Analytic two-loop amplitudes with finite fields

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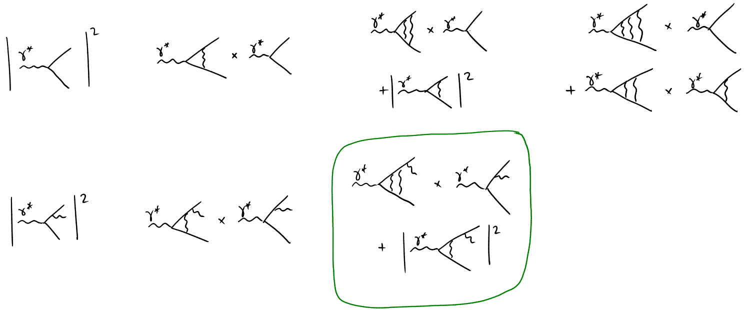
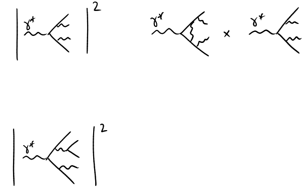
N^{3}LO QED $\gamma^* \to \ell\bar{\ell}$ workstop: RVV

4 Aug 2022

N^{3}LO QED $\gamma^* \to \ell\bar{\ell}$: RVV





- $\gamma^* \to \ell\bar{\ell}\gamma$ at 2-loop?
- Massive ℓ?
- Phenomenology?

Outline

Analytic two-loop amplitudes with finite fields

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Analytic two-loop amplitudes with finite fields

Five-point two-loop massless QCD amplitudes

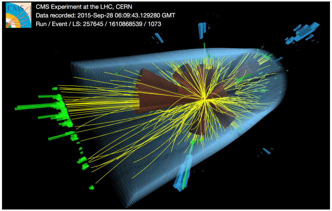
- Major theoretical challenge
  - Precision frontier
  - New methods
- Reconstruct over finite fields

  [Peraro 2019]
- Pentagon function basis

  [Chicherin and Sotnikov 2020]

  [Chicherin, Sotnikov, and Zoia 2021]
- Fast and stable implementations in `NJet`

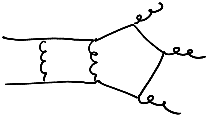
  [Badger, Biedermann, et al. 2021]



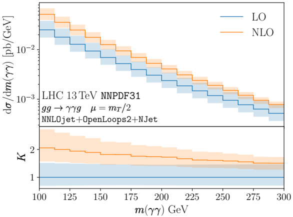
[McCauley et al. 2015]

Five-point two-loop massless QCD amplitudes

- NNLO leading-colour $pp \to jjj$



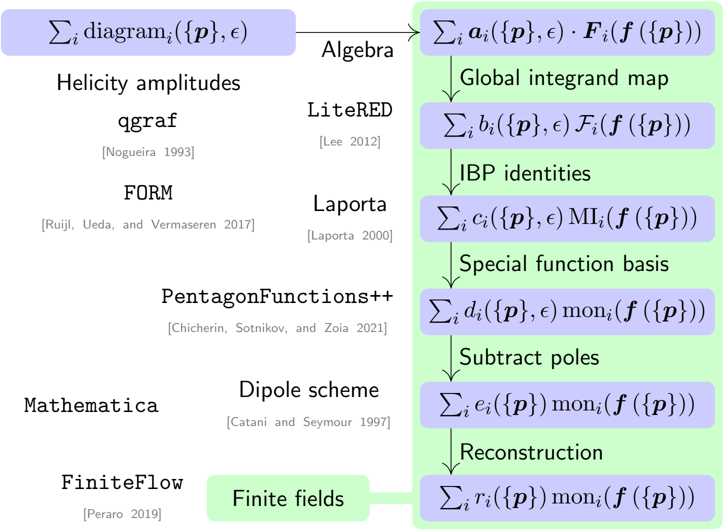
- NLO full-colour $gg \to g\gamma\gamma$ (N^{3}LO $pp \to g\gamma\gamma$)



[Badger, Brønnum-Hansen, et al. 2021]

[Badger, Thomas Gehrmann, Marcoli, and Moodie 2022]

