OpenLoops at 1-loop & 2-loops status and developments

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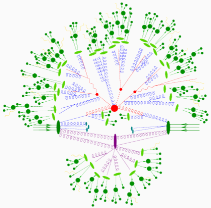
N3LO $\gamma^* \to \ell^+\ell^-$ kick-off workshop
4th August 2022

OpenLoops

- OpenLoops is a numerical tool providing hard scattering amplitudes to Monte Carlo simulations.
- All components to NLO fully automated in OpenLoops for QCD and EW corrections to the SM.

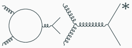


[Schälicke, Gleisberg, Höche, Schumann, Winter, Krauss, Soff]

OpenLoops constructs helicity and color summed scattering probability densities $\mathcal{W}_{LL} = \sum_h \sum_{\text{col}} |\bar{\mathcal{M}}_L(h)|^2$ for $L = 0, 1$ and $\mathcal{W}_{0L} = \sum_h \sum_{\text{col}} 2\,\mathrm{Re}\left[\bar{\mathcal{M}}_L(h)\,\bar{\mathcal{M}}_0^*(h)\right]$ for $L = 1$ from L-loop matrix elements $\bar{\mathcal{M}}_L$.

Example:

$$\mathcal{W}_{01} = \sum_{h} \sum_{\text{col}} 2\mathrm{Re}\Big[$$



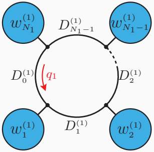
$$ {}^{*} + \ldots \Big]$$

Goals: ultimate for numerical stability for real-virtual applications, automation at NNLO

Components to NLOCalculations

Final result in D-dimensions, numerical tools: construct numerator in 4-dimensions.
For one diagram Γ:

$\mathcal{M}_{1,\Gamma} =$

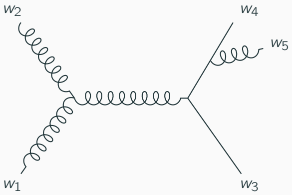


$$= \underbrace{C_{1,\Gamma}}_{\text{color}} \int \mathrm{d}\bar{q}_1 \underbrace{\frac{\mathcal{N}(q_1)}{\mathcal{D}(\bar{q}_1)}}_{\substack{\text{4-dim numerator,}\\ \text{(D-dim denominator)}}} = C_{1,\Gamma} \sum_r \underbrace{\mathcal{N}_{\mu_1 \cdots \mu_r}}_{\text{tensor coefficient}} \underbrace{\int \mathrm{d}\bar{q}_1 \frac{q_1^{\mu_1} \cdots q_1^{\mu_r}}{\mathcal{D}(\bar{q}_1)}}_{\text{tensor integral}}$$

Calculation decomposed into:

- Numerical construction of tensor coefficient in 4-dim → OpenLoops algorithm
  [van Hameren; Cascioli, Maierhöfer, Pozzorini; Buccioni, Lang, Lindert, Maierhöfer, Pozzorini, Zhang, Zoller]
- Renormalization, restoration of (D-4)-dim numerator part → rational counterterms
  $\mathbf{R}\bar{\mathcal{M}}_{1,\Gamma} = \mathcal{M}_{1,\Gamma} + \mathcal{M}^{(\mathrm{CT})}_{0,1,\Gamma}$ [Ossola, Papadopoulos, Pittau]
- Reduction and evaluation of tensor integrals → On-the-fly reduction
  [Buccioni, Pozzorini, Zoller], Collier [Denner, Dittmaier, Hofer], OneLoop [van Hameren]

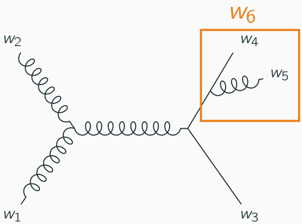
OpenLoops Tree Level Algorithm: Example



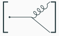
input: external wavefunctions

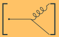
w_1, w_2, w_3, w_4, w_5

OpenLoops Tree Level Algorithm: Example

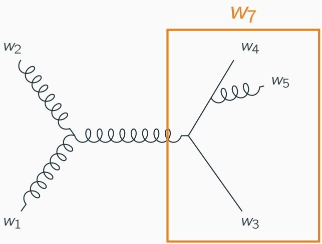


Combine w_4, w_5 into subtree w_6:

$w_6^{\gamma} =$  $^{\gamma}_{\alpha\beta}\, w_4^{\alpha} w_5^{\beta}$

 $^{\gamma}_{\alpha\beta} =$ vertex + propagator,
universal process-independent
Feynman rule

OpenLoops Tree Level Algorithm: Example



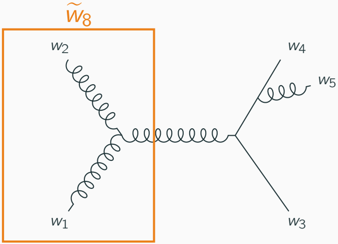
Add next external leg:

$$w_6^{\gamma} = \left[\ \right]^{\gamma}_{\alpha\beta} w_4^{\alpha} w_5^{\beta}$$

$$w_7^{\gamma} = \left[\ \right]^{\gamma}_{\alpha\beta} w_3^{\alpha} w_6^{\beta}$$

$\left[\ \right]^{\gamma}_{\alpha\beta} =$ vertex + propagator, universal process-independent Feynman rule

OpenLoops Tree Level Algorithm: Example



same on the other side:

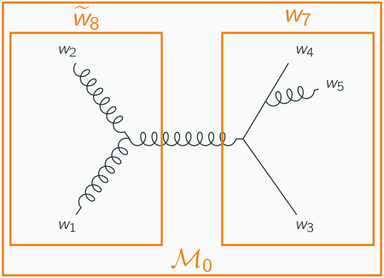
$$w_6^\gamma = [\ ]^\gamma_{\alpha\beta} w_4^\alpha w_5^\beta$$

$$w_7^\gamma = [\ ]^\gamma_{\alpha\beta} w_3^\alpha w_6^\beta$$

$$\widetilde{w}_8^\gamma = [\ ]^\gamma_{\alpha\beta} w_1^\alpha w_2^\beta$$

$[\ ]^\gamma_{\alpha\beta} =$ vertex,
universal process-independent
Feynman rule

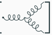
OpenLoops Tree Level Algorithm: Example



combine to full diagram:

$w_6^\gamma = \Big[$  $\Big]^{\gamma}_{\alpha\beta} w_4^\alpha w_5^\beta$

$w_7^\gamma = \Big[$  $\Big]^{\gamma}_{\alpha\beta} w_3^\alpha w_6^\beta$

$\widetilde{w}_8^\gamma = \Big[$  $\Big]^{\gamma}_{\alpha\beta} w_1^\alpha w_2^\beta$

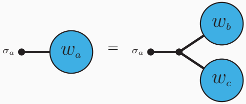
$\mathcal{M}_0 = \Big[$  $\Big]_{\alpha\beta} w_7^\alpha w_8^\beta$

$\Big[$  $\Big]_{\alpha\beta} =$
universal process-independent Feynman rule

OpenLoops Tree Level Algorithm

Recursively construct subtrees starting from external wavefunctions:

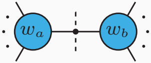
$$w_a^{\sigma_a}(k_a, h_a) =$$



$$= \underbrace{\frac{X^{\sigma_a}_{\sigma_b \sigma_c}(k_b, k_c)}{k_a^2 - m_a^2}}_{\text{model-dependent}} \underbrace{w_b^{\sigma_b}(k_b, h_b)\, w_c^{\sigma_c}(k_c, h_c)}_{\text{process-dependent}}$$

Then contract into full diagram:

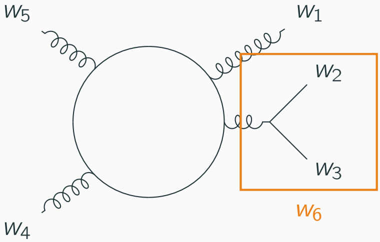
$$\mathcal{M}_{0,\Gamma}(h) =$$



$$= C_{0,\Gamma} \cdot w_a^{\sigma_a}(k_a, h_a)\, \delta_{\sigma_a \sigma_b} \widetilde{w}_b^{\sigma_b}(k_b, h_b)$$

- diagrams constructed using **universal Feynman rules**
- identical subtrees are **recycled** in multiple tree and loop diagrams

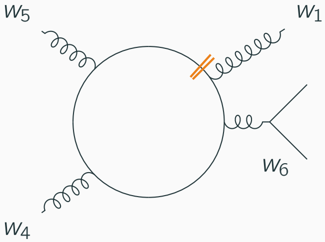
One Loop Algorithm: Example



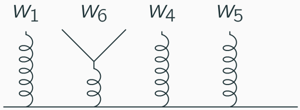
External subtrees constructed in tree level algorithm (together with tree diagrams):

$w_2, w_3 \rightarrow w_6$

One Loop Algorithm: Example



$\downarrow$

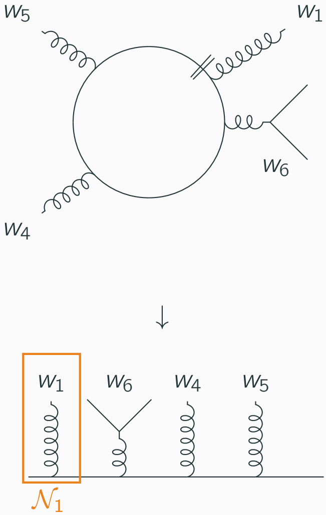


Open Loop:

Diagram factorizes into chain of segments: $\mathcal{N} = S_1 \cdots S_N$

segment = loop vertex + loop propagator + external subtree(s)

One Loop Algorithm: Example

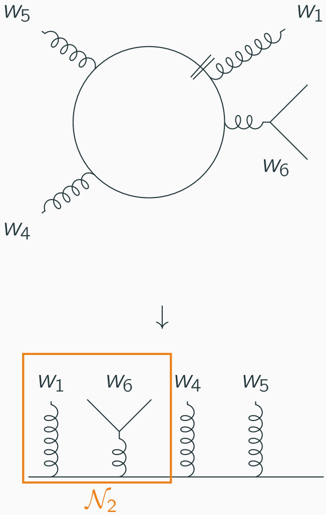


Construct first segment S_1 attaching the external subtree w_1.

$\mathcal{N}_0 = \mathbb{1}$
$\mathcal{N}_1 = \mathcal{N}_0 \cdot S_1(w_1)$

segment = loop vertex + loop propagator + external subtree(s)

One Loop Algorithm: Example

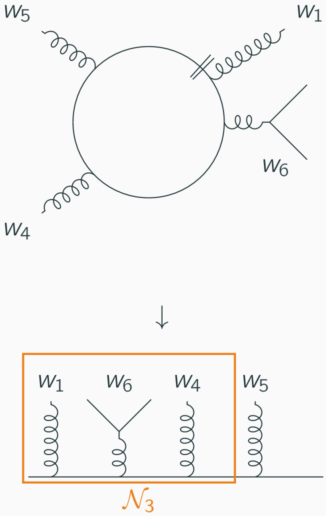


Add second segment attaching the subtree w_6.

$$\mathcal{N}_0 = \mathbb{1}$$
$$\mathcal{N}_1 = \mathcal{N}_0 \cdot S_1(w_1)$$
$$\mathcal{N}_2 = \mathcal{N}_1 \cdot S_2(w_6)$$

segment = loop vertex + loop propagator + external subtree(s)

One Loop Algorithm: Example

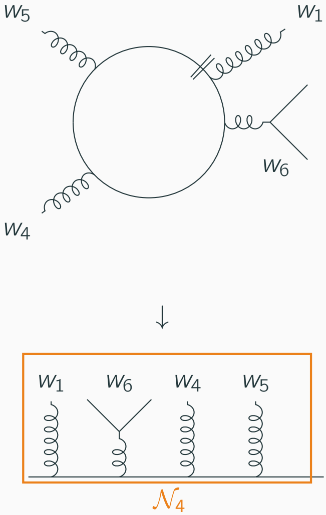


Add third segment.

$$\mathcal{N}_0 = \mathbb{1}$$
$$\mathcal{N}_1 = \mathcal{N}_0 \cdot S_1(w_1)$$
$$\mathcal{N}_2 = \mathcal{N}_1 \cdot S_2(w_6)$$
$$\mathcal{N}_3 = \mathcal{N}_2 \cdot S_3(w_4)$$

segment = loop vertex + loop propagator + external subtree(s)

One Loop Algorithm: Example

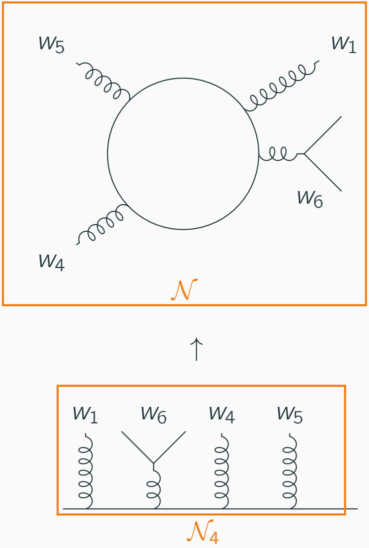


Add last segment.

$$
\begin{aligned}
\mathcal{N}_0 &= \mathbb{1} \\
\mathcal{N}_1 &= \mathcal{N}_0 \cdot S_1(w_1) \\
\mathcal{N}_2 &= \mathcal{N}_1 \cdot S_2(w_6) \\
\mathcal{N}_3 &= \mathcal{N}_2 \cdot S_3(w_4) \\
\mathcal{N}_4 &= \mathcal{N}_3 \cdot S_4(w_5)
\end{aligned}
$$

segment = loop vertex + loop propagator + external subtree(s)

One Loop Algorithm: Example



Close the loop (contract open Lorentz/spinor indices).

$$\mathcal{N}_0 = \mathbb{1}$$
$$\mathcal{N}_1 = \mathcal{N}_0 \cdot S_1(w_1)$$
$$\mathcal{N}_2 = \mathcal{N}_1 \cdot S_2(w_6)$$
$$\mathcal{N}_3 = \mathcal{N}_2 \cdot S_3(w_4)$$
$$\mathcal{N}_4 = \mathcal{N}_3 \cdot S_4(w_5) = {\mathcal{N}_4}_{\beta_0}^{\ \beta_N}$$

$$\mathcal{N} = Tr({\mathcal{N}_4}_{\beta_0}^{\ \beta_N})$$

OpenLoops One Loop Algorithm

One Loop Amplitude:

$$\mathcal{M}_{1,\Gamma} = C_{1,\Gamma} \int \mathrm{d}\bar{q}\, \frac{\mathrm{Tr}[\mathcal{N}(q)]}{D_0 D_1 \cdots D_{N_1 - 1}} =$$

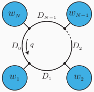
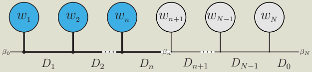


Diagram is cut open resulting in a chain, which **factorizes** into segments:

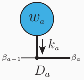
$$\mathcal{N}_n(q) = \prod_{a=1}^{n} S_a(q) =$$



Chain is constructed recursively, recursion step: $\mathcal{N}_n = \mathcal{N}_{n-1} \cdot S_n$.
Implemented at level of tensor coefficients in $\mathcal{N} = \mathcal{N}_{\mu_1 \cdots \mu_r} q_1^{\mu_1} \cdots q_1^{\mu_r}$.

Segment = vertex + propagator + subtree(s)

$$\left[S_a(q)\right]^{\beta_a}_{\beta_{a-1}} =$$

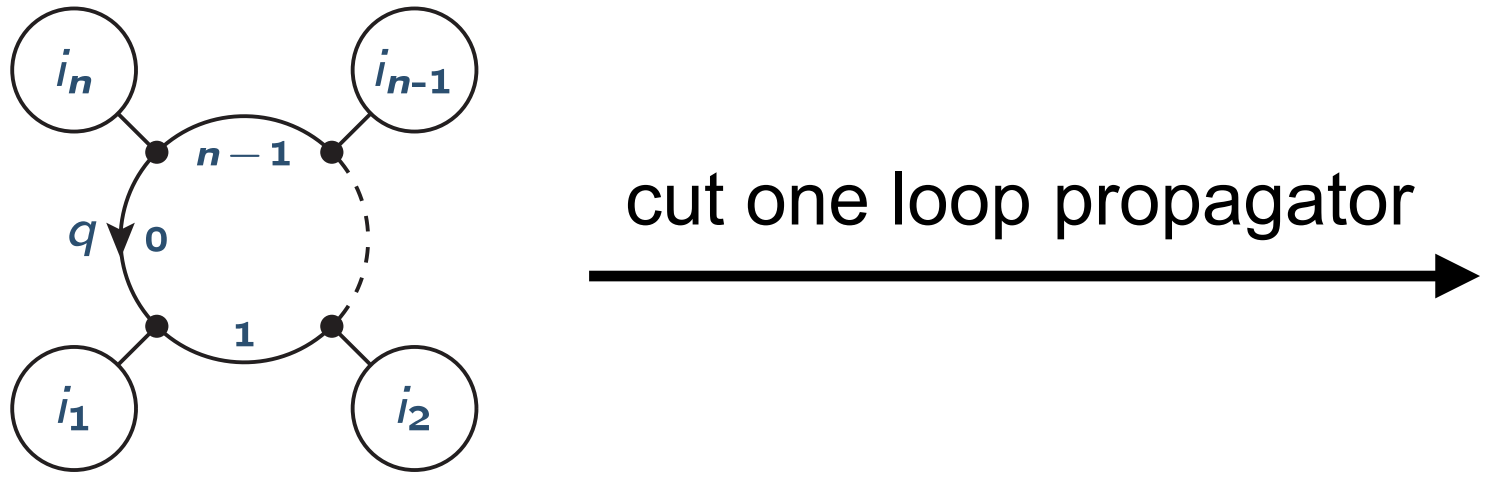


$$= \left[Y_{\sigma_a} + Z_{\sigma_a,\nu}\, q^{\nu}\right]^{\beta_a}_{\beta_{a-1}} w_a^{\sigma_a}(k_a)$$

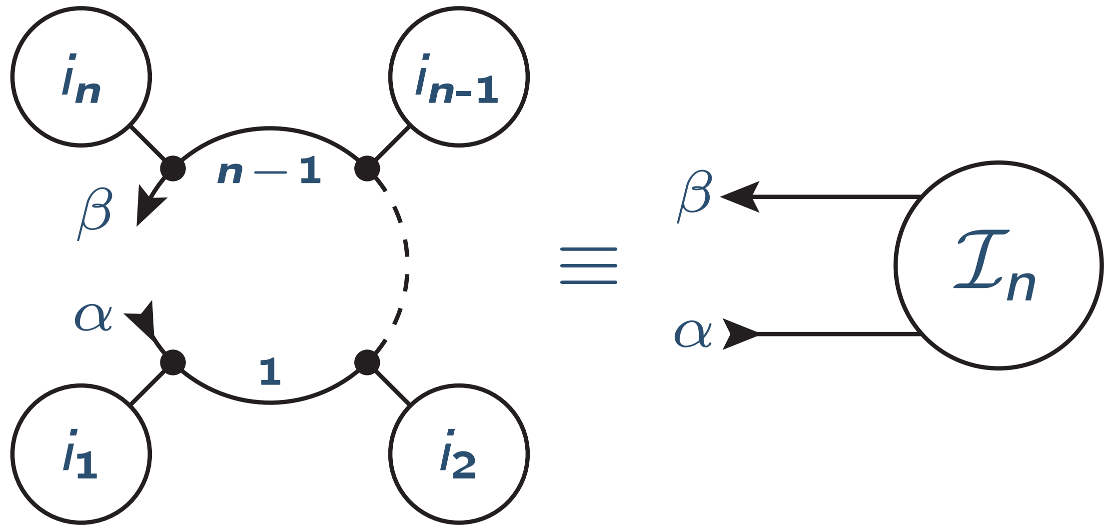
Exploit factorization to construct 1l diagrams from universal process-independent building blocks.

