4th Workstop / Thinkstart

Towards N^3 LO for $\gamma^* \to \ell\bar{\ell}$

McMule approach @ N^{3}LO (pt. 1)

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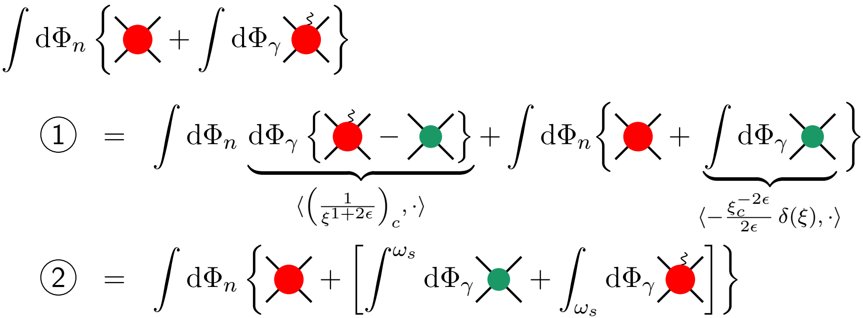
- at $\mathrm{N^3LO}$ the calculation is split into four parts

$$\begin{aligned}\sigma^{(3)} &= \sigma^{(\mathrm{VVV})} + \sigma^{(\mathrm{RVV})} + \sigma^{(\mathrm{RRV})} + \sigma^{(\mathrm{RRR})} \\ &= \int \mathrm{d}\Phi_n \, \mathcal{M}_n^{(3)} + \int \mathrm{d}\Phi_{n+1} \, \mathcal{M}_{n+1}^{(2)} + \int \mathrm{d}\Phi_{n+2} \, \mathcal{M}_{n+2}^{(1)} + \int \mathrm{d}\Phi_{n+3} \, \mathcal{M}_{n+3}^{(0)}\end{aligned}$$

- for each part identify gauge-invariant subsets based on lepton charges (q for electron, Q for muon)
- for now $q^8 \, Q^2$ (emission from e-line only) is enough . . .

(soft-only!) radiative phase space

$$\mathrm{d}\Phi_{n+r} \equiv \mathrm{d}\Phi_n \prod_{i=1}^{r} \mathrm{d}\Phi_{i,\gamma} = \mathrm{d}\Phi_n \prod_{i=1}^{r} \mathrm{d}\Phi_{i,\gamma}^{d=4-2\epsilon}\, \xi^2\, \xi_i^{-1-2\epsilon}\, \mathrm{d}\xi_i\, \mathrm{d}\Upsilon_i^{d=4-2\epsilon}$$



① FKS SUBTRACTION + DR

1. reproduce and isolate IR behaviour from regions of the phase space where (one or more) real photons are soft:

$$\lim_{\xi \to 0} \xi^2 \, \mathcal{M}_{n+1}^{(\ell)} = \mathcal{E} \, M_n^{(\ell)}$$

2. isolate IR-divergent behaviour from virtual amplitudes*:

$$\sum_{\ell=0}^{\infty} \mathcal{M}_n^{(\ell)} = e^{-\alpha \hat{\mathcal{E}}} \sum_{\ell=0}^{\infty} \mathcal{M}_n^{(\ell)\, f}$$

3. cancel analytically IR divergences and then integrate numerically in $d = 4$ over the non-radiative phase space

② SLICING + m_γ

1. choose the resolution parameter for the photon energy, ω_s, as slicing parameter:

   for $E_\gamma < \omega_s$ a real-emission contribution is *degenerate* with the one *without* the emitted photon

2. regulate IR divergences from virtual amplitudes with a non-zero photon mass, λ

3. integrate numerically the contributions regrouped according to rule 1; for soft regions use eikonal approximation*

RRR

$$\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3\, \xi_1^{-1-2\epsilon}\, \xi_2^{-1-2\epsilon}\, \xi_3^{-1-2\epsilon}\, \xi_1^2\, \xi_2^2\, \xi_3^2\, \mathcal{M}_{n+3}^{(0)}$$