Overview of our method



Outline

Analytic two-loop amplitudes with finite fields

Five-point two-loop massless QCD amplitudes

Finite field arithmetic

Reconstruction methods

Implementation and performance

Masses

Conclusion

Finite field arithmetic

- Set of $n \in \mathbb{P}^p$ non-negative integers

$$\mathbb{F}_n = \{0, \ldots, n-1\}$$

- Arithmetic operations
  - Addition modulo n
  - Multiplication modulo n
  - Inverses
- Modular multiplicative inverse $x = a^{-1} \mod n$

$$ax = 1 \mod n \qquad\qquad a \neq 0$$

- One-to-many map $\mathbb{Q} \to \mathbb{F}_n$

$$\frac{a}{b} \to a\, b^{-1} \mod n$$

Numeric representation for computation

- `float`
  - Catastrophic cancellation in large expressions
- $\mathbb{F}_n$
  - Fixed size integer
  - No precision loss (n large)
- $\{\mathbb{F}_n\} \to \mathbb{Q}$
  - Chinese Remainder Theorem
- Amplitudes
  - Large intermediate expressions
  - Bypass with numerical evaluation over $\mathbb{F}_n$
  - Reconstruct analytic expression

Rational on-shell parametrisation

- Momentum twistor variables

$$\langle ij\rangle, [ij] \to x_i$$

$$Z = \begin{pmatrix} \lambda_1 & \cdots & \lambda_n \\ \mu_1(\tilde{\lambda}_{\mathsf{adj}}) & \cdots & \mu_n(\tilde{\lambda}_{\mathsf{adj}}) \end{pmatrix}$$

- Rational functions
  - $p_i(\boldsymbol{x}), s_{ij}(\boldsymbol{x}), \ldots$
- $\boldsymbol{x}$ unconstrained
  - On-shell
  - Momentum conserving

Outline

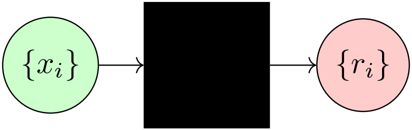
Analytic two-loop amplitudes with finite fields

Reconstruction

- Amplitude components

$$F(\boldsymbol{x}) = \sum_i r_i(\boldsymbol{x}) \, \mathrm{mon}_i(f)$$

- Have numerical algorithm for r_i
- `FiniteFlow` [Peraro 2019]



- Reconstruct expression from sufficient evaluations
- Strategies to optimise and compactify

Linear relations in the coefficients

- Linearise the r_i

$$\{r_i\}_{i\in S} \to \{r_i\}_{i\in T\subseteq S}$$

- Numerically solve:

$$\sum_i a_i\, r_i(\boldsymbol{x}) = 0$$

and choose lowest degrees

- Ansätze, eg permutations of $5g$ r_i for $3g2\gamma$:

$$\sum_i a_i\, r_i(\boldsymbol{x}) + \sum_j b_j\, e_j(\boldsymbol{x}) = 0$$

Matching factors

- Coefficient ansatz

$$r(\boldsymbol{x}) = \frac{n(\boldsymbol{x})}{\prod_k {\ell_k}^{e_k}(\boldsymbol{x})} \qquad \ell_k \in \text{pentagon alphabet}$$

- Determine e_k by reconstructing r on univariate slice

$$\boldsymbol{x} = \boldsymbol{c}_0 + \boldsymbol{c}_1 t \qquad \rightsquigarrow \qquad r(t)$$

  and matching RHS
- Fix denominators

$$\{r_i\} \to \{n_i\}$$

Univariate partial fraction decomposition

- Example in y

$$\frac{N(x,y)}{x^2y^2(x^2+y^2)} = \frac{q_1(x)}{y} + \frac{q_2(x)+q_3(x)y}{y^2} + \frac{q_4(x)+q_5(x)y}{x^2+y^2}$$

  - Needs no knowledge of analytic form of N
  - Choose y by studying one-loop
- Reconstruct r_i directly in decomposed form

$$\{r_i(\overline{\boldsymbol{x}},y)\} \to \{q_i(\overline{\boldsymbol{x}})\}$$

  - Reduce variables by one
  - Lower degrees
- Simplifies reconstruction
  - $\times 10$ point numerical evaluation time (linear fit in $\{q_i\}$ over y)
  - $\times \frac{1}{100}$ samples required