The Open Loops algorithm:
From tree recursion to loop diagrams

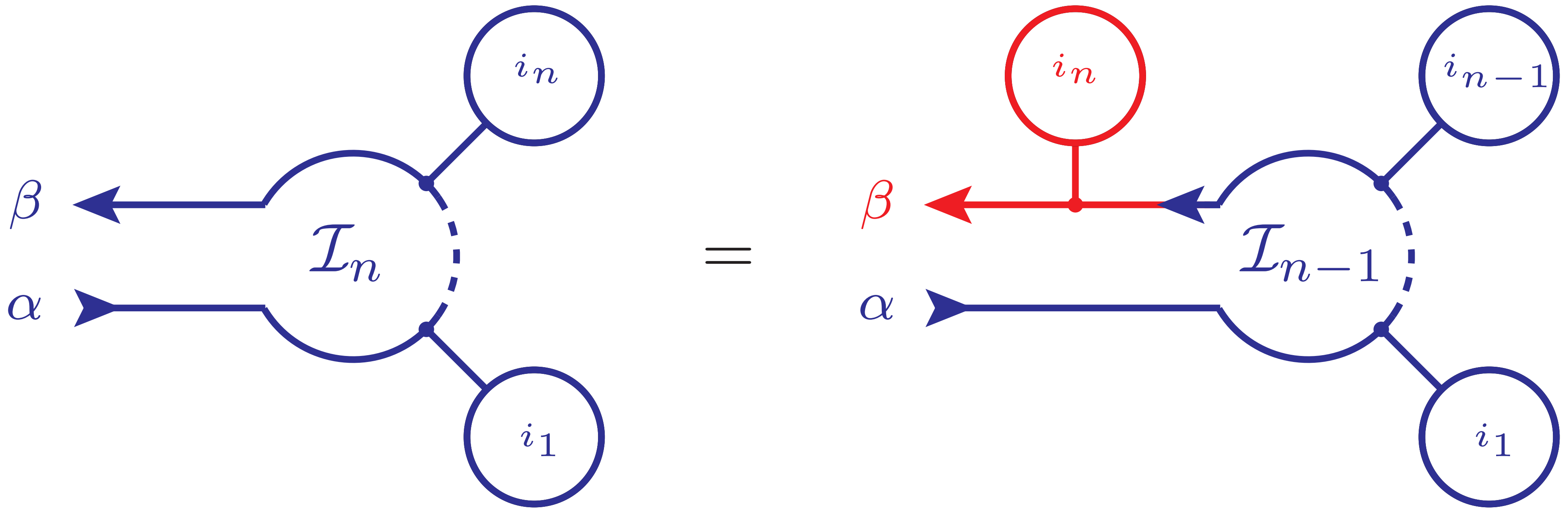
- Treat one-loop diagram as ordered set of sub-trees $\mathcal{I}_n = \{i_1, \ldots, i_n\}$ connected by propagators



$\mathcal{N}_\alpha^\beta(\mathcal{I}_n; q) =$



- Build numerator recursively connecting subtrees along the loop keeping the q dependence



$$\mathcal{N}_\alpha^\beta(\mathcal{I}_n; q) = X_{\gamma\delta}^\beta(q)\, \mathcal{N}_\alpha^\gamma(\mathcal{I}_{n-1}; q)\; w^\delta(i_n)$$

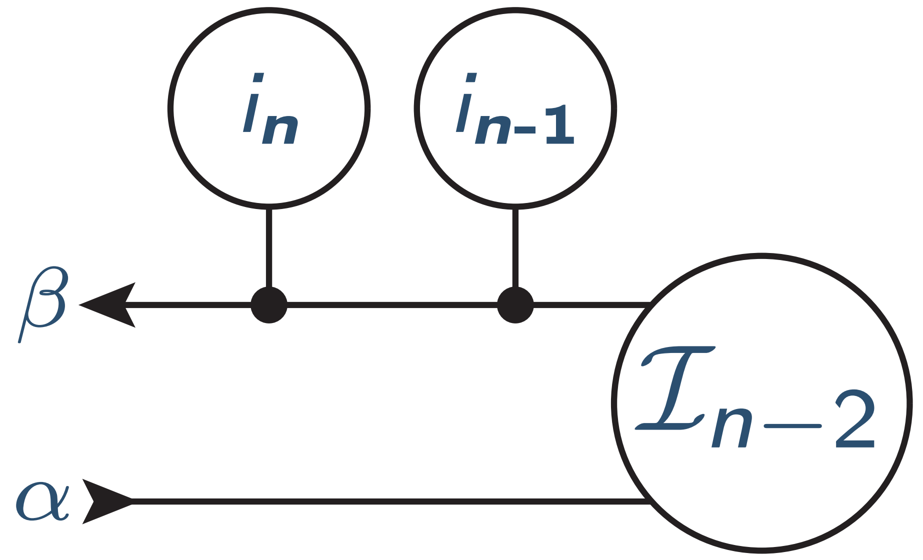
$$X_{\gamma\delta}^\beta = Y_{\gamma\delta}^\beta + q^\nu Z_{\nu;\gamma\delta}^\beta$$

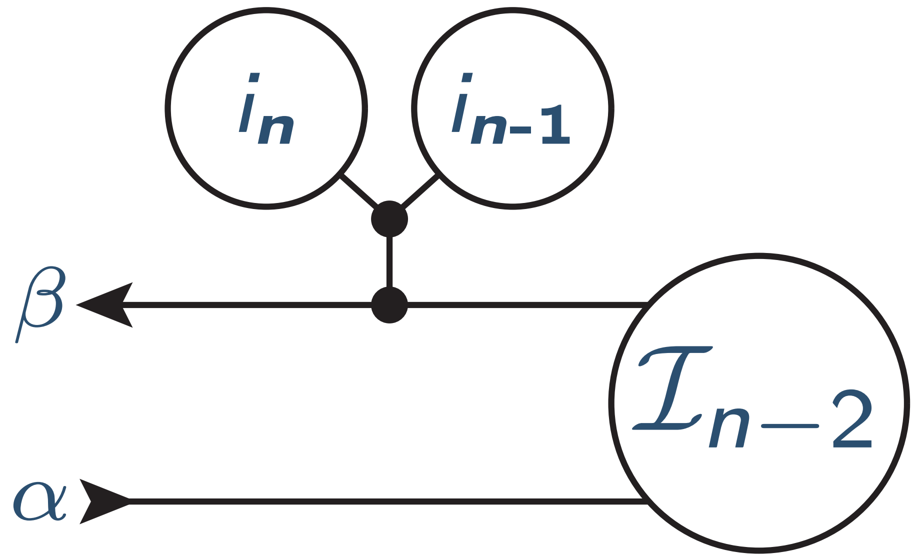
$$\mathcal{N}^\beta_{\mu_1\ldots\mu_r;\alpha}(\mathcal{I}_n) = \left[Y^\beta_{\gamma\delta}\, \mathcal{N}^\gamma_{\mu_1\ldots\mu_r;\alpha}(\mathcal{I}_{n-1}) + Z^\beta_{\mu_1;\gamma\delta}\, \mathcal{N}^\gamma_{\mu_2\ldots\mu_r;\alpha}(\mathcal{I}_{n-1}) \right] w^\delta(i_n)$$

$\Rightarrow$ very fast!

The (original) Open Loops algorithm:
recycle loop structures

OpenLoops recycling:

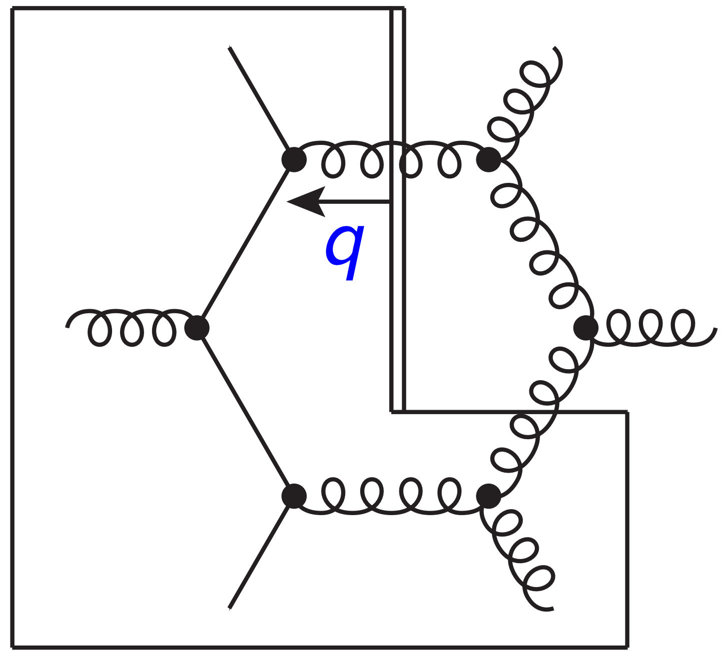




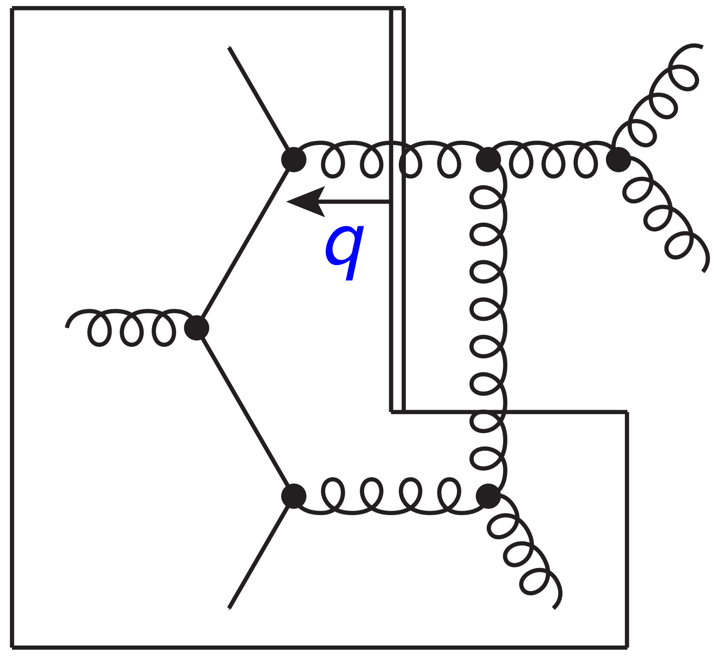
Lower-point open-loops can be shared between diagrams if

- cut is put appropriately
- direction chosen to maximise recyclability

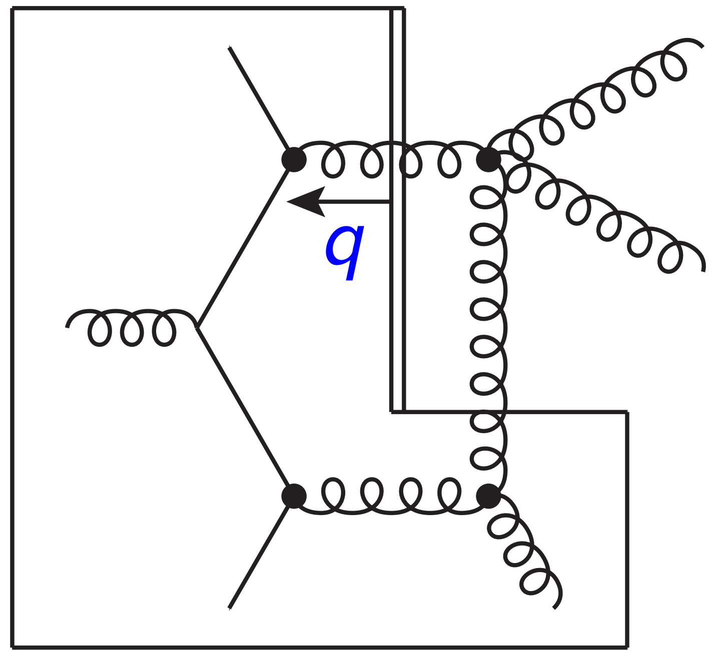
Illustration:



parent



child 1

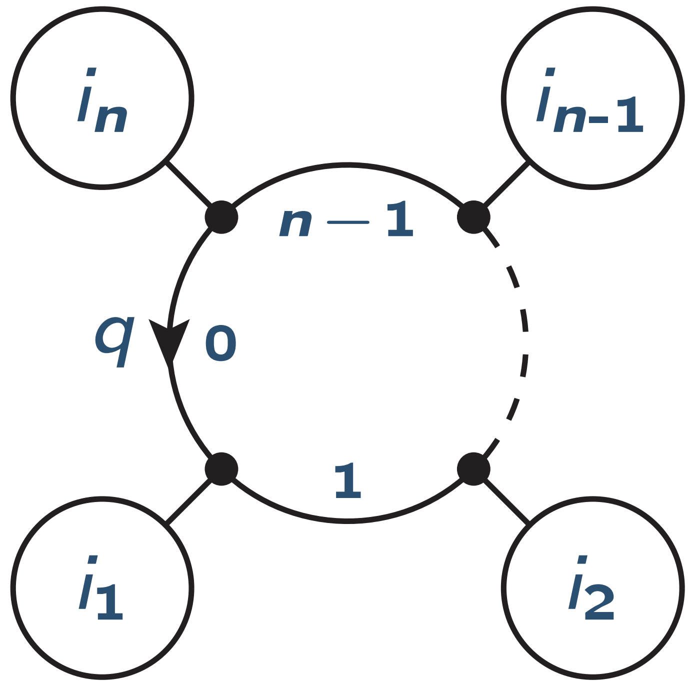


child 2

Complicated diagrams require only “last missing piece”

The (original) Open Loops algorithm: one loop amplitudes

[F. Cascioli, P. Maierhöfer, S. Pozzorini; '12]



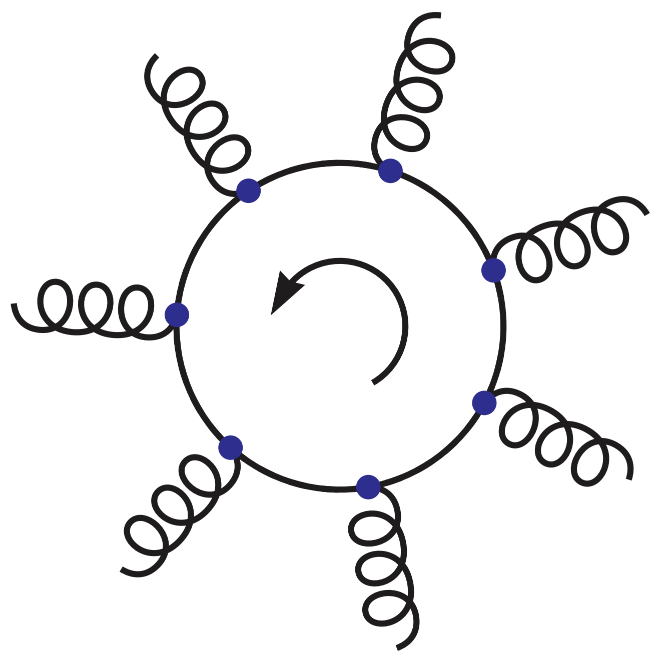
$$= \int \frac{d^D \mathcal{N}(q)}{D_0 D_1 \ldots D_{n-1}} = \sum_{r=0}^{R} \mathcal{N}_{\mu_1 \ldots \mu_r} \underbrace{\int \frac{q^{\mu_1} \ldots q^{\mu_r}}{D_0 D_1 \ldots D_{n-1}}}_{\text{tensor integral}}$$

- Tensorial coefficients $\mathcal{N}^{\alpha}_{\mu_1 \ldots \mu_r; \alpha}$ can directly be contracted with Tensor Integrals evaluated with **COLLIER** [Denner, Dittmaier, Hofer; '16]

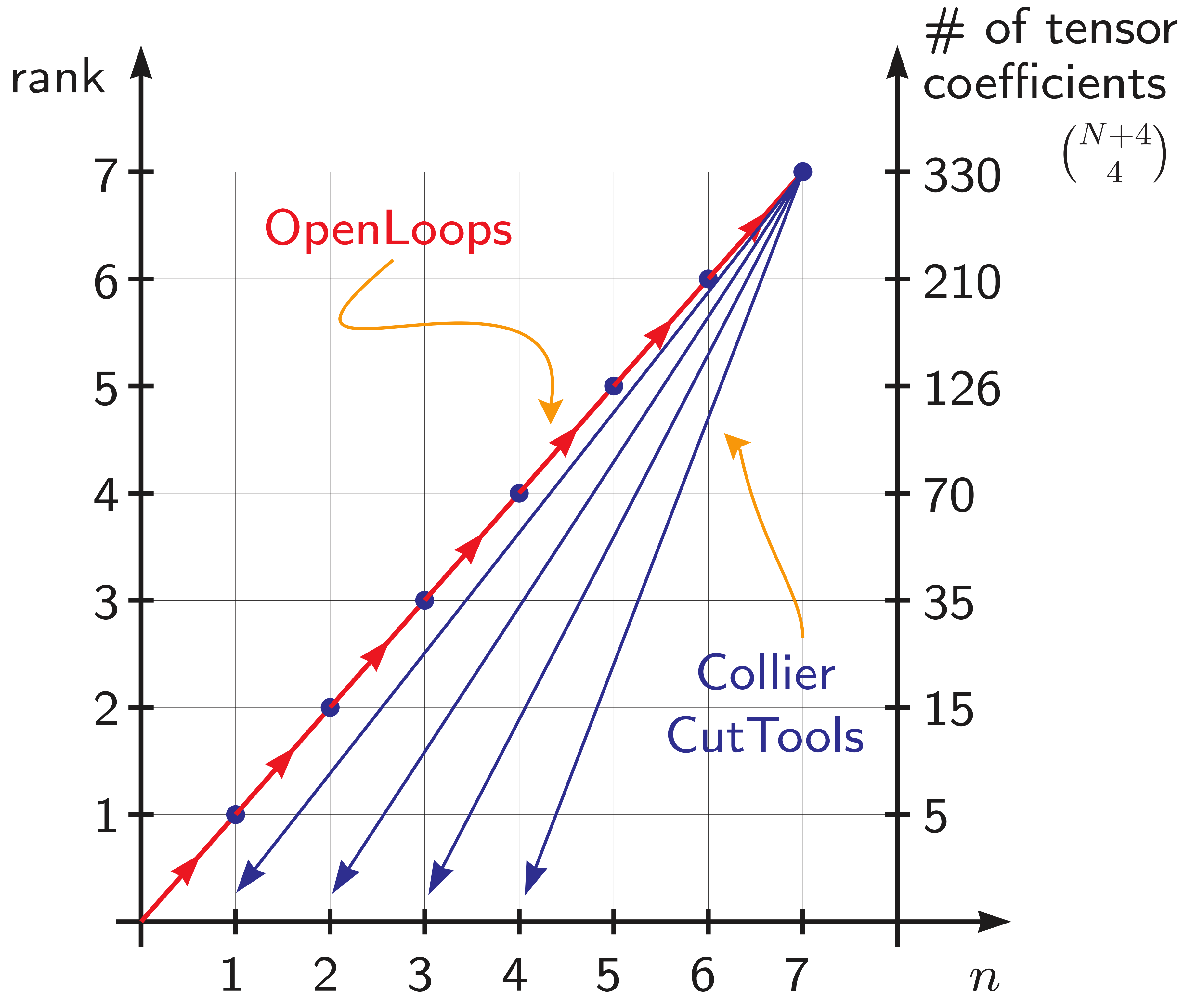
- Fast evaluation of $\mathcal{N}(q) = \sum \mathcal{N}_{\mu_1 \ldots \mu_r} q^{\mu_1} \ldots q^{\mu_r}$ at multiple q-values allows for efficient application of OPP reduction methods e.g. with **CutTools** *[Ossola, Papadopolous, Pittau; '07]*

Standard OpenLoops reduction

Example:



Complexity grows exponential with tensor rank!



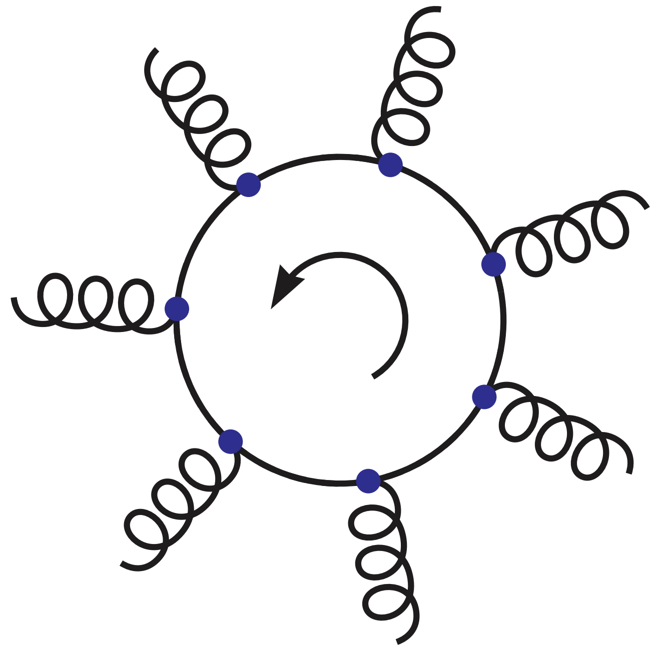
Bottlenecks:

- Large growths of structures prior to reduction
- Evaluation of coefficients required for every helicity h

On-the-fly reduction

[Buccioni, Pozzorini, Zoller, '17]

Example:



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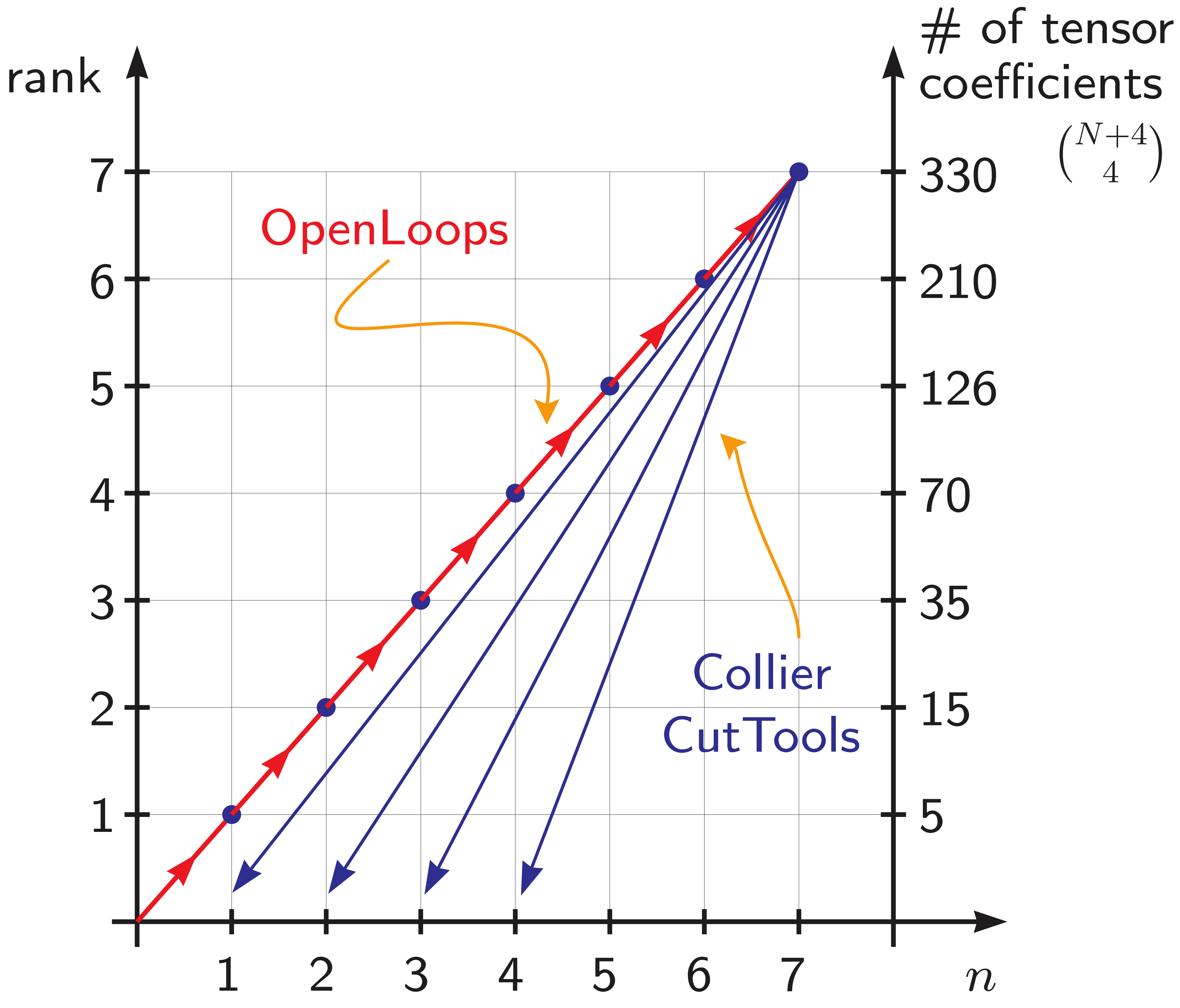