- ① apply distributions to isolate IR behaviour once per photon

$$\begin{aligned}
\text{sss} &= 1 \times \frac{1}{3!}\, \hat{\mathcal{E}}^3(\xi_c)\, \mathcal{M}_n^{(0)} \\
\text{hss} &= 3 \times \frac{1}{3!}\, \hat{\mathcal{E}}^2(\xi_c)\, \mathrm{d}\xi_1 \left(\frac{1}{\xi_1}\right)_c \xi_1^2\, \mathcal{M}_{n+1}^{(0)} \\
\text{hhs} &= 3 \times \frac{1}{3!}\, \hat{\mathcal{E}}(\xi_c)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \left(\frac{1}{\xi_1}\right)_c \left(\frac{1}{\xi_2}\right)_c \xi_1^2\, \xi_2^2\, \mathcal{M}_{n+2}^{(0)} \\
\text{hhh} &= 1 \times \frac{1}{3!}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3 \left(\frac{1}{\xi_1}\right)_c \left(\frac{1}{\xi_2}\right)_c \left(\frac{1}{\xi_3}\right)_c \xi_1^2\, \xi_2^2\, \xi_3^2\, \mathcal{M}_{n+3}^{(0)}
\end{aligned}$$

- ② slice the phase space and keep the region where 3 photons are hard

$$\sigma_{3\gamma,\,\mathrm{h}}(\omega_s) = \sigma^{0\gamma\,\mathrm{virt}}_{0\gamma\,\mathrm{s};\,3\gamma\,\mathrm{h}}(\omega_s)$$

RRV

$$\mathrm{d}\xi_1 \mathrm{d}\xi_2\, \xi_1^{-1-2\epsilon}\, \xi_2^{-1-2\epsilon}\, \xi_1^2\, \xi_2^2\, \mathcal{M}_{n+2}^{(1)}$$

- ① use eikonal subtraction* when matrix elements are IR divergent

$$\begin{aligned}
\mathrm{ss} &= 1 \times \frac{1}{2!}\, \hat{\mathcal{E}}^2(\xi_c)\, \mathcal{M}_n^{(1)} \\
\mathrm{hs} &= 2 \times \frac{1}{2!}\, \hat{\mathcal{E}}(\xi_c)\, \mathrm{d}\xi_1 \left(\frac{1}{\xi_1}\right)_c \xi_1^2\, \mathcal{M}_{n+1}^{(1)} \\
\mathrm{hh} &= 1 \times \frac{1}{2!}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \left(\frac{1}{\xi_1}\right)_c \left(\frac{1}{\xi_2}\right)_c \xi_1^2\, \xi_2^2\, \mathcal{M}_{n+2}^{(1)} \\
&= \frac{1}{2!}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \left(\frac{1}{\xi_1}\right)_c \left(\frac{1}{\xi_2}\right)_c \xi_1^2\, \xi_2^2 \left(\mathcal{M}_{n+2}^{(1),f} - \hat{\mathcal{E}}(\xi_c)\mathcal{M}_{n+2}^{(0)}\right) \equiv \mathrm{hf} + \mathrm{hd}
\end{aligned}$$

- ② sum over degenerate states with 2 real photons

$$\sigma_{2\gamma,\,\mathrm{h}}(\omega_s) = \sigma_{0\gamma\,\mathrm{s};\,2\gamma\,\mathrm{h}}^{1\gamma\,\mathrm{virt}}(\lambda, \omega_s) + \sigma_{1\gamma\,\mathrm{s};\,2\gamma\,\mathrm{h}}^{0\gamma\,\mathrm{virt}}(\lambda, \omega_s)$$

RVV

$$\mathrm{d}\xi_1\, \xi_1^{-1-2\epsilon}\, \xi_1^2\, \mathcal{M}_{n+1}^{(2)}$$

- ① keep isolating pieces until vegas can integrate them

$$\begin{aligned}
\mathrm{s} &= 1 \times \frac{1}{1!}\,\hat{\mathcal{E}}(\xi_c)\,\mathcal{M}_n^{(2)} \\
\mathrm{h} &= 1 \times \frac{1}{1!}\,\mathrm{d}\xi_1 \left(\frac{1}{\xi_1}\right)_c \xi_1^2\, \mathcal{M}_{n+1}^{(2)} \\
&= \mathrm{d}\xi_1 \left(\frac{1}{\xi_1}\right)_c \xi_1^2 \left(\mathcal{M}_{n+1}^{(2),f} - \hat{\mathcal{E}}(\xi_c)\mathcal{M}_{n+1}^{(1)} - \frac{1}{2!}\hat{\mathcal{E}}^2(\xi_c)\mathcal{M}_{n+1}^{(0)}\right) \equiv \mathrm{f} + \mathrm{d1} + \mathrm{d2}
\end{aligned}$$