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Five-point two-loop massless QCD amplitudes

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Implementation

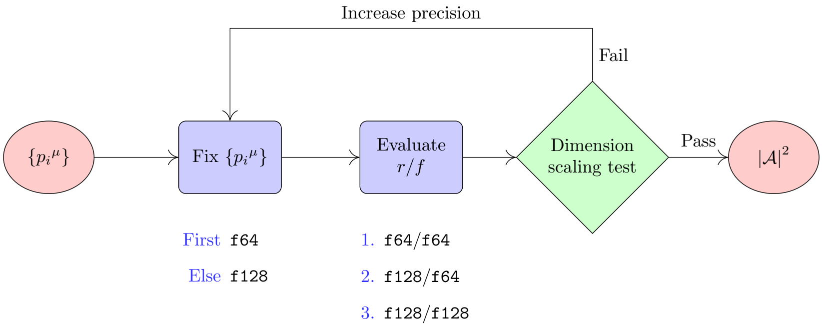
- Finite remainders coded up in `C++` as analytic expressions
- Construct partial amplitudes as

$$F^h = r_i^h(\boldsymbol{x})\, M_{ij}^h \; f_j^h$$

f_j^h	special function monomials	global
M_{ij}^h	rational sparse matrices	partials
r_i^h	independent rational coefficients	helicities
$\boldsymbol{x}$	momentum twistor variables	global

- Independent helicities permuted to all mostly-plus
  - Mostly-minus: ${r_i}^* \, M_{ij} \; P(f_j)$