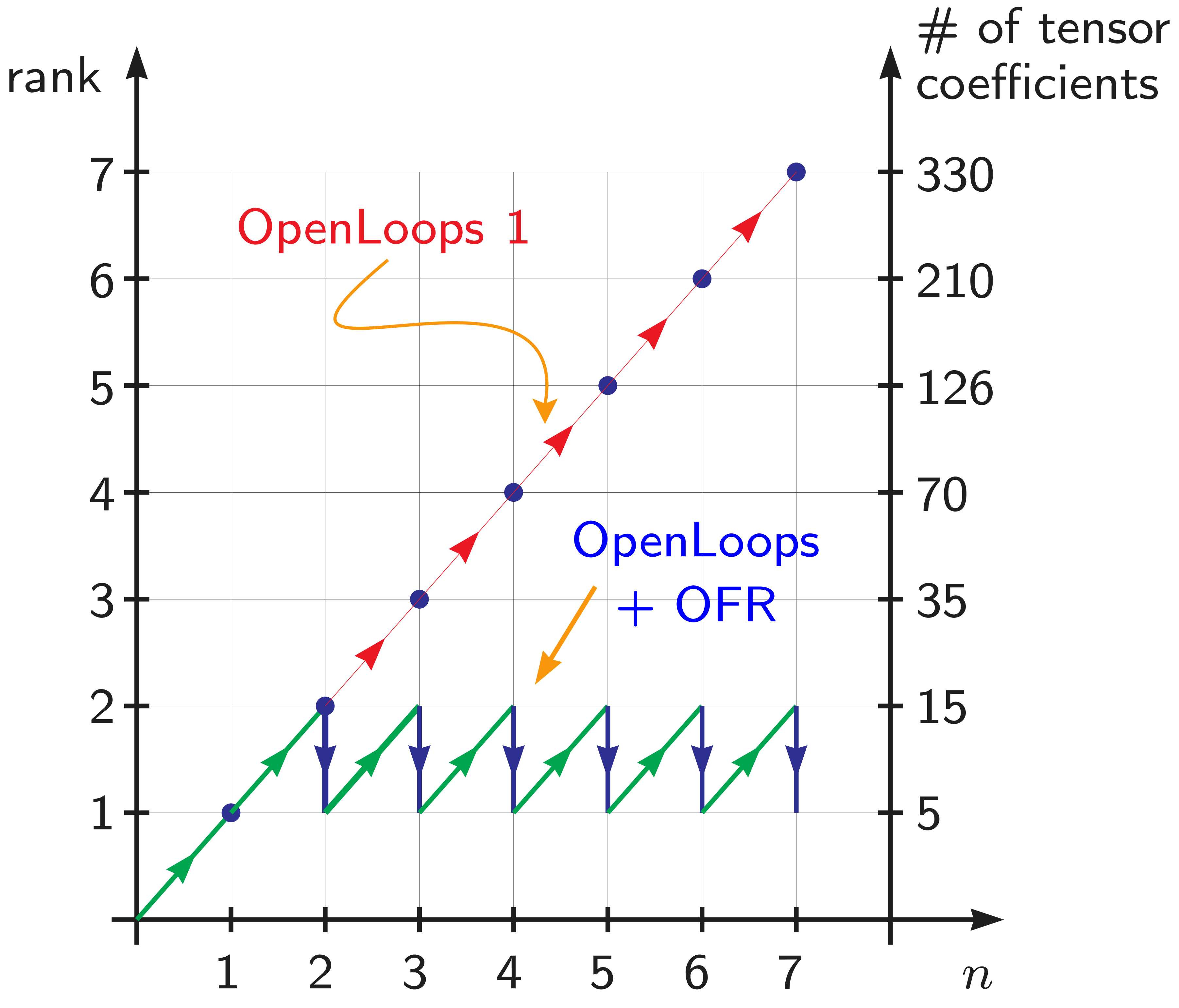

Advantage of OFR:

- one algorithm for construction and reduction of amplitude (less reliance on external codes)
- unprecedented numerical stability (crucial for real-virtual applications)

On-the-fly reduction

- At each Open Loops step that gives rank=2 perform "on-the-fly" 2 -> 1 **integrand-level** reduction:

rank=2 ⟶ rank=1

[F. del Aguila and R. Pittau; '04]

$$q^\mu q^\nu = A^{\mu\nu}_{-1} + A^{\mu\nu}_0 D_0 + \left(B^{\mu\nu}_{-1,\lambda} + \sum_{i=0}^{3} B^{\mu\nu}_{i,\lambda} D_i \right) q^\lambda,$$

$$D_i = (q+p_i)^2 - m_i^2$$

- For N > 3 the reduction identify requires (p_1, p_2, p_3) independent momenta.
- This reduction follows from decomposition:

$$q^\mu = \sum_{i=1}^{4} c_i l_i^\mu, \quad l_i = l_i(p_1, p_2)$$

reduction basis

We can choose this decomposition freely such that we can cancel propagators D_i

- $A_i^{\mu\nu}, B_i^{\mu\nu}$ depend on l_i, e.g.

$$B^{\mu\nu}_{1,\lambda} = \frac{1}{4\gamma^2}\left[\xi_2\left(L^{\mu\nu}_{33}\ell_{4,\lambda} + \frac{1}{\alpha}L^{\mu\nu}_{44}\ell_{3,\lambda}\right) - \left(r_2^\mu L^\nu_{34,\lambda} + r_2^\nu L^\mu_{34,\lambda}\right)\right] + \frac{1}{\gamma}\left(r_2^\mu \delta^\nu_\lambda - A_0^{\mu\nu} r_{2,\lambda}\right)$$

~Gram determinants!

On-the-fly reduction

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We can choose this decomposition freely such that we can cancel propagators D_i

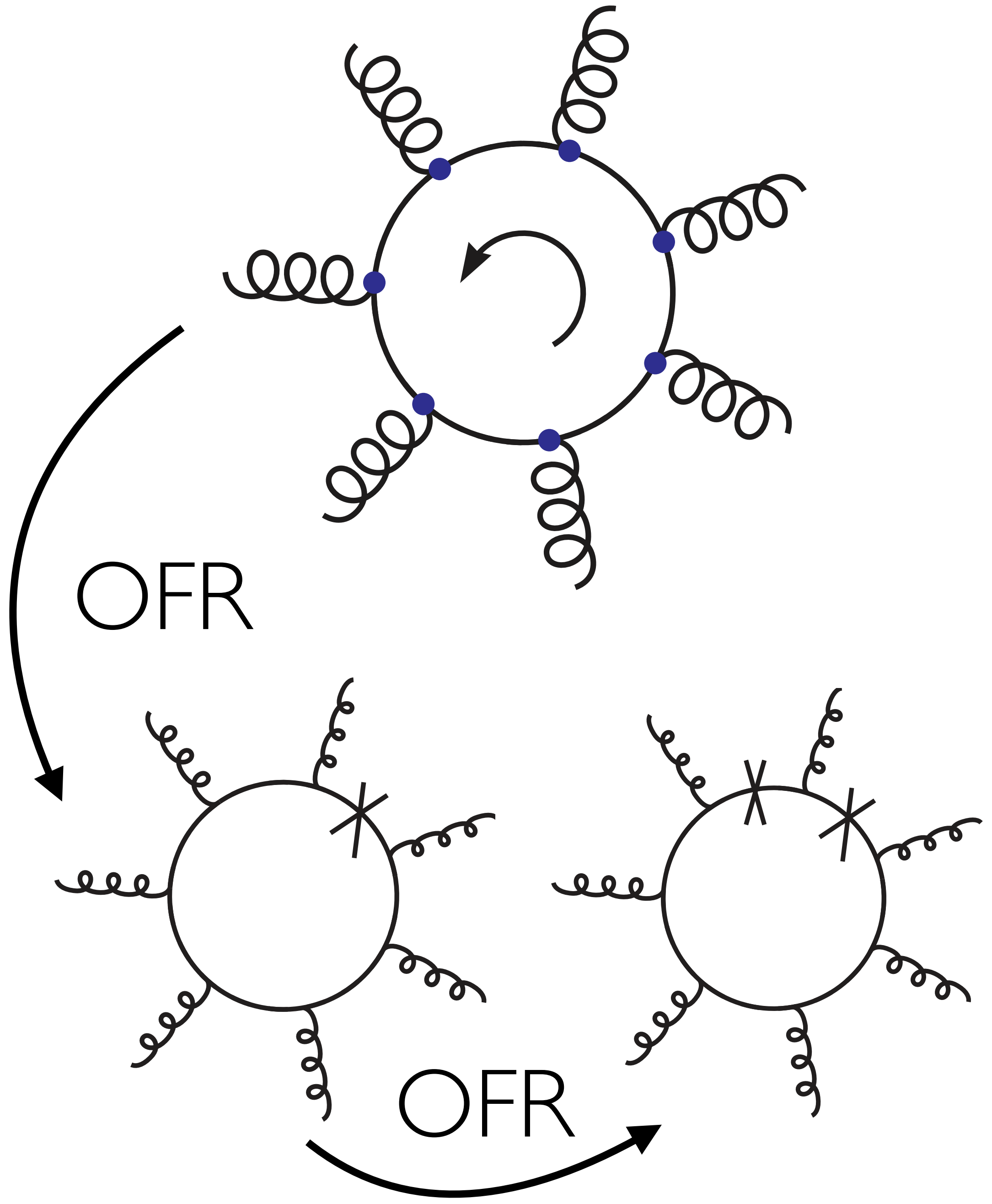
- $A_i^{\mu\nu}, B_i^{\mu\nu}$ depend on l_i, e.g.

$$B^{\mu\nu}_{1,\lambda} = \frac{1}{4\gamma^2}\left[\xi_2 \left(L^{\mu\nu}_{33} \ell_{4,\lambda} + \frac{1}{\alpha} L^{\mu\nu}_{44} \ell_{3,\lambda} \right) - \left(r_2^\mu L^\nu_{34,\lambda} + r_2^\nu L^\mu_{34,\lambda} \right) \right] + \frac{1}{\gamma}\left(r_2^\mu \delta^\nu_\lambda - A_0^{\mu\nu} r_{2,\lambda} \right)$$

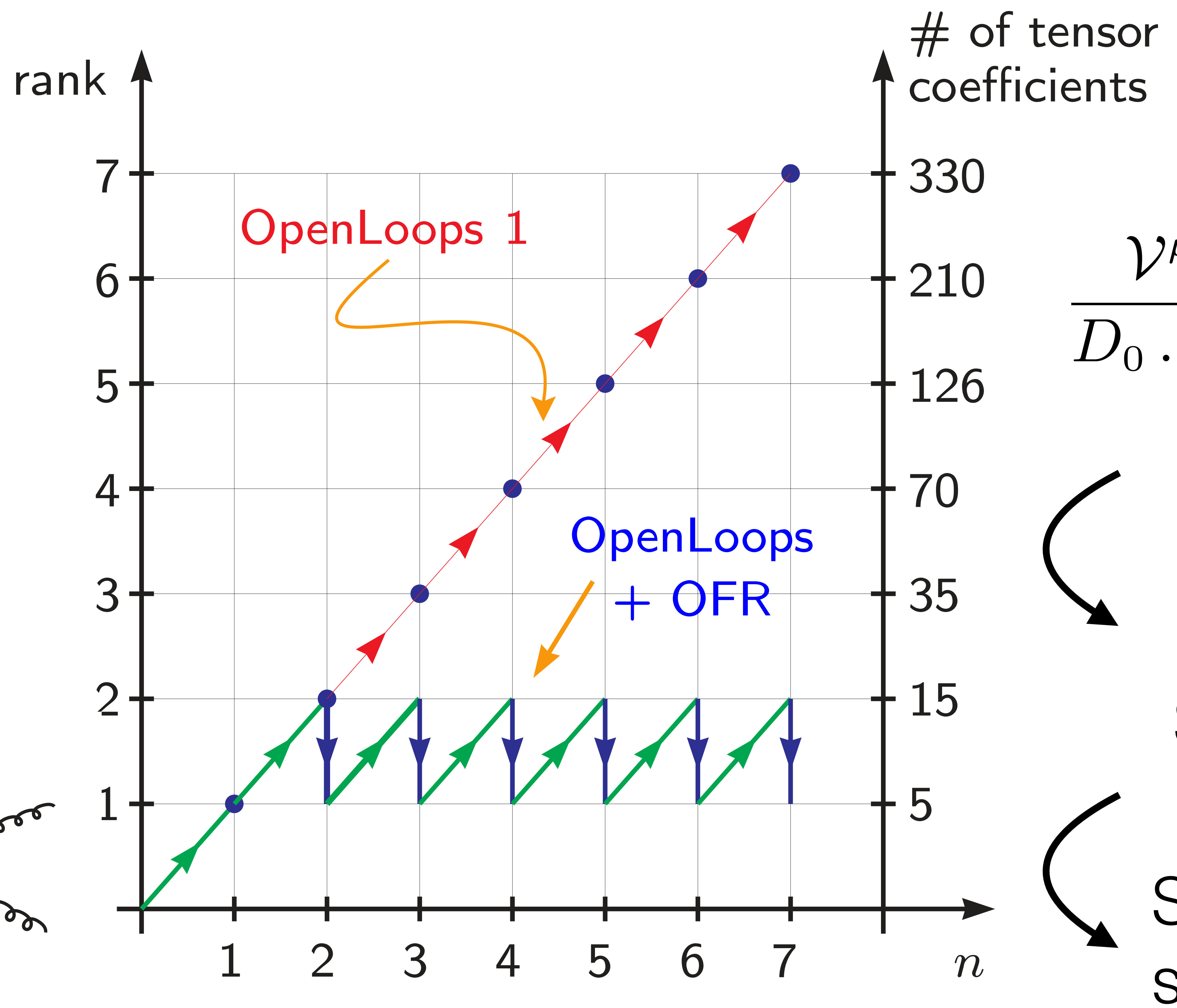
~Gram determinants!

On-the-fly reduction

Example:



4 pinched topologies
generated per reduction step



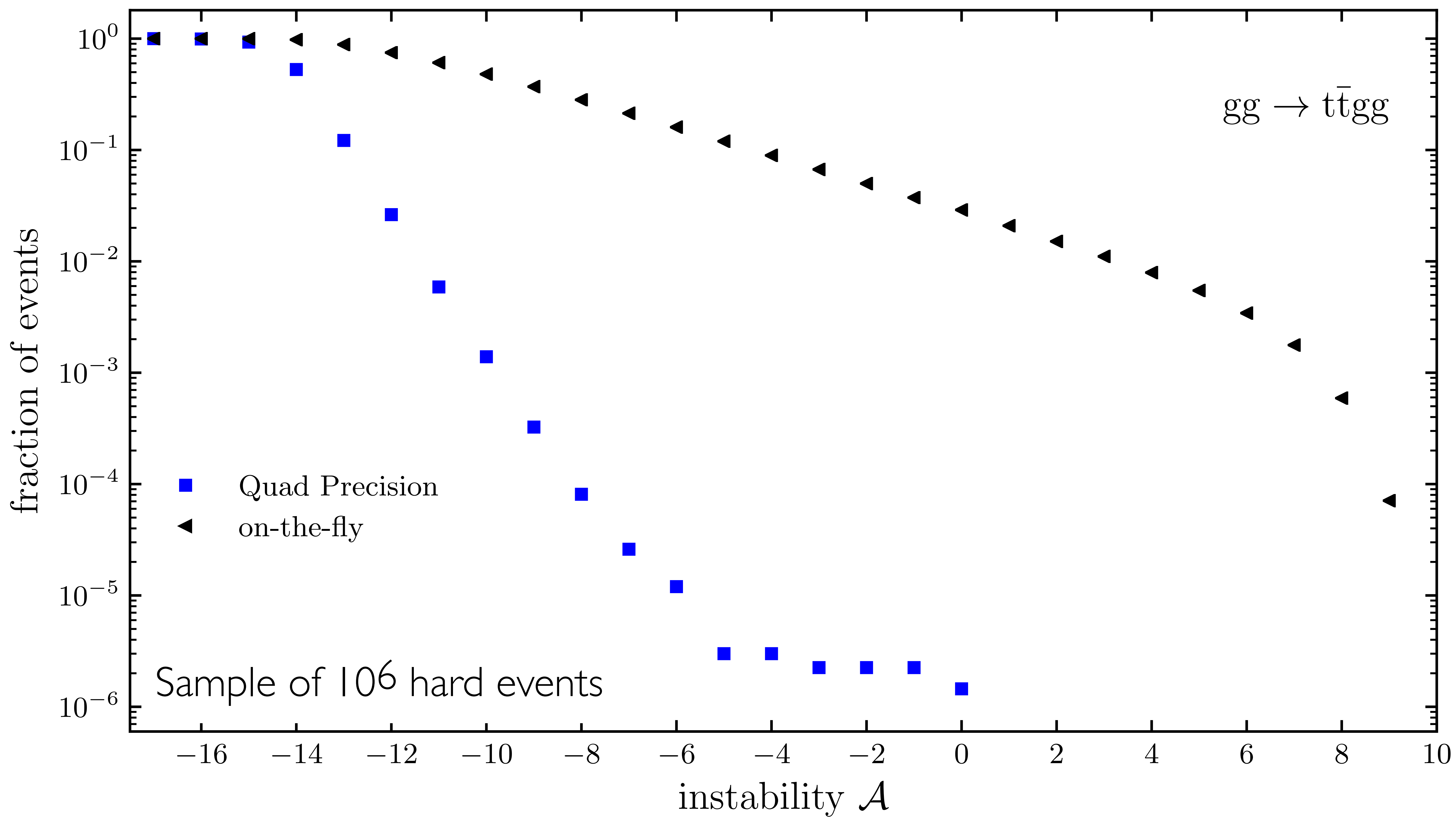
$$\frac{\mathcal{V}^{\mu\nu} q_\mu q_\nu}{D_0 \dots D_{N-1}} = \frac{\mathcal{V}^{\mu}_{-1} q_\mu + \mathcal{V}_{-1}}{D_0 \dots D_{N-1}} + \sum_{i=0}^{3} \frac{\mathcal{V}^{\mu}_{i} q_\mu + \mathcal{V}_{i}}{D_0 \dots \not{D}_i \dots D_{N-1}}$$

Huge proliferation of terms!

Solution: merging (+ OF helicity summation)

Similar CPU performance as
standard OpenLoops

On-the-fly reduction: stability



$$\mathcal{A}_X = \log_{10}\left|\frac{\mathcal{W}_{01}^{X} - \mathcal{W}_{01}^{\mathrm{qp}}}{\mathcal{W}_{01}^{\mathrm{qp}}}\right|$$

$\mathcal{M}_{\mathrm{qp}}$ via OL1 with CutTools

→Huge numerical instabilities in naive OFR implementation

Sources of numerical instabilities in OFR

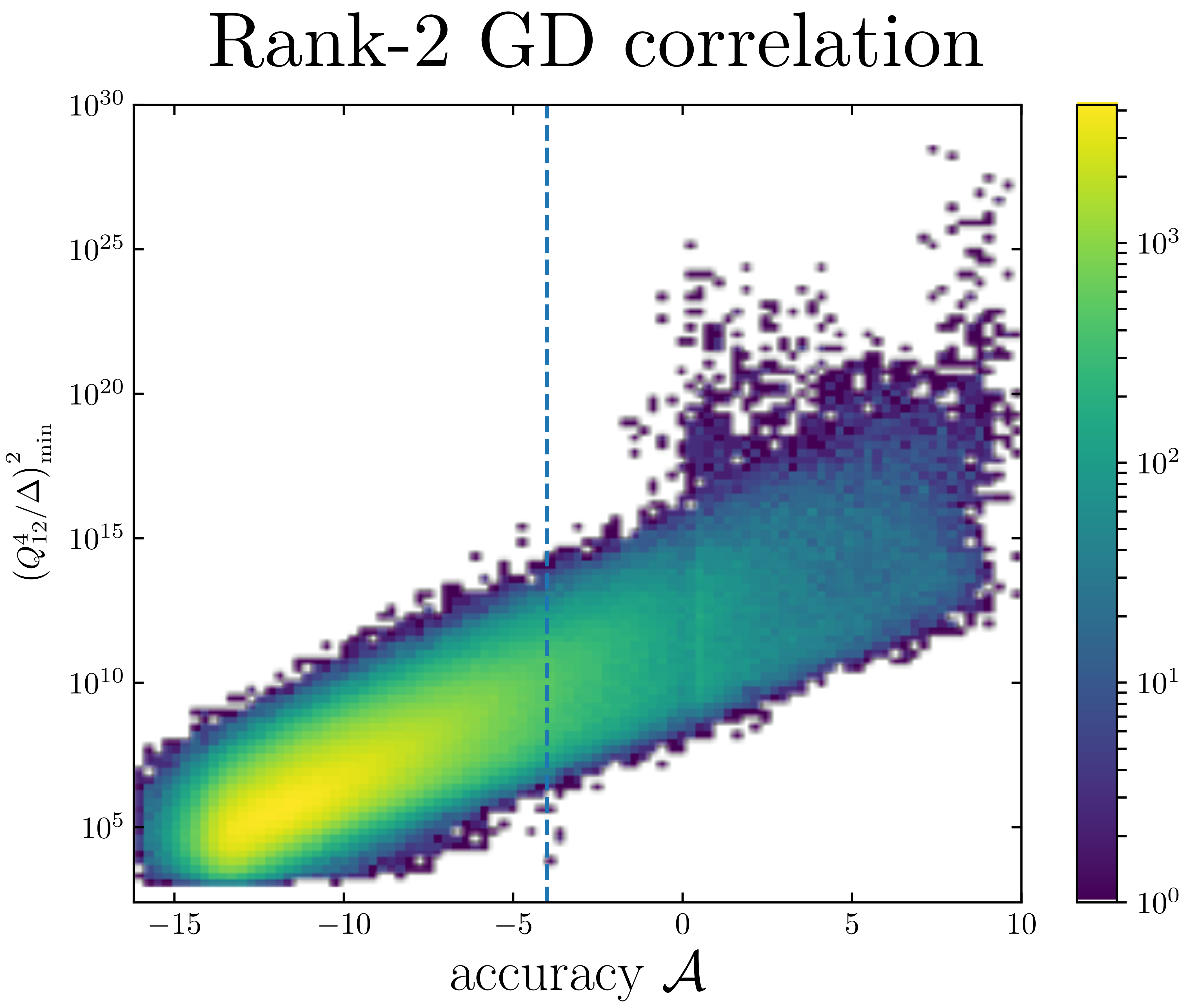
$$q^\mu q^\nu = A^{\mu\nu} + B^{\mu\nu}_\lambda q^\lambda$$

$$A^{\mu\nu}_i = \frac{1}{\Delta_{12}} a^{\mu\nu}_i,$$

$$B^{\mu\nu}_{i,\lambda} = \frac{1}{\Delta_{12}^2}\frac{1}{\sqrt{\Delta_{123}}} b^{(1),\mu\nu}_{i,\lambda} + \frac{1}{\Delta_{12}} b^{(2),\mu\nu}_{i,\lambda}$$