- ② sum over degenerate states with 1 real photon

$$\sigma_{1\gamma,\mathrm{h}}(\omega_s) = \sigma^{2\gamma\,\mathrm{virt}}_{0\gamma\,\mathrm{s};\,1\gamma\,\mathrm{h}}(\lambda,\omega_s) + \sigma^{1\gamma\,\mathrm{virt}}_{1\gamma\,\mathrm{s};\,1\gamma\,\mathrm{h}}(\lambda,\omega_s) + \sigma^{0\gamma\,\mathrm{virt}}_{2\gamma\,\mathrm{s};\,1\gamma\,\mathrm{h}}(\lambda,\omega_s)$$

$$\mathcal{M}_n^{(3)}$$

- ① subtract soft pieces from before (*eikonal subtraction* builds up!)

$$\sigma_n^{(3)}(\xi_c) = \int \mathrm{d}\Phi_n^{d=4}\left(\mathcal{M}_n^{(3)} - \sum_{r=1}^{3} \frac{\hat{\mathcal{E}}^r(\xi_c)}{r!}\,\mathcal{M}_n^{(3-r)}\right) = \int \mathrm{d}\Phi_n^{d=4}\mathcal{M}_n^{(3),f}$$

- ② sum over degenerate states with 0 real photons

$$\sigma_{0\gamma,\,\mathrm{h}}(\omega_s) = \sigma^{3\gamma\,\mathrm{virt}}_{0\gamma\,\mathrm{s};\,0\gamma\,\mathrm{h}}(\lambda) + \sigma^{2\gamma\,\mathrm{virt}}_{1\gamma\,\mathrm{s};\,0\gamma\,\mathrm{h}}(\lambda,\omega_s) + \sigma^{1\gamma\,\mathrm{virt}}_{2\gamma\,\mathrm{s};\,0\gamma\,\mathrm{h}}(\lambda,\omega_s) + \sigma^{0\gamma\,\mathrm{virt}}_{3\gamma\,\mathrm{s};\,0\gamma\,\mathrm{h}}(\lambda,\omega_s)$$

summing up

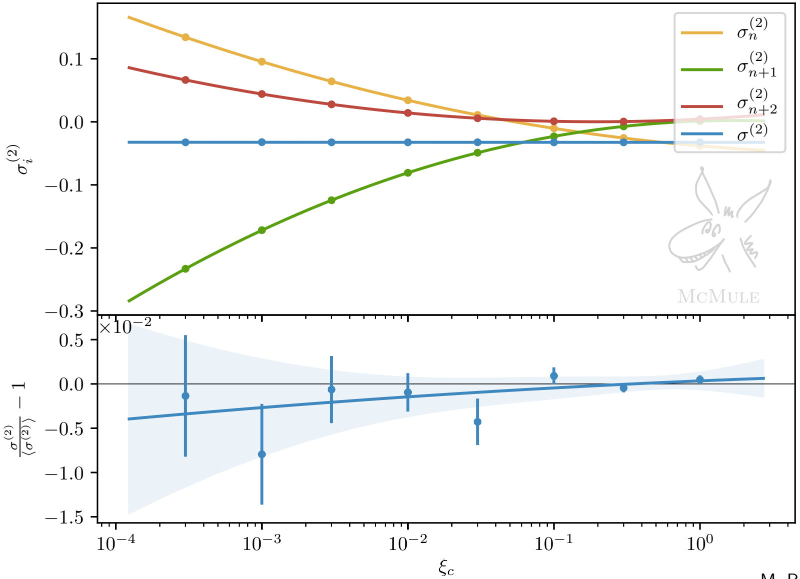
- ① [Engel, Signer, Ulrich, 1909.10244]