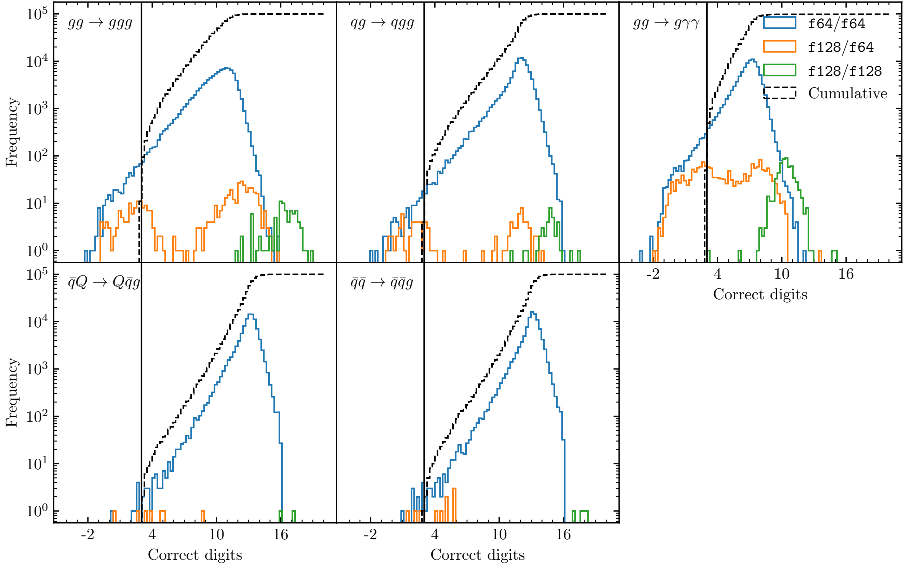
Evaluation strategy



Timing

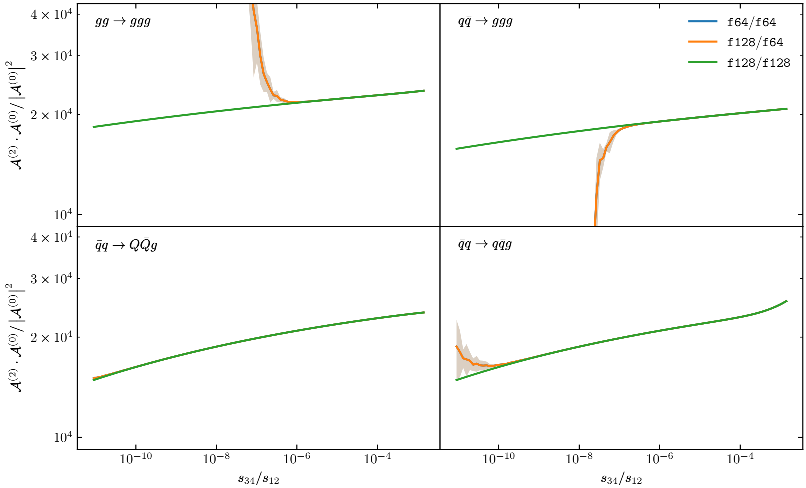
Channel	`f64/f64`		Evaluation strategy	
	Time (s)	s.f. (%)	Time (s)	s.f. (%)
$gg \to ggg$	1.39	69	1.89	77
$gg \to \bar{q}qg$	1.35	91	1.37	91
$gg \to qgg$	1.34	92	1.57	93
$q\bar{q} \to ggg$	1.34	93	1.38	93
$\bar{q}Q \to Q\bar{q}g$	1.14	99	1.16	99
$\bar{q}\bar{Q} \to \bar{q}\bar{Q}g$	1.36	99	1.39	99
$\bar{q}g \to \bar{q}Q\bar{Q}$	1.36	99	1.39	99
$\bar{q}q \to Q\bar{Q}g$	1.14	99	1.14	99
$\bar{q}g \to \bar{q}q\bar{q}$	1.84	99	1.90	99
$\bar{q}\bar{q} \to \bar{q}\bar{q}g$	1.82	99	1.94	99
$\bar{q}q \to q\bar{q}g$	1.71	99	1.77	99
$gg \to \gamma\gamma g$ *	9	99	26	99

Stability



$gg \to ggg$
$qg \to qgg$
$gg \to g\gamma\gamma$
$\bar{q}Q \to Q\bar{q}g$
$\bar{q}\bar{q} \to \bar{q}\bar{q}g$
f64/f64
f128/f64
f128/f128
Cumulative
Frequency
Correct digits
10^0
10^1
10^2
10^3
10^4
10^5
-2
4
10
16

IR performance



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Single external mass at 2-loop (planar)

- Applications of our pipeline

$pp \to Hb\bar{b}$		[Badger, Hartanto, Kryś, and Zoia 2021]
$pp \to Wb\bar{b}$	(on-shell W)	[Badger, Hartanto, and Zoia 2021]
$pp \to W\gamma j$		[Badger, Hartanto, Kryś, and Zoia 2022]
$pp \to W(\ell\nu)b\bar{b}$	$d\sigma$	[Hartanto, Poncelet, Popescu, and Zoia 2022]

- Others

$pp \to (W/Z/\gamma^*)jj$ [Abreu et al. 2022]

- Integrals
  - PentagonFunctions++ [Chicherin, Sotnikov, and Zoia 2021]
  - DiffExp [Hidding 2021]

Internal masses

- Applications of our pipeline (planar)

$gg \to t\bar{t}$	2-loop	[Badger, Chaubey, Hartanto, and Marzucca 2021]
$pp \to t\bar{t}j$	1-loop $\mathcal{O}(\epsilon^2)$	[Badger, Becchetti, et al. 2022]

- Others

$gg \to Hj$	2-loop $d\sigma$	[Bonciani et al. 2022]
$q\bar{q} \to Q\bar{Q}$	2-loop	[Mandal, Mastrolia, Ronca, and Bobadilla Torres 2022]

- Integrals
  - Differential equation [Henn 2013], iterated integrals [Chen 1977]
  - `AMFlow` [Liu and Ma 2022]

Outline

Analytic two-loop amplitudes with finite fields

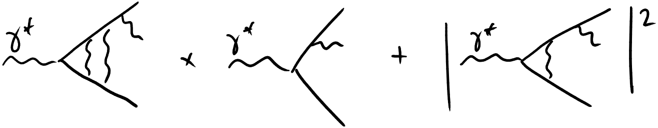
Conclusion

- Two-loop amplitude computation with finite fields
  - Established for massless to $2 \to 3$
  - First massive results appearing
- Promising outlook for RVV $\gamma^* \to \ell\bar{\ell}$
- Integrals

$m_l = 0$ Literature [T. Gehrmann and Remiddi 2001a; T. Gehrmann and Remiddi 2001b]

$m_l \neq 0$ `pySecDec` [Borowka et al. 2018]

`DiffExp` [Hidding 2021]



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