→Clear correlation between severe numerical instabilities and $\Delta_{12} \to 0$

→Instabilities propagate through the reduction and amplify



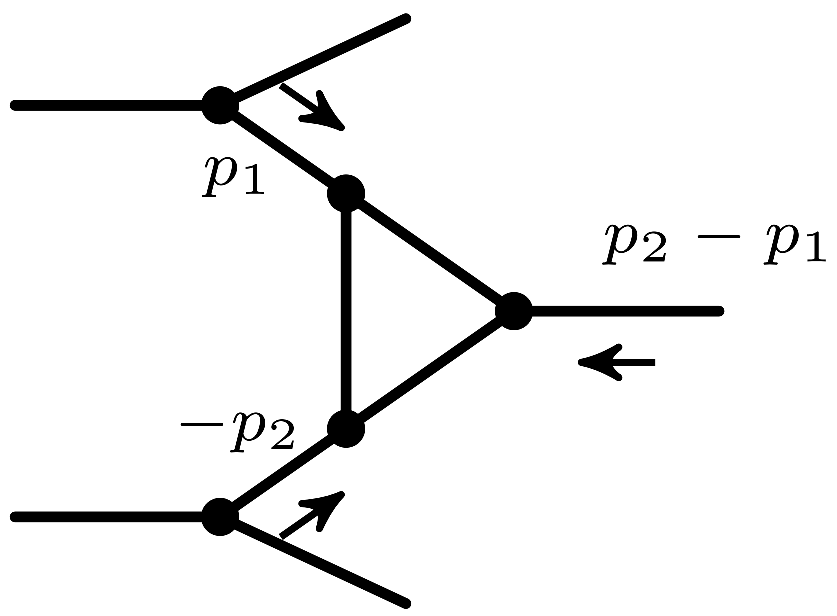
Solutions to numerical instabilities in OFR

1. Use freedom of choice of OFR basis for $N \geq 4$ such that $\Delta_{i_1 i_2} \to \max$.
   This corresponds to permutation of propagators.

$$\frac{\mathcal{V}^{\mu\nu} q_\mu q_\nu}{D_0 D_1 D_2 D_3 \ldots} \to \frac{\mathcal{V}^{\mu\nu} q_\mu q_\nu}{D_0 D_{i_1} D_{i_2} D_{i_3} \ldots}, \quad i_1, i_2, i_3 \in [1, 2, 3]$$

→Avoids small rank=2 Gram-determinant instabilities down to N=3

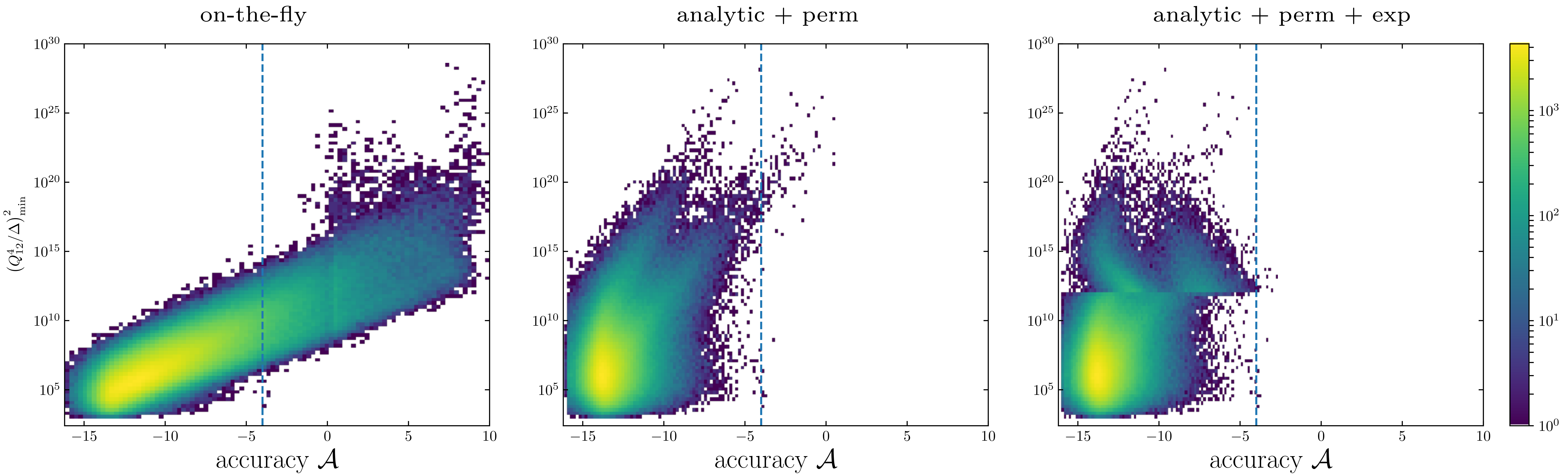
1. For N=3 and hard kinematics: Gram determinant instabilities arise **only** in t-channel topologies



$$\Delta_{12} \to 0 \text{ for } p_1^2 \to p_2^2$$

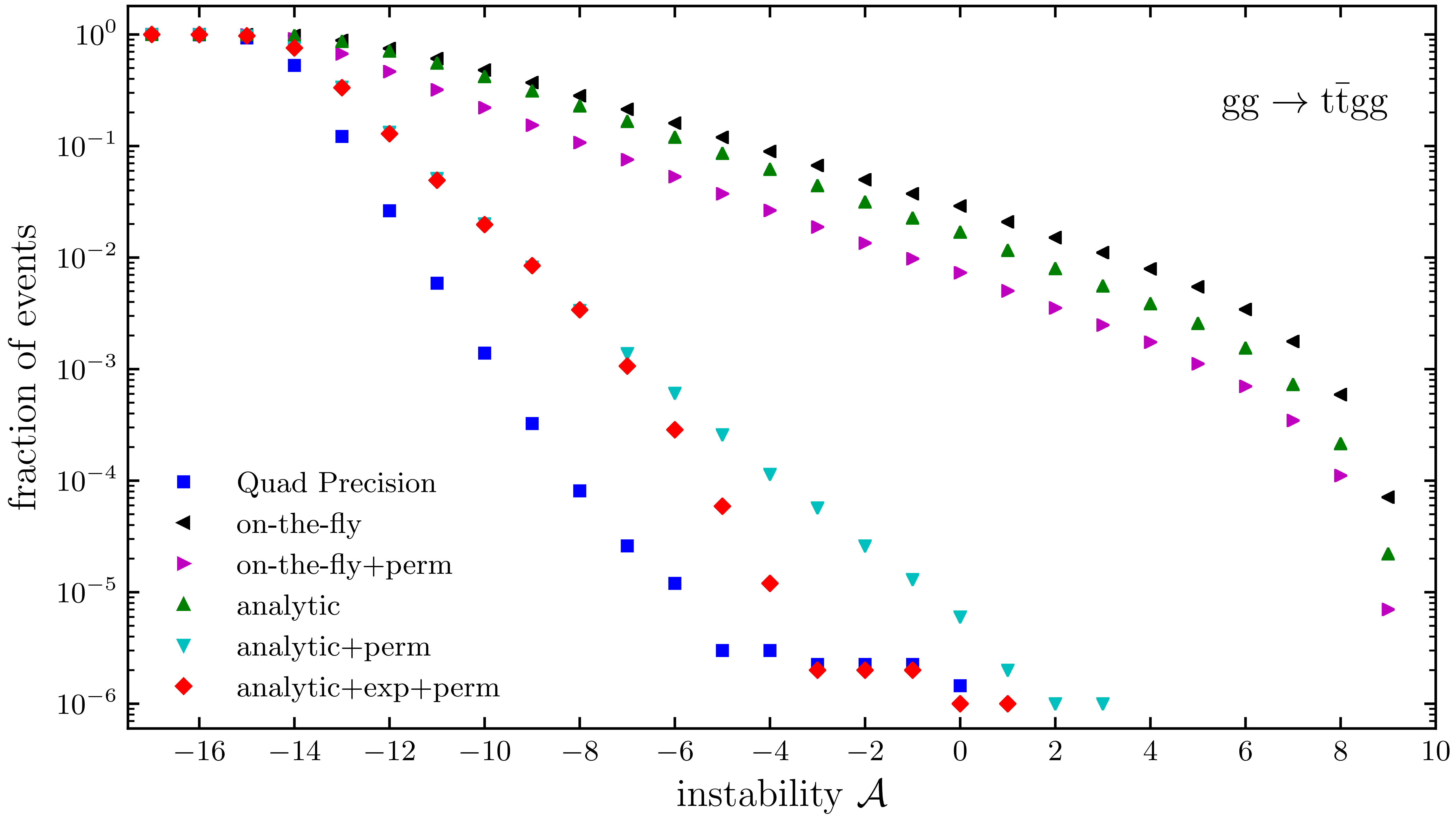
→Can be avoided using analytical reduction to MI plus expansions in Δ_{12}

Solutions to numerical instabilities in OFR



→ No rank=2 Gram determinant instabilities!

Numerical stability with OFR

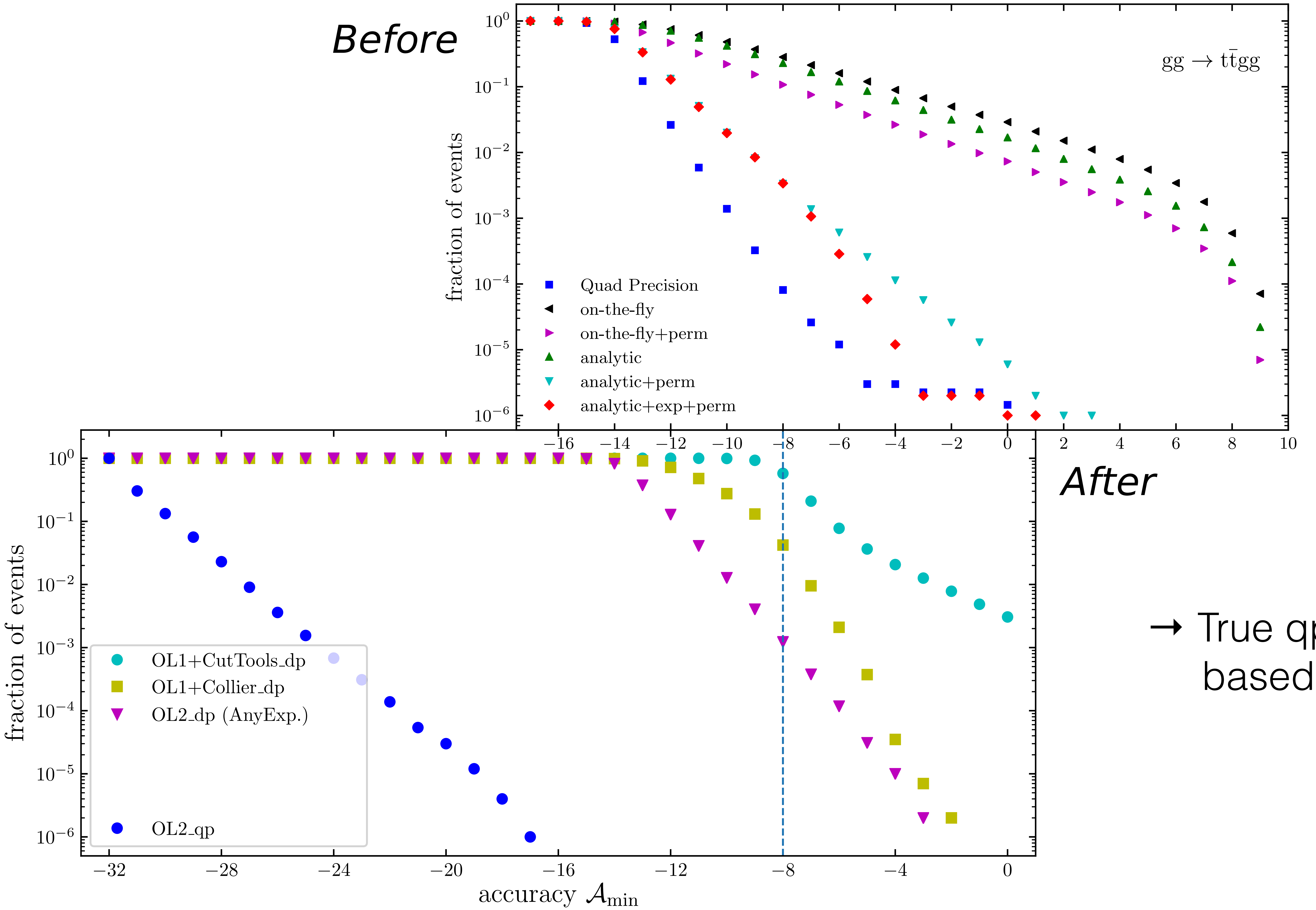


→ For remaining instabilities: use qp

- This also requires true qp benchmark: remove any dp “noise” (inputs, phase-space,…)
- Any-order expansions such that rescaling test is reliable

Numerical stability with OFR

Before



After

→ True qp benchmark based on OFR

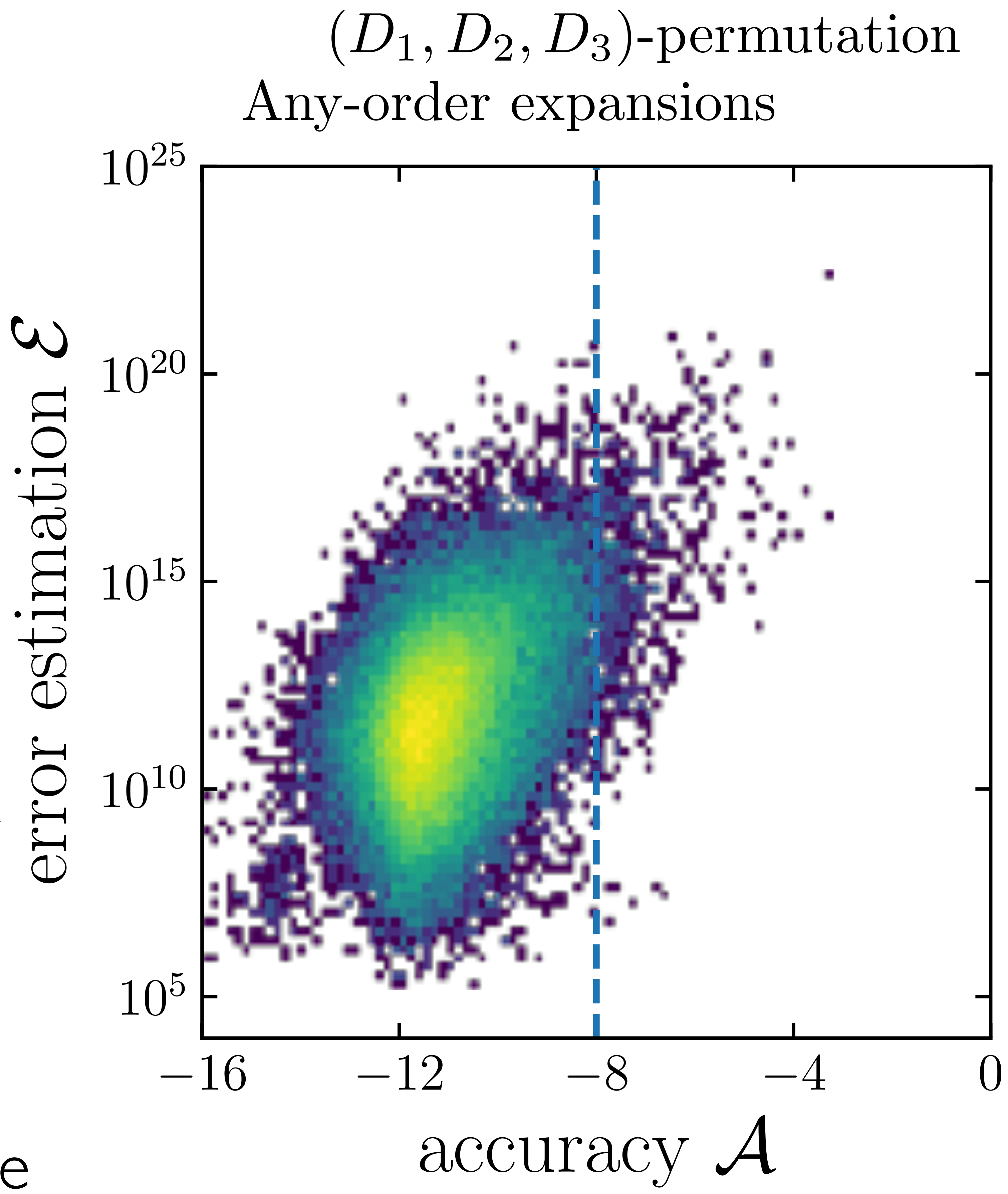
Local estimate of numerical stability

- For each step in the OL+OFR construction we construct and propagate an error estimate

Local error sources

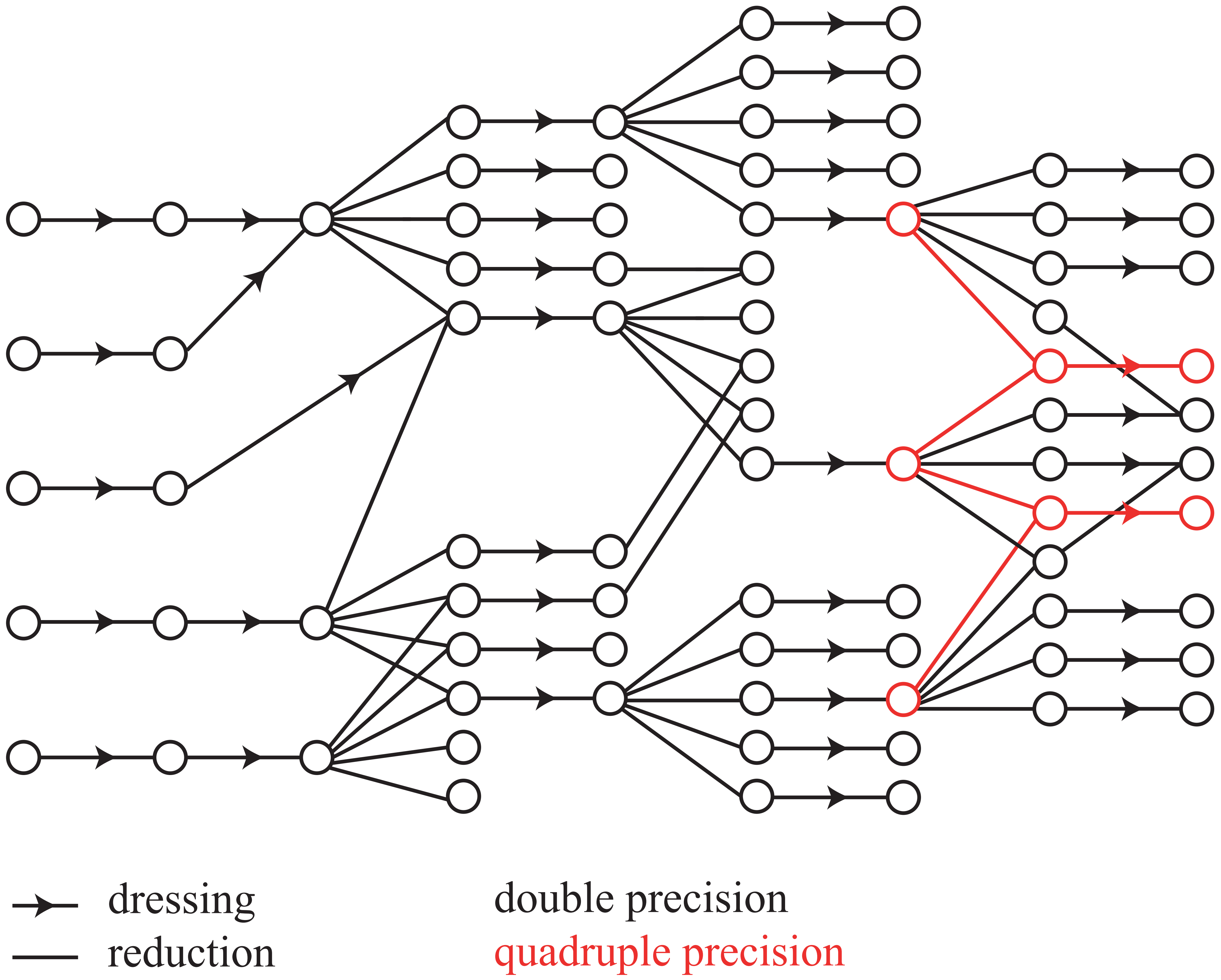
- Reduction basis
  → Estimated via rank=3 Gram determinant
  (no rank=2 Gram determinant instabilities remaining!)
- Reduction steps
  → Estimated via reduction coefficients
- Scalar integrals
  → Estimated using Collier
  (via mod. Cayley determinant)

propagated to global error estimate



Hybrid precision

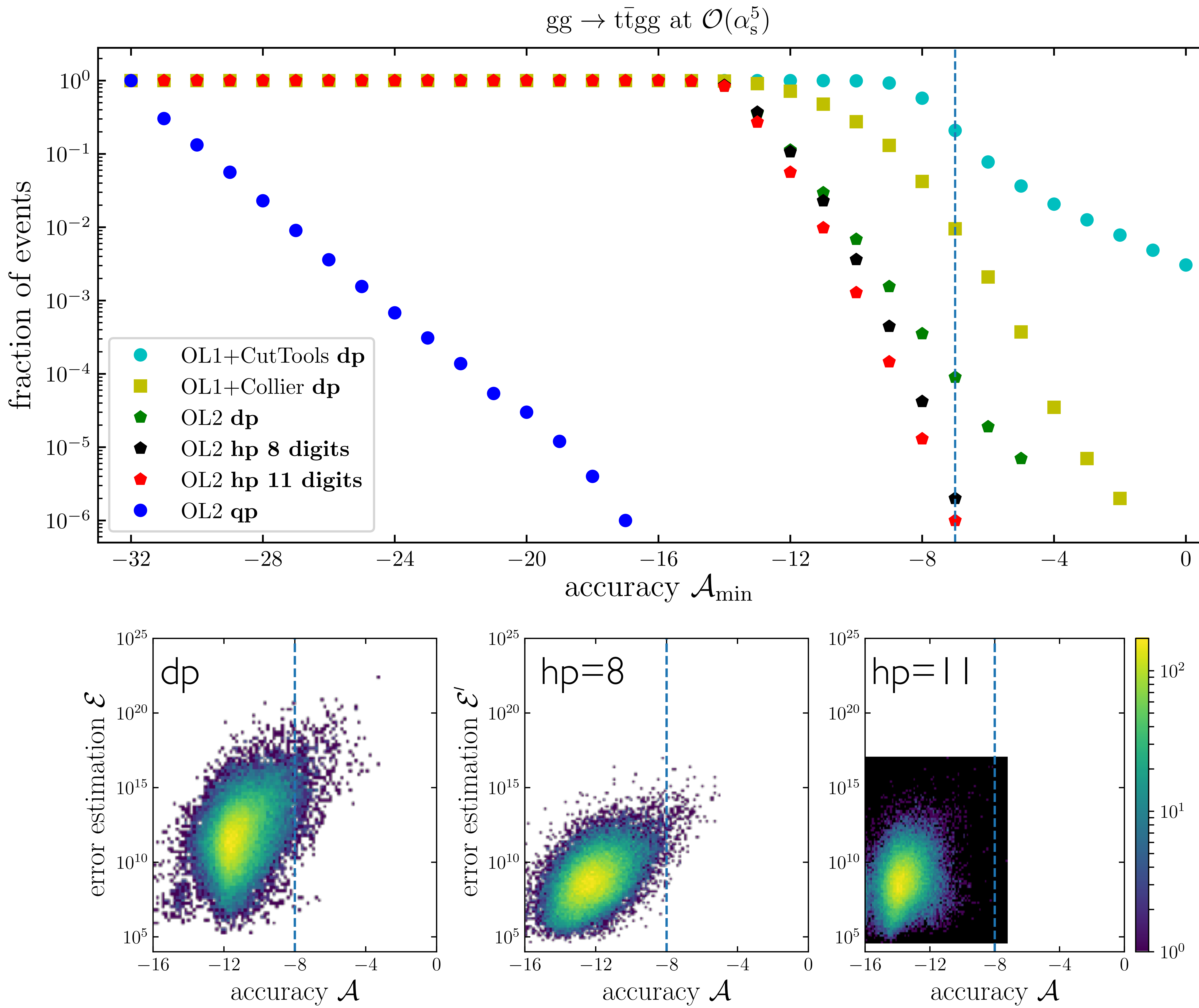
- Trigger qp only where locally necessary, e.g.