$$
\begin{aligned}
\sigma^{(3)} &= \sigma_n^{(3)}(\xi_c) + \sigma_{n+1}^{(3)}(\xi_c) + \sigma_{n+2}^{(3)}(\xi_c) + \sigma_{n+3}^{(3)}(\xi_c) \\
\sigma_n^{(3)}(\xi_c) &= \int \mathrm{d}\Phi_n^{d=4} \mathcal{M}_n^{(3),f} \\
\sigma_{n+1}^{(3)}(\xi_c) &= \int \mathrm{d}\Phi_{n+1}^{d=4} \frac{1}{1!} \left(\frac{1}{\xi_1}\right)_c \xi_1\, \mathcal{M}_{n+1}^{(2),f} \\
\sigma_{n+2}^{(3)}(\xi_c) &= \int \mathrm{d}\Phi_{n+2}^{d=4} \frac{1}{2!} \left(\frac{1}{\xi_1}\right)_c \left(\frac{1}{\xi_2}\right)_c \xi_1\, \xi_2\, \mathcal{M}_{n+2}^{(1),f} \\
\sigma_{n+3}^{(3)}(\xi_c) &= \int \mathrm{d}\Phi_{n+3}^{d=4} \frac{1}{3!} \left(\frac{1}{\xi_1}\right)_c \left(\frac{1}{\xi_2}\right)_c \left(\frac{1}{\xi_3}\right)_c \xi_1\, \xi_2\, \xi_3\, \mathcal{M}_{n+3}^{(0),f}
\end{aligned}
$$

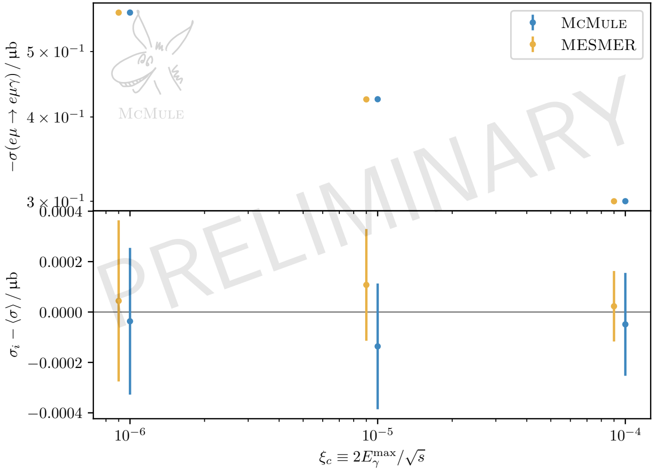
- ②

$$
\sigma^{(3)} = \sigma_{0\gamma,\mathrm{h}}(\omega_s) + \sigma_{1\gamma,\mathrm{h}}(\omega_s) + \sigma_{2\gamma,\mathrm{h}}(\omega_s) + \sigma_{3\gamma,\mathrm{h}}(\omega_s)
$$

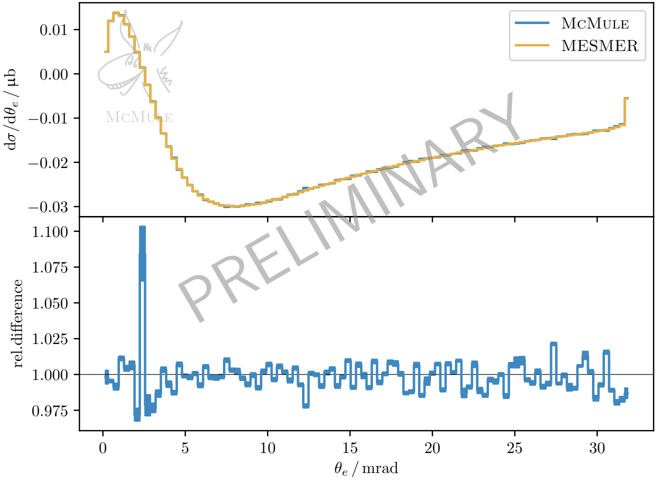
a check for FKS$^\ell$ ($\ell = 2$)



$$\xi_c = \omega_s = 10^{-\{6,5,4\}}$$



comparison FKS vs slicing: θ_e



phase-space integration is done . . .

- FKS$^\ell$: (1) isolate IR behaviour, (2) cancel IR poles analytically before integration, (3) integrate with exact matrix elements
- slicing: (1) split the radiative phase space into soft and hard regions, (2) use eikonal approximation for soft regions, (3) integrate

. . . but don't forget about matrix elements!

- RRR is (hopefully) trivial
- we have discussed RRV (NTS stabilisation for 2 photons?)
- Tim will talk about RVV
- VVV has been already discussed as well