} = (S_\epsilon)^2 \int \prod_{i=1}^{2} \frac{d^d k_i}{(2\pi)^d} \sum_G \frac{N_G}{\prod_{\sigma \in G} D_\sigma}$$

Adaptive Integrand Decomposition

Idea: $$d = d_{||} + d_{\perp}$$
$$k_i = k_{||i} + k_{\perp i}$$

D_σ will not depend on transverse directions

$\Downarrow$

Direct integration

$\Downarrow$

$$\frac{N_G}{\prod_{\sigma \in G} D_\sigma} = \sum_{\tau \in P(G)} \frac{\Delta_\tau}{\prod_{j \in \tau} D_j}$$

Integration-by-parts Identities

$$\int \prod_{i=1}^{2} \frac{d^d k_i}{(2\pi)^d} \frac{\partial}{\partial k_l^\mu} \left(q^\mu \frac{\mathcal{N}}{\prod_{\sigma \in G} D_\sigma} \right) = 0$$

- IBPs generate a linear system of Eqs.
- Relation between integrals
- Coefficient depending on scales and d

# of independent Eqs. = # of **Master Integrals**

$$\mathcal{M}_b^{(2)} = \mathbb{C}^{(2)} \cdot \mathbf{I}^{(2)}$$

[Mastrolia, Peraro, Primo, Torres (2016)]
[Mastrolia, Peraro, Primo, Torres, JR (To be published)]

[Chetyrkin, Tkachev (1981)] [Laporta (2000)]