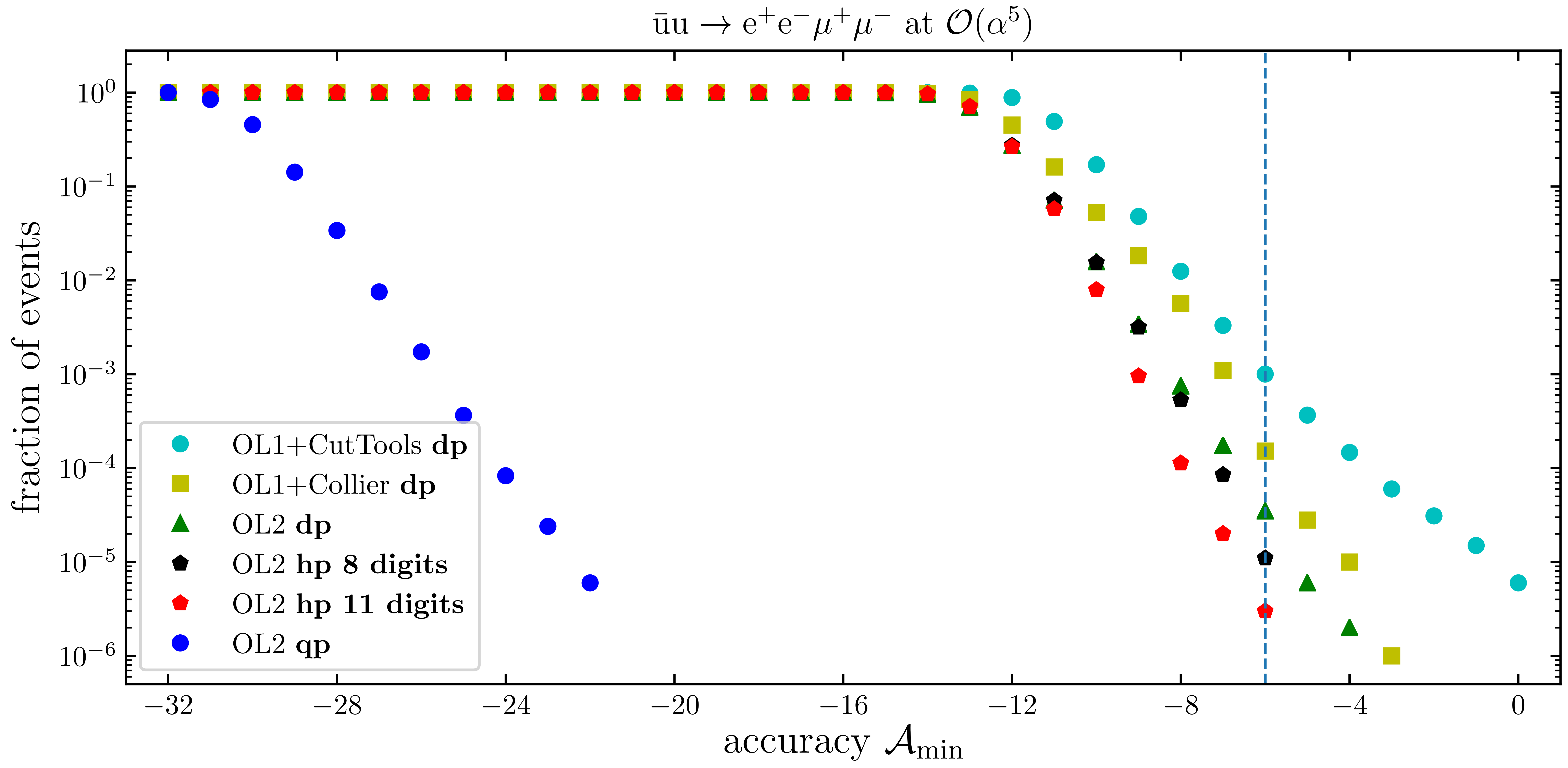
→ CPU cost: O(1%) of full qp evaluation

→ for hard kinematics: excellent numerical stability at only O(10%) cost with respect to pure dp

Hybrid precision performance



Hybrid precision performance



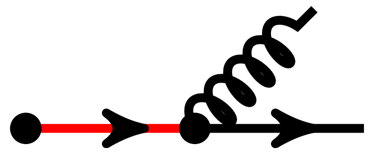
Numerical instabilities in the IR

- Frequent appearance of double small rank 2 GD instabilities

$$\Delta_{ij} \approx 0, \qquad \Delta_{kl} \approx 0$$

→ change of basis is futile

- IR kinematics

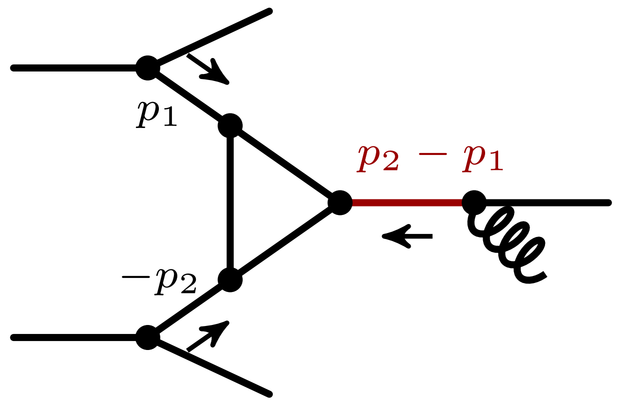


$$\sim \frac{1}{(p+k)^2 - m^2} = \frac{1}{2p \cdot k}$$

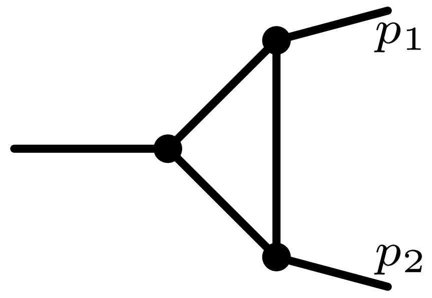
→ ensure stable invariants

- Unstable triangle reductions

  - IR t-channel $(p_2 - p_1)^2 \approx 0$

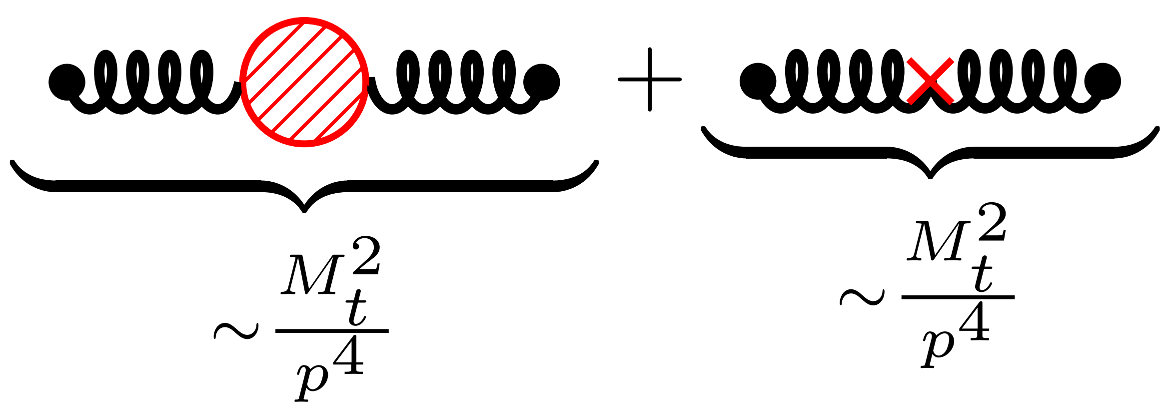


  - IR triangles $\Delta_{12} \approx 0$



- Cancellations: virtual + CT

  - *Gluon self-energy*

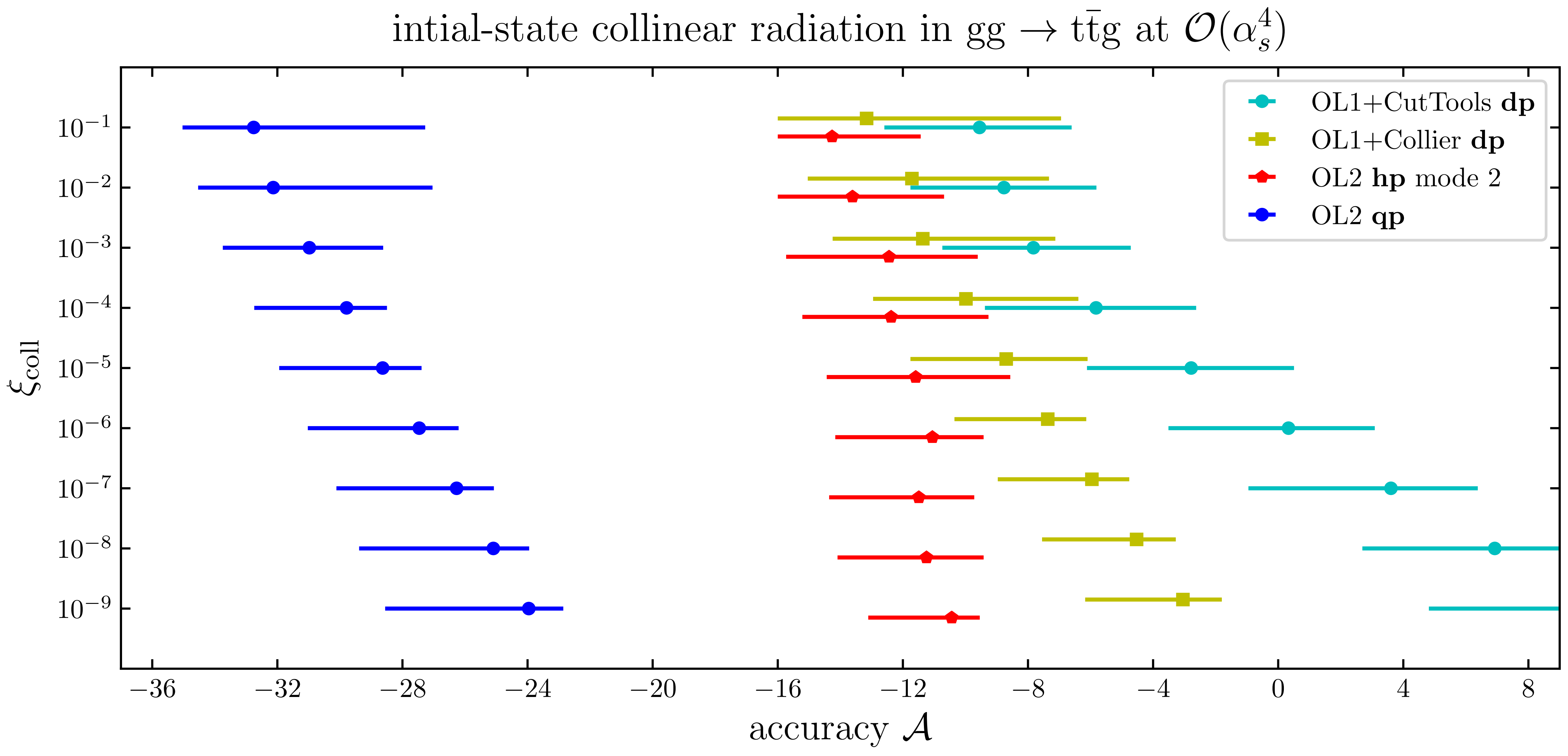


$$+ \quad \overset{!}{\propto} \frac{1}{p^2}$$

→ allow for analytical cancelation: reorganise contributions

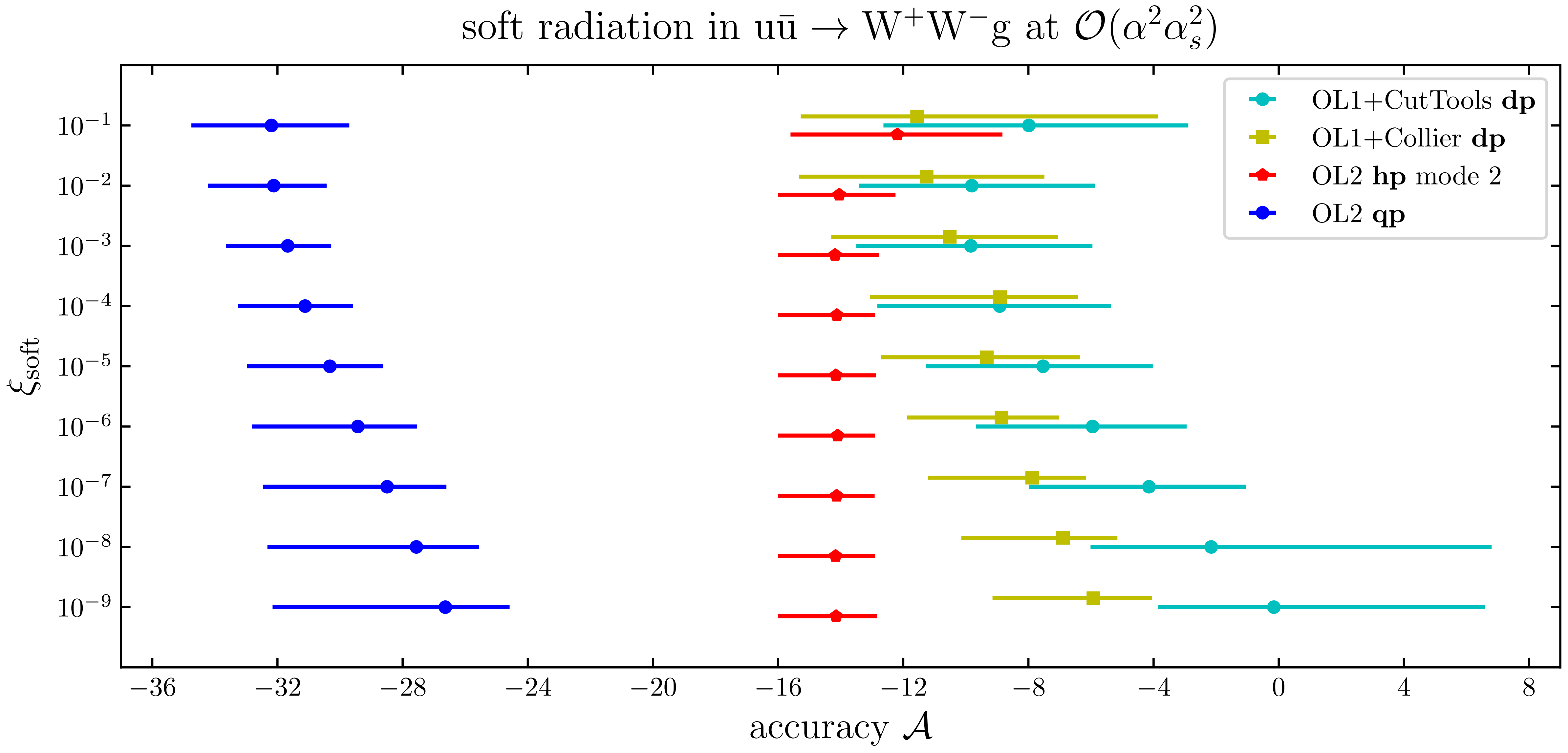
> → IR features and dedicated IR qp triggers via hp_mode=2
>
> ◉ currently only fully consistent for NLO QCD
>
> ◉ extension to NLO QED trivial

Numerical stability in the IR



$$\xi_{\rm soft} = E_{\rm soft}/Q$$

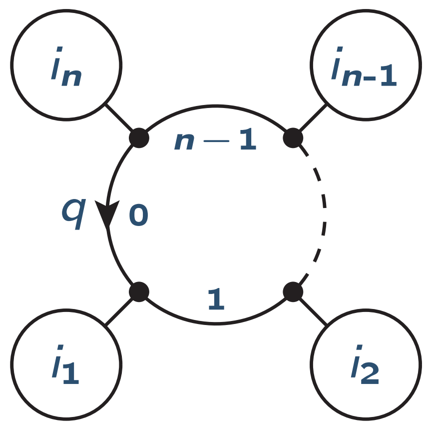
$$\xi_{\rm coll} = \arccos\left(\frac{\boldsymbol{p}_i \cdot \boldsymbol{p}_j}{|\boldsymbol{p}_i||\boldsymbol{p}_j|}\right)^2$$



New: On-The-fly TEnsor Reduction (OTTER)

- Perform OFR directly at the level of tensor integrals

$$T_N^{\mu_1\cdots\mu_r} = \int \mathrm{d}^D\bar{q}\,\frac{q^{\mu_1}\cdots q^{\mu_r}}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{N-1}}$$



$$= \int \frac{d^D\mathcal{N}(q)}{D_0D_1\ldots D_{n-1}} = \sum_{r=0}^{R} \mathcal{N}_{\mu_1\ldots\mu_r} \underbrace{\int \frac{q^{\mu_1}\ldots q^{\mu_r}}{D_0D_1\ldots D_{n-1}}}_{\text{tensor integral}}$$

→ targeted stability improvements as in OFR: change of basis, expansions, hp, …

→ Most important advantages:

1. for the first time OFR including hp for loop^2 processes (game-changer for loop-induced processes)
2. qp/dp can be restricted to tensor integrals. Coefficients can be determined in dp only

New: On-The-fly TEnsor Reduction (OTTER)

Details of OTTER reduction strategy:

- N >4
  - rank=2…N: dAP
  - rank=0,1: OPP
- N=4
  - rank=2,3,4: dAP
  - rank=1: special case
- N=3
  - rank=1,2,3: dAP or PV

N=2

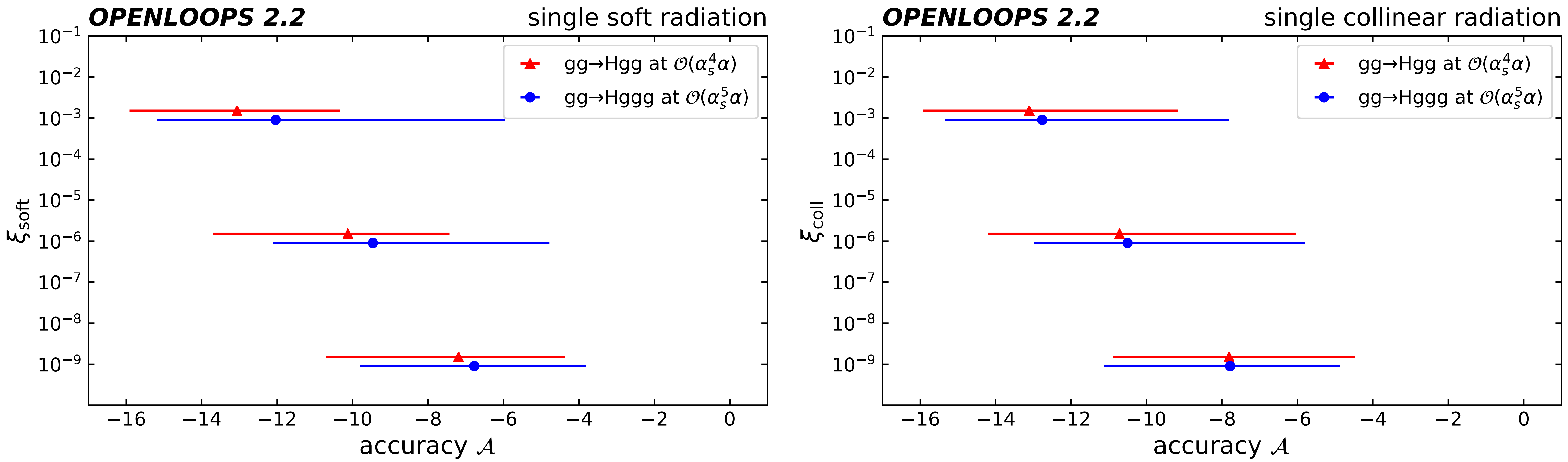
- rank=1,2: PV

Implementation:

1. determination of reduction dependences: top-down (large N to small N)
2. evaluation of tensor integrals: bottom-up (small N to large N)

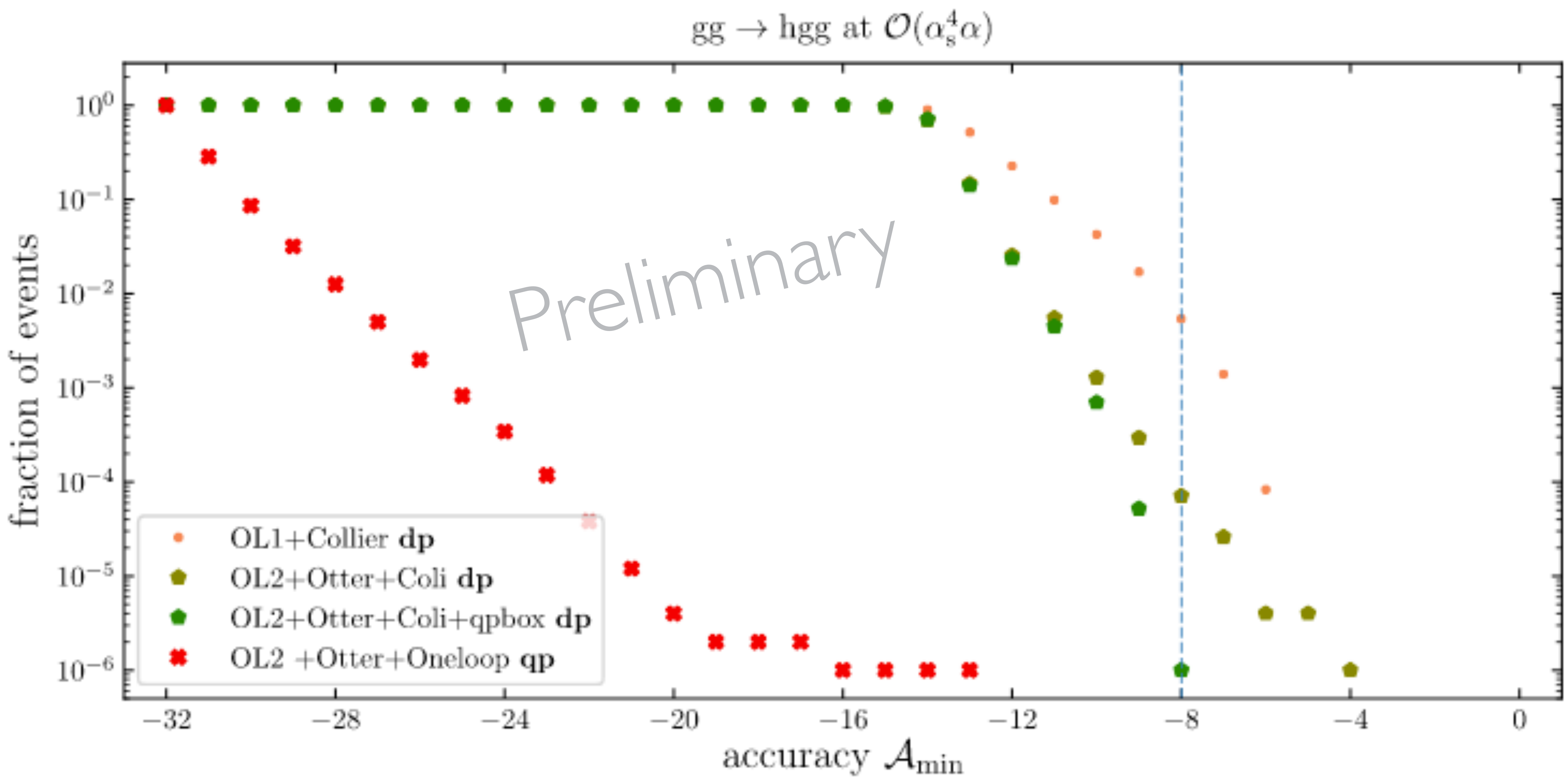
OTTER performance

$$\xi_{\rm soft} = E_{\rm soft}/\sqrt{s}, \quad \xi_{\rm coll} = \theta_{ij}^2$$



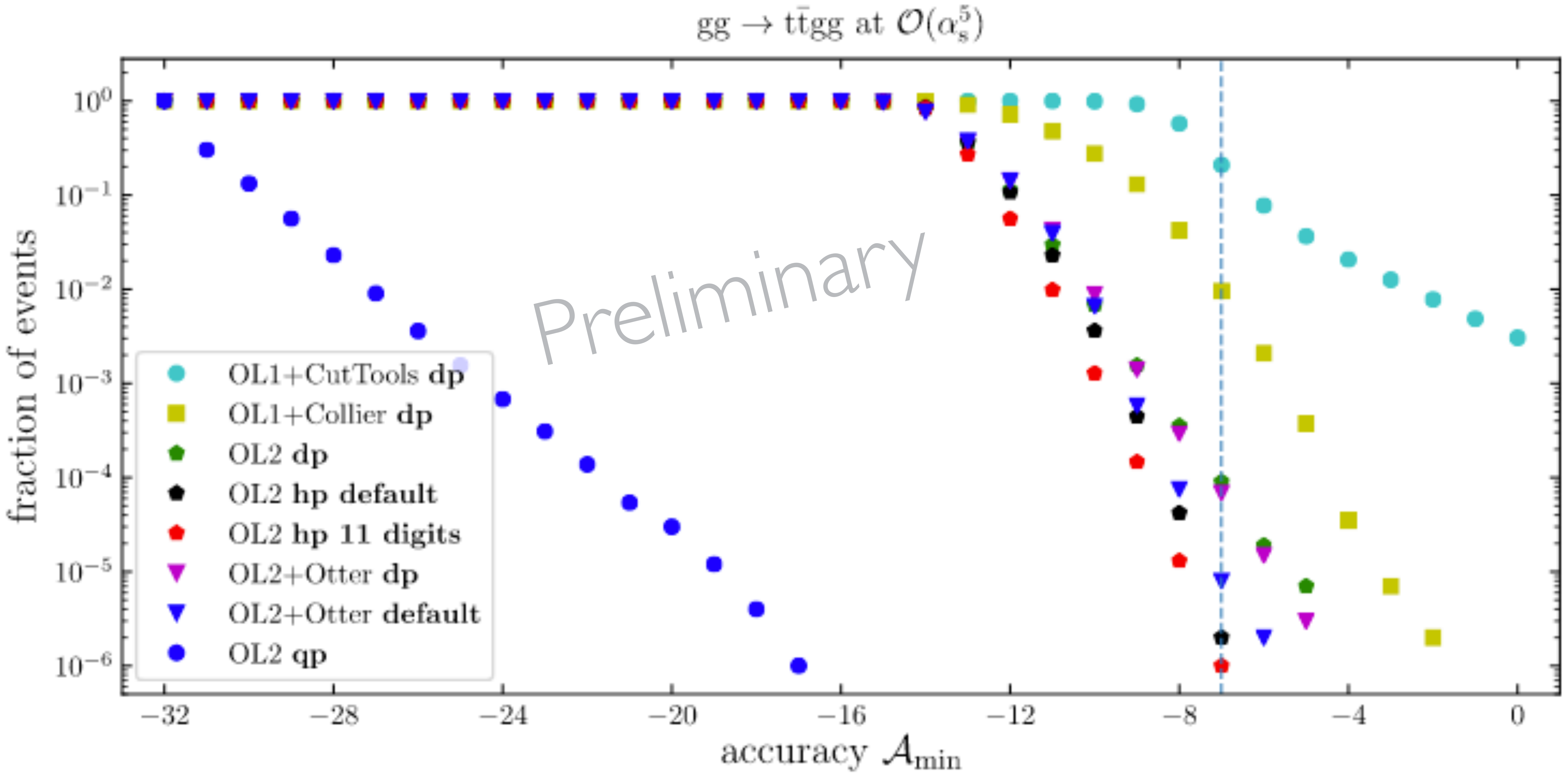
Mode	$gg \to Hgg$ (time/psp)	$gg \to Hggg$ (time/psp)
OL2.1+Collier DP	13ms	0.56s
OL2.1+Collier DP + error estimation	19ms	0.89s
OL2.1+CutTools QP	43000ms	2300s
OL2.2+Otter DP	8.9ms	0.29s
OL2.2+Otter DP + error estimation	11ms	0.32s
OL2.2+Otter DP+QP tensor integrals	68ms	0.87s
OL2.2+Otter QP	740ms	23s

OTTER performance



→ stability of scalar integrals becomes relevant

OTTER performance



OTTER performance: RRV to $\gamma^* \to e^+e^-$

CPU performance for ee~aaa at NLO QED:

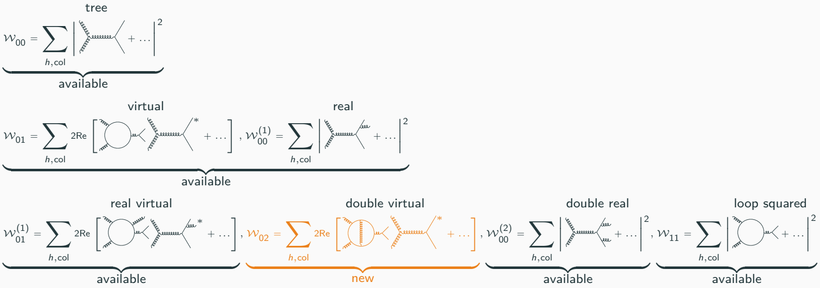
OL+OFR dp	OL+OFR qp	
4.4 ms	125ms	
OL+Otter dp	OL+Otter qp (full)	OL+Otter qp (only TI)
4.0 ms	78ms	47ms

Conclusions: real-virtual stability

- OpenLoops provides very fast and stable one-loop amplitudes in the SM at NLO QCD, NLO EW and NLO QED up to high multiplicities
- Systematic stability improvements thanks to OFR techniques
- New/upcoming: On-The-fly TEnsor Reduction (OTTER)
- OL+OTTER: new standard for one-loop real-virtual applications

Automation at NNLO

The public OpenLoops [Buccioni, Lang, Lindert, Maierhöfer, Pozzorini, Zhang, Zoller] already delivers some components to NNLO:

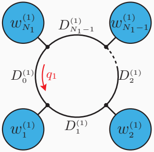


- OpenLoops is already being used in NNLO calculations in particular for the real virtual components in e.g. MATRIX [Grazzini, Kallweit, Wiesemann], NNLOJET [Gehrmann-De Ridder, Gehrmann, Glover, Huss, Walker], McMule [Banerjee, Engel, Signer, Ulrich].
- **NNLO in OpenLoops: require double virtual**

Components to NLO Calculations

Final result in D-dimensions, numerical tools: construct numerator in 4-dimensions.
For one diagram Γ:

$$\mathcal{M}_{1,\Gamma} =$$



$$= \underbrace{C_{1,\Gamma}}_{\text{color}} \int \mathrm{d}\bar{q}_1 \, \underbrace{\frac{\mathcal{N}(q_1)}{\mathcal{D}(\bar{q}_1)}}_{\substack{\text{4-dim numerator,} \\ \text{(D-dim denominator)}}} = C_{1,\Gamma} \sum_r \underbrace{\mathcal{N}_{\mu_1\cdots\mu_r}}_{\text{tensor coefficient}} \underbrace{\int \mathrm{d}\bar{q}_1 \frac{q_1^{\mu_1} \cdots q_1^{\mu_r}}{\mathcal{D}(\bar{q}_1)}}_{\text{tensor integral}}$$

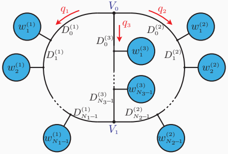
Calculation decomposed into:

- Numerical construction of tensor coefficient in 4-dim → OpenLoops algorithm
  [van Hameren; Cascioli, Maierhöfer, Pozzorini; Buccioni, Lang, Lindert, Maierhöfer, Pozzorini, Zhang, Zoller]
- Renormalization, restoration of (D-4)-dim numerator part → rational counterterms
  $\mathbf{R}\bar{\mathcal{M}}_{1,\Gamma} = \mathcal{M}_{1,\Gamma} + \mathcal{M}_{0,1,\Gamma}^{(\mathrm{CT})}$ [Ossola, Papadopoulos, Pittau]
- Reduction and evaluation of tensor integrals → On-the-fly reduction
  [Buccioni, Pozzorini, Zoller], Collier [Denner, Dittmaier, Hofer], OneLoop [van Hameren]

Components to NNLO Calculations

Final result in D-dimensions, numerical tools: construct numerator in 4-dimensions.
For one diagram Γ:

$\mathcal{M}_{2,\Gamma} =$

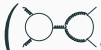


$$= \underbrace{\mathcal{C}_{2,\Gamma}}_{\text{color}} \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \; \underbrace{\frac{\mathcal{N}(q_1, q_2)}{\mathcal{D}(\bar{q}_1, \bar{q}_2)}}_{\substack{\text{4-dim numerator,} \\ \text{(D-dim denominator)}}} = \mathcal{C}_{2,\Gamma} \sum_{r,s} \underbrace{\mathcal{N}_{\mu_1 \cdots \mu_r \nu_1 \cdots \nu_s}}_{\text{tensor coefficient}} \underbrace{\int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \frac{q_1^{\mu_1} \cdots q_1^{\mu_r} \, q_2^{\nu_1} \cdots q_2^{\nu_s}}{\mathcal{D}(\bar{q}_1, \bar{q}_2)}}_{\text{tensor integral}}$$

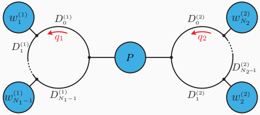
Calculation decomposed into:

- Numerical construction of tensor coefficient in 4-dim → this talk, **complete**
- Renormalization, restoration of (D-4)-dim numerator part → rational counterterms
  $\mathbf{R}\bar{\mathcal{M}}_{2,\Gamma} = \mathcal{M}_{2,\Gamma} + \mathcal{M}^{(\mathrm{CT})}_{1,1,\Gamma} + \mathcal{M}^{(\mathrm{CT})}_{0,2,\Gamma}$ [Lang, Pozzorini, Zhang, Zoller], **implementation ongoing**
- Reduction and evaluation of tensor integrals → todo

Two Loop Algorithm: Reducible Diagrams

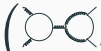
Distinguish irreducible () and reducible (, ) diagrams.

Exploit numerator factorization:

$\mathcal{N}(q_1, q_2) =$ 

$$= \underbrace{\left[\mathcal{N}^{(1)}(q_1)\right]^{\alpha_1}}_{\text{chain 1}} \underbrace{P_{\alpha_1 \alpha_2}}_{\text{bridge}} \underbrace{\left[\mathcal{N}^{(2)}(q_2)\right]^{\alpha_2}}_{\text{chain 2}}$$

Two Loop Algorithm: Reducible Diagrams

Distinguish irreducible () and reducible (, ) diagrams.

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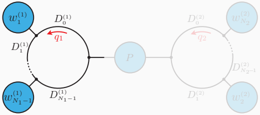
$\mathcal{N}(q_1, q_2) =$ 

$$= \underbrace{\left[\mathcal{N}^{(1)}(q_1)\right]^{\alpha_1}}_{\text{chain 1}} \underbrace{P_{\alpha_1\alpha_2}}_{\text{bridge}} \underbrace{\left[\mathcal{N}^{(2)}(q_2)\right]^{\alpha_2}}_{\text{chain 2}}$$

1. Construct chain 1 using extension of one-loop algorithm, perform first loop integration.

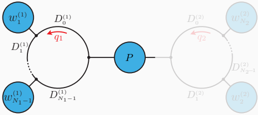
$$\mathcal{N}_n^{(1)} = \mathcal{N}_{n-1}^{(1)} S_n^{(1)}, \qquad \mathcal{N}_0^{(1)} = \mathbb{1}, \qquad \left[\mathcal{M}^{(1)}\right]^{\alpha_1} = \int \mathrm{d}\bar{q}_1 \frac{\mathrm{Tr}\left[\mathcal{N}_{N_1}^{(1)}(q_1)\right]^{\alpha_1}}{\mathcal{D}^{(1)}(\bar{q}_1)}$$

Two Loop Algorithm: Reducible Diagrams

Distinguish irreducible () and reducible (, ) diagrams.

Exploit numerator factorization:

$\mathcal{N}(q_1, q_2) =$

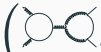


$$= \underbrace{\left[\mathcal{N}^{(1)}(q_1)\right]^{\alpha_1}}_{\text{chain 1}} \underbrace{P_{\alpha_1\alpha_2}}_{\text{bridge}} \underbrace{\left[\mathcal{N}^{(2)}(q_2)\right]^{\alpha_2}}_{\text{chain 2}}$$

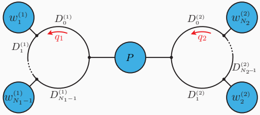
1. Construct chain 1 using extension of one-loop algorithm, perform first loop integration.
2. Connect bridge using tree algorithm
   $\rightarrow$ treat first loop as external "subtree".

$$P_n = P_{n-1} s_n^{(B)}(w_n^{(B)}), \qquad w_0^{(B)} = \left[\mathcal{M}^{(1)}\right]^{\alpha_1}, \qquad P_{-1} = \mathbb{1}$$

Two Loop Algorithm: Reducible Diagrams

Distinguish irreducible () and reducible (, ) diagrams.

Exploit numerator factorization:

$\mathcal{N}(q_1, q_2) =$ 

$= \underbrace{\left[\mathcal{N}^{(1)}(q_1)\right]^{\alpha_1}}_{\text{chain 1}} \underbrace{P_{\alpha_1\alpha_2}}_{\text{bridge}} \underbrace{\left[\mathcal{N}^{(2)}(q_2)\right]^{\alpha_2}}_{\text{chain 2}}$

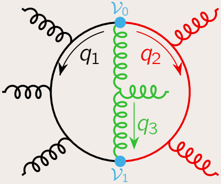
1. Construct chain 1 using extension of one-loop algorithm, perform first loop integration.
2. Connect bridge using tree algorithm
   → treat first loop as external "subtree".
3. Construct chain 2 using extension of one-loop algorithm
   → treat first loop + bridge as external "subtree".

$$\mathcal{N}_n^{(2)} = \mathcal{N}_{n-1} S_n^{(2)}(w_n^{(2)}), \qquad w_1^{(2)} = \left[\mathcal{M}^{(1)}\right]^{\alpha_1} P_{\alpha_1\alpha_2}, \qquad \mathcal{N}_0^{(2)} = \mathbb{1}$$

Two Loop Algorithm: Irreducible Diagrams

Two-loop numerator factorizes:

$$\mathcal{N}(q_1, q_2) = \mathcal{N}^{(1)}(q_1)\ \mathcal{N}^{(2)}(q_2)\ \mathcal{N}^{(3)}(q_3)\ \mathcal{V}_0(q1, q2)\ \mathcal{V}_1(q1, q2)\big|_{q_3 \to -(q_1+q_2)}$$
$$\mathcal{N}^{(i)}(q_i) = s_0^{(i)}(q_i)\ s_1^{(i)}(q_i) \cdots s_{N_i-1}^{(i)}(q_i)$$



Building blocks $\mathcal{K}_n$ for algorithm:

- $\mathcal{N}^{(1)}, \mathcal{N}^{(2)}, \mathcal{N}^{(3)}$ 3 chains
- $s_a^{(1)}, s_a^{(2)}, s_a^{(3)}$ their segments
- $\mathcal{V}_0, \mathcal{V}_1$ vertices connecting chains
- $\mathcal{U}_0 = 2 \sum_{\text{col}} c\mathcal{M}_0^*$ Born and color

⇒ Construct Born-loop interference recursively from building blocks:

$$\mathcal{U}_n = \mathcal{U}_{n-1}\mathcal{K}_n, \quad \mathcal{K}_n \in \{\mathcal{U}_0, \mathcal{N}^{(i)}, s_a^{(i)}, \mathcal{V}_j\}$$

Factorization results in freedom of choice for two-loop algorithm.

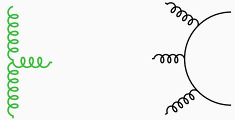
- CPU cost ∼ # multiplications
- determine most efficient variant through cost simulation

Two Loop Algorithm: Irreducible Diagrams

1. Construct shortest chain $\mathcal{N}^{(3)}(q_3)$.

$$\mathcal{N}_n^{(3)}(q_3) = \mathcal{N}_{n-1}^{(3)} S_n^{(3)}, \qquad \mathcal{N}_0^{(3)} = \mathbb{1}$$

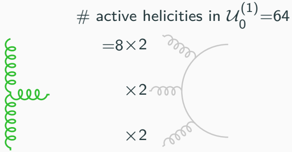
Two Loop Algorithm: Irreducible Diagrams



1. Construct shortest chain $\mathcal{N}^{(3)}(q_3)$.
2. Construct longest chain $\mathcal{N}^{(1)}(q_1)$ using $\mathcal{U}_0 = 2\sum_{col} C\mathcal{M}_0^*(h)$ as the initial condition.

$$\mathcal{U}_n^{(1)} = \mathcal{U}_{n-1}^{(1)} S_n^{(1)}, \qquad \mathcal{U}_0^{(1)} = 2\sum_{col} C\mathcal{M}_0^*$$

Two Loop Algorithm: Irreducible Diagrams



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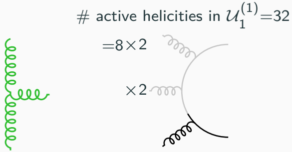
$$\mathcal{U}_n^{(1)}(h_{n+1}, h_{n+2}, \ldots) = \sum_{h_n} \mathcal{U}_{n-1}^{(1)}(h_n, h_{n+1}, h_{n+2} \ldots) S_n^{(1)}(h_n), \qquad \mathcal{U}_0^{(1)} = \mathcal{U}_0^{(1)}(h_1, h_2, \ldots, h_{N_1+N_2+N_3})$$

Two Loop Algorithm: Irreducible Diagrams



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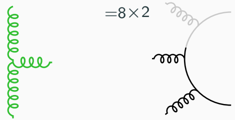
Two Loop Algorithm: Irreducible Diagrams



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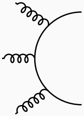
Two Loop Algorithm: Irreducible Diagrams

active helicities in $\mathcal{U}_3^{(1)}$=8



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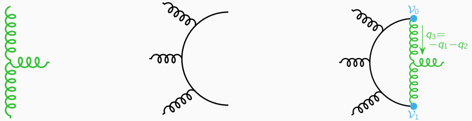
Two Loop Algorithm: Irreducible Diagrams



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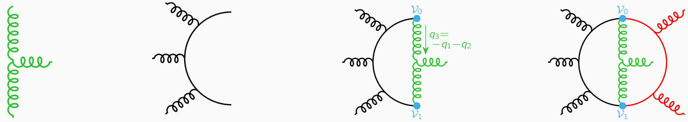
$$[\mathcal{U}^{(13)}]^{\beta_{N_2}^{(2)}}_{\beta_0^{(2)}} = [\mathcal{U}^{(1)}]^{\beta_{N_1}^{(1)}}_{\beta_0^{(1)}} [\mathcal{N}^{(3)}]^{\beta_{N_3}^{(3)}}_{\beta_0^{(3)}} \left[\mathcal{V}_0(q1,\, q2)\right]^{\beta_0^{(1)}\beta_0^{(2)}\beta_0^{(3)}} \left[\mathcal{V}_1(q1,\, q2)\right]_{\beta_{N_1}^{(1)}\beta_{N_2}^{(2)}\beta_{N_3}^{(3)}} \Big|_{q_3 \to -(q_1+q_2)}$$

Two Loop Algorithm: Irreducible Diagrams



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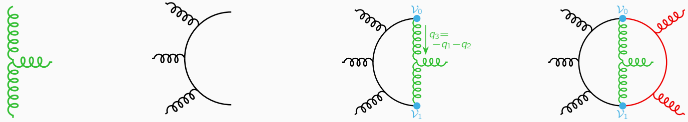
4. Attach $\mathcal{N}^{(2)}(q_2)$ segments to previously constructed object, sum helicities on-the-fly.

$$\mathcal{U}_n^{(123)} = \mathcal{U}_{(n-1)}^{(123)} S_n^{(2)}, \qquad \mathcal{U}_0^{(123)} = \mathcal{U}^{(13)} = \mathcal{U}^{(1)}(q_1)\mathcal{N}^{(3)}(q_3)\mathcal{V}_1(q1,\, q2)\mathcal{V}_0(q1,\, q2)$$

Two Loop Algorithm: Irreducible Diagrams



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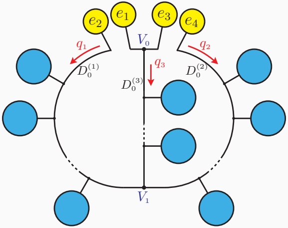
4. Attach $\mathcal{N}^{(2)}(q_2)$ segments to previously constructed object, sum helicities on-the-fly.

Completely general and highly efficient algorithm.
Fully implemented for QED and QCD corrections to the SM.

Numerical Stability

Validate and measure numerical stability of two-loop algorithm without computing tensor integrals using **pseudotree test**.



- Cut two propagators of two-loop diagram
- Insert random wavefunctions e_1, e_2, e_3, e_4 saturating indices
- Set q_1, q_2 to random constant values, contract tensor coefficients $\mathcal{N}_{\mu_1\ldots\mu_r\nu_1\ldots\nu_s}$with fixed-value tensor integrand $\frac{q_1^{\mu_1}\cdots q_1^{\mu_r} q_2^{\nu_1}\cdots q_1^{\nu_s}}{\mathcal{D}(q_1,q_2)}$
- Compare to computation with well-tested tree level algorithm

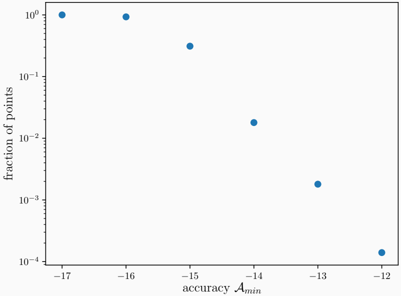
Typical accuracy around 10^{-15} in double (DP) and 10^{-30} in quad (QP) precision, always much better than 10^{-17} in QP $\Rightarrow$ **Establish QP as benchmark for DP**

Numerical Stability: Irreducible Diagrams

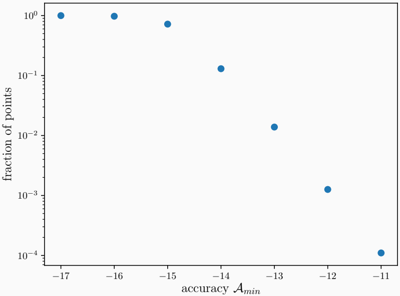
Numerical stability of scattering probability density $\mathcal{W}_{02}^{(\text{2L,pr})}$ in double (pr=DP) vs quad (pr=QP) precision in pseudotree mode.

$$\mathcal{A}_{\mathrm{DP}} = \log_{10}\left(\frac{|\mathcal{W}_{02}^{(\mathrm{2L,DP})} - \mathcal{W}_{02}^{(\mathrm{2L,QP})}|}{\mathrm{Min}(|\mathcal{W}_{02}^{(\mathrm{2L,DP})}|, |\mathcal{W}_{02}^{(\mathrm{2L,QP})}|)}\right)$$

Process: $gg \to \bar{t}t$



Process: $d\bar{d} \to u\bar{u}g$



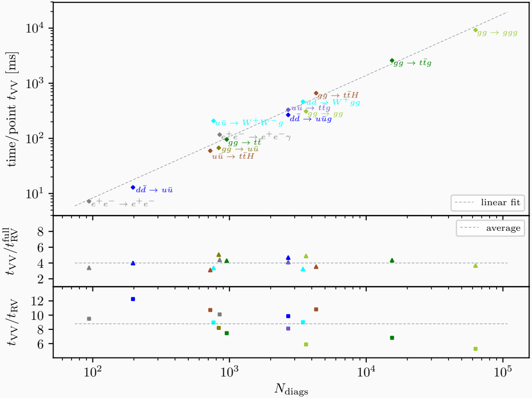
The plot shows the fraction of points with $\mathcal{A}_{\mathrm{DP}} > \mathcal{A}_{\mathrm{min}}$ for 10^5 uniform random points.

Excellent numerical stability. Essential for full calculation, tensor integrals will be main source of instabilities.

Efficiency: Irreducible Diagrams

Construction of tensor coefficients for QED, QCD and SM (NNLO QCD) processes

(single intel i7-6600U, 2.6 GHz, 16GB RAM, 1000 points)



- $2 \to 2$ process: 10-300ms/psp
- $2 \to 3$ process: 65-9200ms/psp

Runtime $\propto$ # diagrams

time/psp/diagram $\sim$ 150 μs

Constant ratios between NNLO double virtual (VV) and real-virtual (RV):

$$\frac{t_{VV}}{t_{RV}^{full}} \approx 4 \pm 1 \quad \text{(full RV)}$$

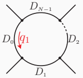
$$\frac{t_{VV}}{t_{RV}} \approx 9 \pm 3 \quad \text{(tensor coefficients)}$$

Strong CPU performance, comparable to real-virtual corrections in OpenLoops.

One-loop rational terms

Amputated one-loop diagram γ:[1]

$$\bar{\mathcal{M}}_{1,\gamma} = C_{1,\gamma}\int \mathrm{d}\bar{q}_1 \frac{\bar{\mathcal{N}}(q_1)}{\mathcal{D}(\bar{q}_1)} = C_{1,\gamma}\int \mathrm{d}\bar{q}_1 \frac{\overbrace{\mathcal{N}(q_1)}^{\text{4-dim}} + \overbrace{\tilde{\mathcal{N}}(\bar{q}_1)}^{\text{(D-4)-dim}}}{\mathcal{D}(\bar{q}_1)} =$$



$$\Rightarrow \delta\mathcal{R}_{1,\gamma} = C_{1,\gamma}\int \mathrm{d}\bar{q}_1 \frac{\tilde{\mathcal{N}}(\bar{q}_1)}{\mathcal{D}(\bar{q}_1)}$$

The ε-dim numerator parts $\tilde{\mathcal{N}}(\bar{q}_1) = \bar{\mathcal{N}}(\bar{q}_1) - \mathcal{N}(q_1)$ contribute only via interaction with $\frac{1}{\varepsilon}$ UV poles

$\Rightarrow$ Can be restored through **rational counterterm** $\delta\mathcal{R}_{1,\gamma}$ [Ossola, Papadopoulos, Pittau]

$$\underbrace{\mathbf{R}\,\bar{\mathcal{M}}_{1,\gamma}}_{D-\text{dim, renormalised}} = \underbrace{\mathcal{M}_{1,\gamma}}_{4-\text{dim numerator}} + \underbrace{\delta Z_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}}_{\text{UV and rational counterterm}}$$

Finite set of process-independent rational terms in renormalisable models.

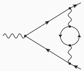
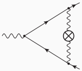
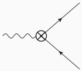
[1] Bar denotes quantities in D dimensions.

Two-loop rational terms

Renormalised D-dim amplitudes from amplitudes with 4-dim numerator [Pozzorini, Zhang, Zoller]

$$\mathbf{R}\,\bar{\mathcal{M}}_{2,\Gamma} = \mathcal{M}_{2,\Gamma} + \sum_{\gamma}\Big(\underbrace{\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma}}_{\text{subtract subdivergences}} + \underbrace{\delta\mathcal{R}_{1,\gamma}}_{\text{restore }\tilde{\mathcal{N}}\text{-terms from subdiagrams}}\Big)\cdot\mathcal{M}_{1,\Gamma/\gamma} + \Big(\underbrace{\delta Z_{2,\Gamma}}_{\text{subtract remaining local divergence}} + \underbrace{\delta\mathcal{R}_{2,\Gamma}}_{\text{restore remaining }\tilde{\mathcal{N}}\text{-term}}\Big)$$

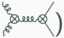
Example:

$\mathbf{R}\,\bar{\mathcal{M}}_{2,\Gamma} = \Big[$  $+$  $\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right) +$  $\left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)\Big]_{\text{4-dim numerators}}$

- Divergences from subdiagrams γ and remaining local one subtracted by usual UV counterterms $\delta Z_{1,\gamma}, \delta Z_{2,\Gamma}$.
- Additional UV counterterm $\delta\tilde{Z}_{1,\gamma} \propto \frac{\tilde{q}_1^{\,2}}{\varepsilon}$ for subdiagrams with mass dimension 2.
- $\delta\mathcal{R}_{2,\Gamma}$ is a **two-loop rational term** stemming from the interplay of $\tilde{\mathcal{N}}$ with UV poles.
- **Finite set of process-independent rational terms of UV origin.**
- **Available for QED and QCD corrections to the SM.** [Lang, Pozzorini, Zhang, Zoller,2021]
- Rational terms of IR origin currently under investigation.

Implementation of Renormalization, Rational Terms at NNLO

Status:

- Implementation of new tree (e.g. ) and one-loop (e.g. ) universal Feynman rules, **complete**
- Validation of new 1l tensor structures using pseudotree-test, **complete**
- **Ongoing**: **Validation of implementation of two-loop rational terms** by pole-cancellation check, computation of **first full amplitudes for simple processes** $\rightarrow$ **require tensor integrals**

Currently working on twored, small in-house tensor integral library for 2 and 3 point topologies with off-shell external legs and massless propagators.

Approach:

- Covariant decomposition: express tensor integrals in terms of scalar integrals and their coefficients.
- Reduce scalar integrals to master integrals using FIRE[Smirnov, Chukharev].
- Implement analytic master integrals from literature in twored.

Summary

New algorithm for two loop tensor coefficients:

- **Fully general algorithm**
- **Excellent numerical stability**
- **Highly efficient, comparable to real virtual contribution**
  - Exploit factorization for ideal order of building blocks.
  - Efficient treatment of helicities and ranks in loop momenta.
- **Fully implemented for NNLO QED and QCD Corrections to SM**

Current and future projects

- Implementation of two-loop UV and rational counterterms
- Tensor integrals (in-house framework and or external tool or mixture thereof)

Backup

On-The-Fly Helicity Summation at NLO

Final result: $\mathcal{W}_{01} = \sum_h \sum_{\mathrm{col}} 2\,\mathrm{Re}\left[\bar{\mathcal{M}}_1(h)\,\bar{\mathcal{M}}_0^*(h)\right]$

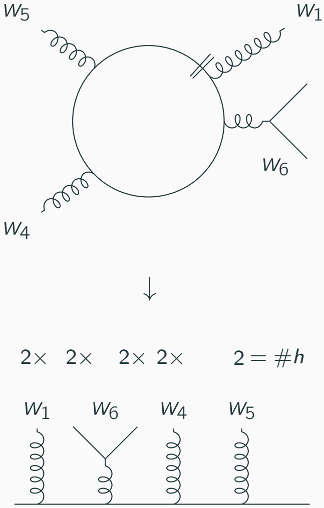
Instead of $\mathcal{N}(q, h) = \prod_a S_a(q, h)$, construct $\mathcal{U}(q) = \sum_h \left[2 \sum_{col} C\mathcal{M}_0^*(h)\right] \mathcal{N}(q, h)$

Perform **on-the-fly helicity summation** [Buccioni, Pozzorini, Zoller], for each diagram:

- Use Born-color interfernce $\mathcal{U}_0 = 2\sum_{col} C\mathcal{M}_0^*(h)$ as initial condition, begin the recursion with maximal helicities.
- **Exploit factorization to sum helicities in each recursion step:**
$$\sum_h \mathcal{U}_0(h)\,\mathcal{N}(q, h) = \sum_{h_N}\left[\cdots \sum_{h_2}\left[\sum_{h_1} \mathcal{U}_0(h_1, h_2, \ldots) S_1(h_1)\right] S_2(h_2) \cdots\right] S_N(h_N)$$
- (in renormalizable theories) each segment:
  - increases rank by 1 (or 0)
  - decreases total helicities by a factor of # helicities of subtree in the segment

Minimal helicities with maximal rank, complexity is kept low in final recursion steps.

On-The-Fly Helicity Summation: Example

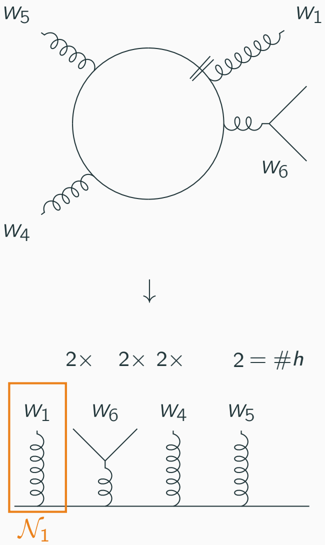


In each recursion step:

- increase rank by 1
- decrease total helicities by a factor of # helicities of wavefunction in the segment

helicities=32,
rank=0

On-The-Fly Helicity Summation: Example

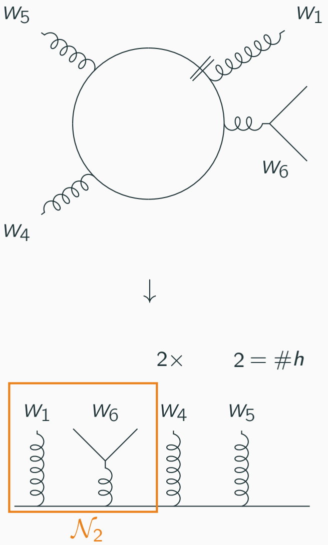


In each recursion step:

- increase rank by 1
- decrease total helicities by a factor of # helicities of wavefunction in the segment

helicities=16,
rank=1

On-The-Fly Helicity Summation: Example

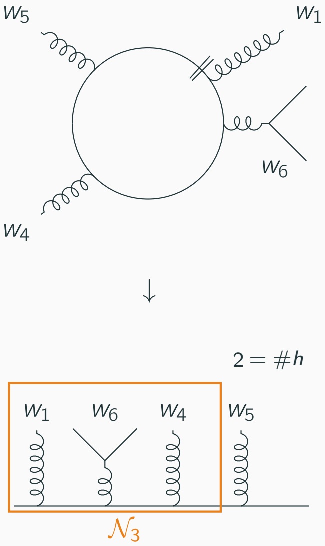


In each recursion step:

- increase rank by 1
- decrease total helicities by a factor of # helicities of wavefunction in the segment

helicities=4,
rank=2

On-The-Fly Helicity Summation: Example

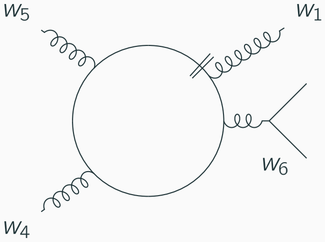


In each recursion step:

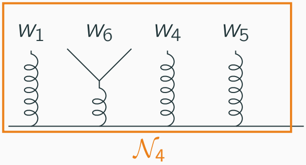
- increase rank by 1
- decrease total helicities by a factor of # helicities of wavefunction in the segment

helicities=2,
rank=3

On-The-Fly Helicity Summation: Example



$\downarrow$

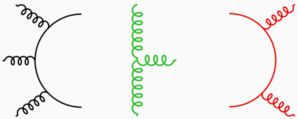


In each recursion step:

- increase rank by 1
- decrease total helicities by a factor of # helicities of wavefunction in the segment

helicities=1,
rank=4

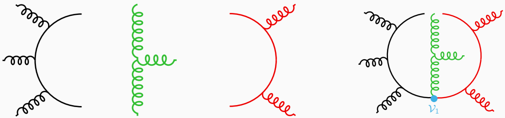
Two Loop Algorithm: Naive Approach



1. construct chains $\mathcal{N}^{(1)}(q_1)$, $\mathcal{N}^{(2)}(q_2)$, $\mathcal{N}^{(3)}(q_3)$ using one-loop algorithm.

$$\left[\mathcal{N}^{(1)}(q_1)\right]_{\beta_0^{(1)}}^{\beta_{N_1}^{(1)}} \left[\mathcal{N}^{(2)}(q_2)\right]_{\beta_0^{(2)}}^{\beta_{N_2}^{(2)}} \left[\mathcal{N}^{(3)}(q_3)\right]_{\beta_0^{(3)}}^{\beta_{N_3}^{(3)}}$$

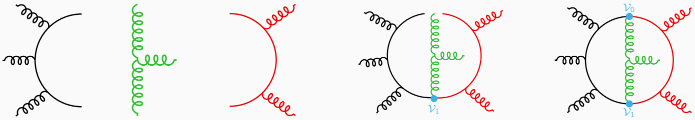
Two Loop Algorithm: Naive Approach



1. construct chains $\mathcal{N}^{(1)}(q_1)$, $\mathcal{N}^{(2)}(q_2)$, $\mathcal{N}^{(3)}(q_3)$ using one-loop algorithm.
2. combine with vertex $\mathcal{V}_1$, closing indices $\beta^{(1)}_{N_1}, \beta^{(2)}_{N_2}, \beta^{(3)}_{N_3}$

$$\left[\mathcal{N}^{(1)}(q_1)\right]_{\beta^{(1)}_0}^{\beta^{(1)}_{N_1}} \left[\mathcal{N}^{(2)}(q_2)\right]_{\beta^{(2)}_0}^{\beta^{(2)}_{N_2}} \left[\mathcal{N}^{(3)}(q_3)\right]_{\beta^{(3)}_0}^{\beta^{(3)}_{N_3}} \left[\mathcal{V}_1(q1,\, q2)\right]_{\beta^{(1)}_{N_1}\beta^{(2)}_{N_2}\beta^{(3)}_{N_3}}$$

Two Loop Algorithm: Naive Approach



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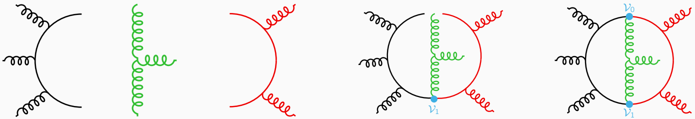
2. combine with vertex $\mathcal{V}_1$, closing indices $\beta^{(1)}_{N_1}, \beta^{(2)}_{N_2}, \beta^{(3)}_{N_3}$
3. combine with vertex $\mathcal{V}_0$, closing indices $\beta^{(1)}_{0}, \beta^{(2)}_{0}, \beta^{(3)}_{0}$

$$\left[\mathcal{N}^{(1)}(q_1)\right]_{\beta_0^{(1)}}^{\beta_{N_1}^{(1)}} \left[\mathcal{N}^{(2)}(q_2)\right]_{\beta_0^{(2)}}^{\beta_{N_2}^{(2)}} \left[\mathcal{N}^{(3)}(q_3)\right]_{\beta_0^{(3)}}^{\beta_{N_3}^{(3)}} \left[\mathcal{V}_1(q1,\, q2)\right]_{\beta_{N_1}^{(1)}\beta_{N_2}^{(2)}\beta_{N_3}^{(3)}} \left[\mathcal{V}_0(q1,\, q2)\right]^{\beta_0^{(1)}\beta_0^{(2)}\beta_0^{(3)}}$$

Two Loop Algorithm: Naive Approach



1. construct chains $\mathcal{N}^{(1)}(q_1)$, $\mathcal{N}^{(2)}(q_2)$, $\mathcal{N}^{(3)}(q_3)$ using one-loop algorithm.
2. combine with vertex $\mathcal{V}_1$, closing indices $\beta^{(1)}_{N_1}, \beta^{(2)}_{N_2}, \beta^{(3)}_{N_3}$
3. combine with vertex $\mathcal{V}_0$, closing indices $\beta^{(1)}_{0}, \beta^{(2)}_{0}, \beta^{(3)}_{0}$
4. multiply Born-color interference, sum over helicities, map momenta

$$\sum_h \mathcal{U}_0(h) \left[\mathcal{N}^{(1)}(q_1, h)\right] \left[\mathcal{N}^{(2)}(q_2, h)\right] \left[\mathcal{N}^{(3)}(q_3, h)\right] \left[\mathcal{V}_1(q1, q2, h)\right] \left[\mathcal{V}_0(q1, q2, h)\right] \Big|_{q_3 \to -(q_1+q_2)}$$

Two Loop Algorithm: Observations and Challenges

$$\sum_h \mathcal{U}_0(h) \left[\mathcal{N}^{(1)}(q_1\,,\,h)\right] \left[\mathcal{N}^{(2)}(q_2\,,\,h)\right] \left[\mathcal{N}^{(3)}(q_3\,,\,h)\right] \left[\mathcal{V}_1(q1,\,q2,\,h)\right] \left[\mathcal{V}_0(q1,\,q2,\,h)\right] \Big|_{q_3 \to -(q_1+q_2)}$$

1. construct chains $\mathcal{N}^{(1)}(q_1)$, $\mathcal{N}^{(2)}(q_2)$, $\mathcal{N}^{(3)}(q_3)$ using one-loop algorithm
2. combine with vertex $\mathcal{V}_1$, closing indices $\beta^{(1)}_{N_1}$,$\beta^{(2)}_{N_2}$,$\beta^{(3)}_{N_3}$
3. combine with vertex $\mathcal{V}_0$, closing indices $\beta^{(1)}_{0}$,$\beta^{(2)}_{0}$,$\beta^{(3)}_{0}$
4. sum over helicities, map momenta, multiply Born-color interference

Observations:

- complexitiy of each step depends on ranks in q_1, q_2 and helicities
- step 2, 3 are performed for 6, 3 open spinor/Lorentz indices
- step 2, 3 are performed at maximal ranks
- all steps are performed for all helicities

Very inefficient: most expensive steps performed for maximal number of components and helicities.

Helicity Bookkeeping

For a set of particles $\mathcal{E} = \{1, 2, \ldots, N\}$ the helicity configurations are identified as:

$$\lambda_p = \begin{cases} 1, 3 & \text{for fermions with helicity } s = -1/2, 1/2 \\ 1, 2, 3 & \text{for gauge bosons with } s = -1, 0, 1 \\ 0 & \text{for scalars with } s = 0 \text{ or unpolarized particles} \end{cases} \qquad \forall\, p \in \mathcal{E}$$

Each particle is assigned a base 4 helicity label

$$\bar{h}_p = \lambda_p \, 4^{p-1},$$

which can be used to define a similar numbering scheme for a set of particles:

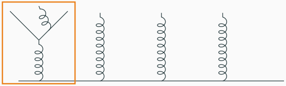
$\mathcal{E}_a = \{p_{a_1}, \ldots, p_{a_n}\}$ has the helicity label,

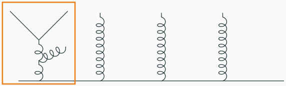
$$h_a = \sum_{p \in \mathcal{E}_a} \bar{h}_p.$$

Merging

Example:

- After one dressing step subsequent dressing steps are identical.
- Topology (scalar propagators) is identical for both diagrams.
- Diagrams can be merged.





For diagrams A,B with identical segments after n dressing steps (exploit factorization):

$$\mathcal{U}_{A,B} = \mathcal{U}_0 \, Tr(\mathcal{N}_{A,B}) = \text{numerator} \cdot \text{Born} \cdot \text{color}$$

$$\begin{aligned} \mathcal{U}_A + \mathcal{U}_B &= (\mathcal{U}_{n,A} \cdot S_{n+1} \cdots S_N) + (\mathcal{U}_{n,B} \cdot S_{n+1} \cdots S_N) \\ &= (\mathcal{U}_{n,A} + \mathcal{U}_{n,B}) \cdot S_{n+1} \cdots S_N \end{aligned}$$

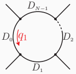
Only perform dressing steps n+1 to N once.

Highly efficient way of dressing a large number of diagrams for complicated processes.

One-loop rational terms

Amputated one-loop diagram γ (1PI)

$$\bar{\mathcal{M}}_{1,\gamma} = \underbrace{C_{1,\gamma}}_{\text{color factor}} \int \mathrm{d}\bar{q}_1 \frac{\mathcal{N}(q_1) + \tilde{\mathcal{N}}(\bar{q}_1)}{\mathcal{D}(\bar{q}_1)} =$$



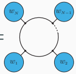
$$\Rightarrow \delta\mathcal{R}_{1,\gamma} = C_{1,\gamma} \int \mathrm{d}\bar{q}_1 \frac{\tilde{\mathcal{N}}(\bar{q}_1)}{\mathcal{D}(\bar{q}_1)}$$

The ε-dim numerator parts $\tilde{\mathcal{N}}(\bar{q}_1) = \bar{\mathcal{N}}(\bar{q}_1) - \mathcal{N}(q_1)$ contribute only via interaction with $\frac{1}{\varepsilon}$ UV poles $\Rightarrow$ Can be restored through **rational counterterm** $\delta\mathcal{R}_{1,\gamma}$ [Ossola, Papadopoulos, Pittau]

$$\Rightarrow \quad \boxed{\underbrace{\mathbf{R}\,\bar{\mathcal{M}}_{1,\gamma}}_{D-\text{dim, renormalised}} = \underbrace{\mathcal{M}_{1,\gamma}}_{4-\text{dim numerator}} + \underbrace{\delta Z_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}}_{\text{UV and rational counterterm}}}$$

Generic one-loop diagram Γ factorises into 1PI subdiagram γ and external subtrees w_i (4-dim):

$$\bar{\mathcal{M}}_{1,\Gamma} =$$



$$= \left[\bar{\mathcal{M}}_{1,\gamma}\right]^{\sigma_1\ldots\sigma_N} \prod_{i=1}^{N} [w_i]_{\sigma_i} \Rightarrow \boxed{\mathbf{R}\,\bar{\mathcal{M}}_{1,\Gamma} = \mathcal{M}_{1,\Gamma} + \left(\delta Z_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right) \underbrace{\prod_{i=1}^{N} w_i}_{\text{tree diagram}}}$$

Finite set of process-independent rational terms in renormalisable models
computed from UV divergent vertex functions

Status of two-loop rational terms

Renormalised D-dim amplitudes can be computed from amplitudes with 4-dim numerators and a **finite set of universal UV and rational counterterms** inserted lower-loop amplitudes

$$\mathbf{R}\,\bar{\mathcal{M}}_{2,\Gamma} \quad = \quad \mathcal{M}_{2,\Gamma} + \sum_{\gamma} \left(\delta Z_{1,\gamma} + \delta \tilde{Z}_{1,\gamma} + \delta \mathcal{R}_{1,\gamma} \right) \cdot \mathcal{M}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma} + \delta \mathcal{R}_{2,\Gamma} \right)$$

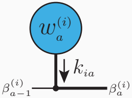
Status of two-loop rational terms

- **General method for the computation** of rational counterterms of UV origin from simple tadpole integrals in any renormalisable model [Pozzorini, Zhang, Zoller,2020]
- **Complete renormalisation scheme dependence** [Lang, Pozzorini, Zhang, Zoller,2020]
- **Rational Terms for Spontaneously Broken Theories** [Lang, Pozzorini, Zhang, Zoller,2021]
- **Full set of two-loop rational terms** computed for
  - QED with full dependence on the gauge parameter [Pozzorini, Zhang, Zoller,2020]
  - $SU(N)$ and $U(1)$ in any renormalisation scheme [Lang, Pozzorini, Zhang, Zoller,2020]
  - **QED and QCD corrections to the full SM** [Lang, Pozzorini, Zhang, Zoller,2021]
- Rational terms of IR origin currently under investigation

Explicit dressing steps

Triple vertex loop segment:

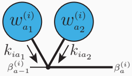
$$\left[S_a^{(i)}(q_i, h_a^{(i)})\right]_{\beta_{a-1}^{(i)}}^{\beta_a^{(i)}} =$$



$$= \left\{ \left[Y_{ia}^{\sigma}\right]_{\beta_{a-1}^{(i)}}^{\beta_a^{(i)}} + \left[Z_{ia,\nu}^{\sigma}\right]_{\beta_{a-1}^{(i)}}^{\beta_a^{(i)}} q_i^{\nu} \right\} w_{a\sigma}^{(i)}(k_{ia}, h_a^{(i)})$$

Quartic vertex segments:

$$\left[S_a^{(i)}(q_i, h_a^{(i)})\right]_{\beta_{a-1}^{(i)}}^{\beta_a^{(i)}} =$$



$$= \left[Y_{ia}^{\sigma_1\sigma_2}\right]_{\beta_{a-1}^{(i)}}^{\beta_a^{(i)}} w_{a_1\sigma_1}^{(i)}(k_{ia_1}, h_{a_1}^{(i)})\, w_{a_2\sigma_2}^{(i)}(k_{ia_2}, h_{a_2}^{(i)})$$

with $h_a^{(i)} = h_{a_1}^{(i)} + h_{a_2}^{(i)}$ and $k_{ia} = k_{ia_1} + k_{ia_2}$.

Dressing step for a segment with a triple vertex:

$$\left[\mathcal{N}_{n;\,\mu_1\ldots\mu_r}^{(1)}(\hat{h}_n^{(1)})\right]_{\beta_0^{(1)}}^{\beta_n^{(1)}} = \left\{ \left[\mathcal{N}_{n-1;\,\mu_1\ldots\mu_r}^{(1)}(\hat{h}_{n-1}^{(1)})\right]_{\beta_0^{(1)}}^{\beta_{n-1}^{(1)}} \left[Y_{1n}^{\sigma}\right]_{\beta_{n-1}^{(1)}}^{\beta_n^{(1)}} \right.$$

$$\left. + \left[\mathcal{N}_{n-1;\,\mu_2\ldots\mu_r}^{(1)}(\hat{h}_{n-1}^{(1)})\right]_{\beta_0^{(1)}}^{\beta_{n-1}^{(1)}} \left[Z_{1n,\mu_1}^{\sigma}\right]_{\beta_{n-1}^{(1)}}^{\beta_n^{(1)}} \right\} w_{n\sigma}^{(1)}(k_n, h_n^{(1)}).$$

Processes considered in performance tests

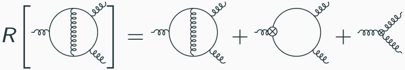
corrections	process type	massless fermions	massive fermions	process
QED	$2 \to 2$	e	–	$e^+e^- \to e^+e^-$
	$2 \to 3$	e	–	$e^+e^- \to e^+e^-\gamma$
QCD	$2 \to 2$	u	–	$gg \to u\bar{u}$
		u, d	–	$d\bar{d} \to u\bar{u}$
		u	–	$gg \to gg$
		u	t	$u\bar{u} \to t\bar{t}g$
		u	t	$gg \to t\bar{t}$
		u	t	$gg \to t\bar{t}g$
	$2 \to 3$	u, d	–	$d\bar{d} \to u\bar{u}g$
		u	–	$gg \to ggg$
		u, d	–	$u\bar{d} \to W^+gg$
		u, d	–	$u\bar{u} \to W^+W^-g$
		u	t	$u\bar{u} \to t\bar{t}H$
		u	t	$gg \to t\bar{t}H$

Memory usage of the two-loop algorithm

	virtual–virtual memory [MB]		real–virtual [MB]	
hard process	segment-by-segment	diagram-by-diagram	coefficients	full
$e^+e^- \to e^+e^-$	18	8	6	23
$e^+e^- \to e^+e^-\gamma$	154	25	22	54
$gg \to u\bar{u}$	75	31	10	26
$gg \to t\bar{t}$	94	35	15	34
$gg \to t\bar{t}g$	2000	441	152	213
$u\bar{d} \to W^+gg$	563	143	54	90
$u\bar{u} \to W^+W^-g$	264	67	36	67
$u\bar{u} \to t\bar{t}H$	82	28	14	40
$gg \to t\bar{t}H$	604	145	50	90
$u\bar{u} \to t\bar{t}g$	323	83	41	74
$gg \to gg$	271	94	41	55
$d\bar{d} \to u\bar{u}$	18	10	9	20
$d\bar{d} \to u\bar{u}g$	288	85	39	68
$gg \to ggg$	6299	1597	623	683

Pole Cancellation Check

Renormalized two-loop diagram Γ (assuming off-shell external legs):



(from `arxiv:2007.03713v2`)

$$\mathbf{R}\bar{\mathcal{M}}_{2,\Gamma} = \mathcal{M}_{2,\Gamma} + \sum_{\gamma}\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta R_{1,\gamma}\right)\mathcal{M}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma} + \delta R_{2,\Gamma}\right)$$

Pole Cancellation Check

Renormalized two-loop diagram Γ (assuming off-shell external legs):
$\mathbf{R}\bar{\mathcal{M}}_{2,\Gamma} = \mathcal{M}_{2,\Gamma} + \sum_\gamma \left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta R_{1,\gamma}\right)\mathcal{M}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma} + \delta R_{2,\Gamma}\right)$

In terms of ϵ:

$$\begin{aligned}
\mathcal{M}_{2,\Gamma} &= \frac{1}{\epsilon^2} M_{2,\Gamma}^{(2)} + \frac{1}{\epsilon} M_{2,\Gamma}^{(1)} + M_{2,\Gamma}^{(0)} + \epsilon M_{2,\Gamma}^{(-1)} + \mathcal{O}(\epsilon) \\
\mathcal{M}_{1,\Gamma/\gamma} &= \frac{1}{\epsilon} M_{1,\Gamma/\gamma}^{(1)} + M_{1,\Gamma/\gamma}^{(0)} + \epsilon M_{1,\Gamma/\gamma}^{(-1)} + \mathcal{O}(\epsilon^2) \\
\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta R_{1,\gamma}\right) &= \frac{1}{\epsilon} Z_{1,\gamma}^{(1)} + Z_{1,\gamma}^{(0)} \\
\left(\delta Z_{2,\Gamma} + \delta R_{2,\Gamma}\right) &= \frac{1}{\epsilon^2} Z_{2,\Gamma}^{(2)} + \frac{1}{\epsilon} Z_{2,\Gamma}^{(1)} + Z_{2,\Gamma}^{(0)}
\end{aligned}$$

Pole Cancellation Check

Renormalized two-loop diagram Γ (assuming off-shell external legs):
$\mathbf{R}\bar{\mathcal{M}}_{2,\Gamma} = \mathcal{M}_{2,\Gamma} + \sum_\gamma \left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta R_{1,\gamma}\right)\mathcal{M}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma} + \delta R_{2,\Gamma}\right)$

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\mathcal{M}_{1,\Gamma/\gamma} &= \frac{1}{\epsilon}M_{1,\Gamma/\gamma}^{(1)} + M_{1,\Gamma/\gamma}^{(0)} + \epsilon M_{1,\Gamma/\gamma}^{(-1)} + \mathcal{O}(\epsilon^2) \\
\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta R_{1,\gamma}\right) &= \frac{1}{\epsilon}Z_{1,\gamma}^{(1)} + Z_{1,\gamma}^{(0)} \\
\left(\delta Z_{2,\Gamma} + \delta R_{2,\Gamma}\right) &= \frac{1}{\epsilon^2}Z_{2,\Gamma}^{(2)} + \frac{1}{\epsilon}Z_{2,\Gamma}^{(1)} + Z_{2,\Gamma}^{(0)}
\end{aligned}$$

then poles should cancel:

- $\frac{1}{\epsilon}Z_{2,\Gamma}^{(1)} + \frac{1}{\epsilon}\sum_\gamma\left(Z_{1,\gamma}^{(1)}M_{1,\Gamma/\gamma}^{(0)} + Z_{1,\gamma}^{(0)}M_{1,\Gamma/\gamma}^{(1)}\right) + \frac{1}{\epsilon}M_{2,\Gamma}^{(1)}$

Pole Cancellation Check

Renormalized two-loop diagram Γ (assuming off-shell external legs):
$\mathbf{R}\bar{\mathcal{M}}_{2,\Gamma} = \mathcal{M}_{2,\Gamma} + \sum_{\gamma} \left(\delta Z_{1,\gamma} + \delta \tilde{Z}_{1,\gamma} + \delta R_{1,\gamma}\right) \mathcal{M}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma} + \delta R_{2,\Gamma}\right)$

In terms of ϵ:

$$\mathcal{M}_{2,\Gamma} = \frac{1}{\epsilon^2} M_{2,\Gamma}^{(2)} + \frac{1}{\epsilon} M_{2,\Gamma}^{(1)} + M_{2,\Gamma}^{(0)} + \epsilon M_{2,\Gamma}^{(-1)} + \mathcal{O}(\epsilon)$$

$$\mathcal{M}_{1,\Gamma/\gamma} = \frac{1}{\epsilon} M_{1,\Gamma/\gamma}^{(1)} + M_{1,\Gamma/\gamma}^{(0)} + \epsilon M_{1,\Gamma/\gamma}^{(-1)} + \mathcal{O}(\epsilon^2)$$

$$\left(\delta Z_{1,\gamma} + \delta \tilde{Z}_{1,\gamma} + \delta R_{1,\gamma}\right) = \frac{1}{\epsilon} Z_{1,\gamma}^{(1)} + Z_{1,\gamma}^{(0)}$$

$$\left(\delta Z_{2,\Gamma} + \delta R_{2,\Gamma}\right) = \frac{1}{\epsilon^2} Z_{2,\Gamma}^{(2)} + \frac{1}{\epsilon} Z_{2,\Gamma}^{(1)} + Z_{2,\Gamma}^{(0)}$$

then poles should cancel:

- $\frac{1}{\epsilon} Z_{2,\Gamma}^{(1)} + \frac{1}{\epsilon} \sum_{\gamma} \left(Z_{1,\gamma}^{(1)} M_{1,\Gamma/\gamma}^{(0)} + Z_{1,\gamma}^{(0)} M_{1,\Gamma/\gamma}^{(1)} \right) + \frac{1}{\epsilon} M_{2,\Gamma}^{(1)}$
- $\frac{1}{\epsilon^2} M_{2,\Gamma}^{(2)} + \frac{1}{\epsilon^2} \sum_{\gamma} Z_{1,\gamma}^{(1)} M_{1,\Gamma/\gamma}^{(1)} + \frac{1}{\epsilon^2} Z_{2,\Gamma}^{(2)}$

Pole Cancellation Check

Renormalized two-loop diagram Γ (assuming off-shell external legs):
$\mathbf{R}\bar{\mathcal{M}}_{2,\Gamma} = \mathcal{M}_{2,\Gamma} + \sum_\gamma \left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta R_{1,\gamma}\right)\mathcal{M}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma} + \delta R_{2,\Gamma}\right)$

In terms of ϵ:

$$\mathcal{M}_{2,\Gamma} = \frac{1}{\epsilon^2}M^{(2)}_{2,\Gamma} + \frac{1}{\epsilon}M^{(1)}_{2,\Gamma} + M^{(0)}_{2,\Gamma} + \epsilon M^{(-1)}_{2,\Gamma} + \mathcal{O}(\epsilon)$$

$$\mathcal{M}_{1,\Gamma/\gamma} = \frac{1}{\epsilon}M^{(1)}_{1,\Gamma/\gamma} + M^{(0)}_{1,\Gamma/\gamma} + \epsilon M^{(-1)}_{1,\Gamma/\gamma} + \mathcal{O}(\epsilon^2)$$

$$\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta R_{1,\gamma}\right) = \frac{1}{\epsilon}Z^{(1)}_{1,\gamma} + Z^{(0)}_{1,\gamma}$$

$$\left(\delta Z_{2,\Gamma} + \delta R_{2,\Gamma}\right) = \frac{1}{\epsilon^2}Z^{(2)}_{2,\Gamma} + \frac{1}{\epsilon}Z^{(1)}_{2,\Gamma} + Z^{(0)}_{2,\Gamma}$$

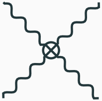
then poles should cancel:

- $\frac{1}{\epsilon}Z^{(1)}_{2,\Gamma} + \frac{1}{\epsilon}\sum_\gamma\left(Z^{(1)}_{1,\gamma}M^{(0)}_{1,\Gamma/\gamma} + Z^{(0)}_{1,\gamma}M^{(1)}_{1,\Gamma/\gamma}\right) + \frac{1}{\epsilon}M^{(1)}_{2,\Gamma}$
- $\frac{1}{\epsilon^2}M^{(2)}_{2,\Gamma} + \frac{1}{\epsilon^2}\sum_\gamma Z^{(1)}_{1,\gamma}M^{(1)}_{1,\Gamma/\gamma} + \frac{1}{\epsilon^2}Z^{(2)}_{2,\Gamma}$

This would validate $\delta R_{2,\Gamma}$ (contains $\frac{1}{\epsilon}$ pole) as well as implementation of $\delta\tilde{Z}_{1,\gamma}$, $\delta Z_{2,\Gamma}$

Implementation of Renormalization, Rational Terms

Example (from `arXiv:2001.11388v3`) :



$$\sim ie^2 \underbrace{\left(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}\right)}_{\text{tensor structure}} \sum_{k=1}^{2} \left(\frac{\alpha}{4\pi}\right)^k \underbrace{\delta R^{(s)}_{k,4\gamma}}_{\substack{\text{rational} \\ \text{counterterms}}}$$

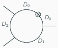
where k=1,2 is the loop order.

For NNLO need to implement:

- universal Feynman rules for new tensor structures
- new rational counterterms

Tensor Integrals

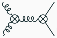
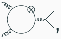
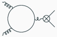
For NNLO need:

- 1l TI for
  - 1l diagrams with ct insertions: up to $\mathcal{O}(\epsilon)$, new topolgies due to squared propagator,
    e.g. 

    $= \int \mathrm{d}\bar{q}_1 \frac{q_1^{\mu_1} \cdots q_1^{\mu_r}}{\bar{D}_0 \bar{D}_0 \bar{D}_1 \bar{D}_2} = I^{\mu_1 \cdots \mu_r}$
  - VV reducible, V, RV, L2: exists
- 2l TI
  - VV irreducible:
    $$\int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \frac{q_1^{\mu_1} \cdots q_1^{\mu_r}\; q_2^{\nu_1} \cdots q_2^{\nu_s}}{\mathcal{D}^{(1)}(\bar{q}_1)\, \mathcal{D}^{(2)}(\bar{q}_2)\, \mathcal{D}^{(3)}(\bar{q}_3)} \Bigg|_{q_3 \to -(q_1+q_2)} = I^{\mu_1 \cdots \mu_r \nu_1 \cdots \nu_s}$$

Implementation of Renormalization, Rational Terms

for NNLO need the following UV rational/counterterms:

- 1l ct in 0l diagrams (ct and tensor structures exist)
  renormalization of:
  - 1l diagrams (V, RV, L2): , exists
  - reducible 2l diagrams (VV): , new
- 1l ct in 1l diagrams (ct exist, new tensor structures $\rightarrow$ implemented and tested with pseudotree test)
  renormalization of:
  - irreduclible 2l diagrams (VV): , new
  - reducible 2l diagrams (VV): , new
- 2l ct in 0l diagrams (new ct, tensor structures exists)
  renormalization of:
  - irreducible 2l diagrams (VV): , (new)

Tensor Integrals

Currently working on interfacing and extending twored:
an in-house tensor integral library for 2 and 3 point topologies (possibly extend to 4 point) with off-shell external legs and massless propagators.

Approach:
For a given topology with tensor integral $I^{\mu_1 \cdots \mu_r}$

- covariant decomposition: $I^{\mu_1 \cdots \mu_r} = T_i^{\mu_1 \cdots \mu_r} \cdot C_i$, generate all possible tensor structures $T^{\mu_1 \cdots \mu_r}$ from ext. momenta metric tensors
- express coefficients in terms of scalar integrals C_i using projectors $P_{\mu_1 \cdots \mu_r}$,
  $C_i = (P_{j,\mu_1 \cdots \mu_r} T_i^{\mu_1 \cdots \mu_r})^{-1} P_{j,\mu_1 \cdots \mu_r} I^{\mu_1 \cdots \mu_r}$
- reduce scalar integrals to master integrals G_k using FIRE
  $C_i = \alpha_{ik} G_k \Rightarrow I^{\mu_1 \cdots \mu_r} = T_i^{\mu_1 \cdots \mu_r} \cdot \alpha_{ik} \cdot G_k$